p-adic Hodge Theory (Spring 2023): Week 11

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This week: crystalline Representations
The goal is to study the period ring B_{cris} and crystalline representations. So far, the only result we assumed was the Tate–Sen Theorem (and one smaller result about the new topology on B_{dR}^+). In this section, we will starting assumingmore results without proof.

1 The crystalline period ring B_{cris}

Throughout this section, we write $W(k)$ for the ring of Witt vectors over k, and K_0 for its fraction field. Recall that we have fixed an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$ and set $\xi = [p^\flat] - p \in A_{inf}$.

Definition 1.1. *We define the integral crystalline period ring by*

$$A_{cris} : \{\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \in B_{dR}^+ : a_n \in A_{inf} \text{ with } \lim_{n\to\infty} a_n = 0\},$$

and write $B_{cris}^+ := A_{cris}[1/p]$.

Remark. In the definition of A_{cris} above, it is vital to consider the refinement of the discrete valuation topology on $B+_{dR}$. While the convergence of the infinite sum $\sum_{n\geq 0} a_n \frac{\xi_n}{n!}$ relies on the discrete valuation topology on B_{dR}^+, the limit of the coefficients an should be taken with respect to the p-adic topology on A_{inf}.

We warn the readers that the terminology given in Definition 1.1 is not standard at all. In fact, most authors do not give a separate name for the ring A_{cris}. Our choice of the terminology comes from the fact that A_{cris} plays the role of the crystalline period ring in the integral p-adic Hodge theory.

Proposition 1.2. *We have* $t \in A_{cris}$ *and* $t^{p-1} \in pA_{cris}$.

Proof. By previous lemma we write $[\varepsilon] - 1 = \xi c$ for some $c \in A_{inf}$. Then we obtain

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! c^n \cdot \frac{\xi^n}{n!}. \tag{1}$$

We thus find $t \in A_{cris}$ as we have $\lim_{n\to\infty} (n-1)! c^n = 0$ in A_{inf} relative to the p-adic topology.

It remains to show $t^{p-1} \in pA_{cris}$. Let us set

$$\breve{t} := \sum_{n=1}^{p} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}. \tag{2}$$

Since $(n-1)!$ is divisible by p for all $n > p$, we find $t - \breve{t} \in pA_{cris}$ by (1). Hence it suffices to prove $\breve{t}^{p-1} \in pA_{cris}$.

The terms for $n < p$ in (2) are all divisible by $[\varepsilon] - 1$ in A_{cris}, whereas the term for $n = p$ in (2) can be written as

$$(-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \cdot ([\varepsilon] - 1).$$

In other words, we may write

$$\breve{t} = ([\varepsilon]-1) \left(a + (-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \right)$$

for some $a \in A_{cris}$. It is therefore enough to show $([\varepsilon]-1)^{p-1} \in pA_{cris}$. In addition, by we have

$$\nu^\flat((\varepsilon-1)^{p-1}) = p = \nu^\flat((p^\flat)^p).$$

and consequently find that $[(\varepsilon-1)^{p-1}]$ is divisible by $[p^\flat]^p = (\xi+p)^p$. We thus deduce the desired assertion by observing that $\xi^p = p \cdot (p-1)! \cdot (\xi^p/p!)$ is divisible by p in A_{cris}. □

Remark. As a consequence, we find that

$$\frac{t^p}{p!} = \frac{t^{p-1}}{p} \cdot \frac{t}{(p-1)!} \in A_{cris}.$$

In fact, it is not hard to prove that for every $a \in A_{cris}$ with $\theta_{dR}^+(a) = 0$ we have a $n/n! \in A_{cris}$ for all $n \geq 1$.

Corollary 1.3. *We have an identification* $B_{cris}^+[1/t] = A_{cris}[1/t]$.

Proof. Proposition 1.2 implies that p is a unit in $A_{cris}[1/t]$, thereby yielding $B_{cris}^+[1/t] = A_{cris}[1/p, 1/t] = A_{cris}[1/t]$, as desired. □

Definition 1.4. *We define the crystalline period ring by*

$$B_{cris} := B_{cris}^+[1/t] = A_{cris}[1/t]$$

Remark. Let us briefly explain Fontaine's insight behind the construction of B_{cris}. The main motivation for constructing the crystalline period ring B_{cris} is to obtain the Grothendieck mysterious functor as described in Chapter I, Conjecture 1.2.3. Recall that, for a proper smooth variety $\mathcal{X}$ over K with a proper smooth integral model X over $\mathcal{O}_K$, the crystalline cohomology $H_{cris}^n(\mathcal{X}_k, W(k))$ admits a natural Frobenius action and refines the de Rham cohomology $H_d^n R(X/K)$ via a canonical isomorphism

$$H_{cris}^n(\mathcal{X}_k, W(k))[1/p] \otimes_{K_0} K \cong H_{dR}^n(X/K).$$

In addition, since A_{inf} is by construction the ring of Witt vectors over a perfect $\mathbb{F}_p$-algebra $\mathcal{O}_F$, it admits the Frobenius automorphism φ_{inf} as noted in Chapter II, Example 2.3.2. Fontaine sought to construct B_{cris} as a sufficiently large subring of B_{dR} on which φ_{inf} naturally extends. For B_{dR} there is no natural extension of φ_{inf} since $\ker(\theta[1/p])$ is not stable under φ_{inf}. Fontaine's key observation is that by adjoining to A_{inf} the elements of the form $\xi^n/n!$ for $n \geq 1$ we obtain a subring of $A_{inf}[1/p]$ such that the image of $\ker(\theta[1/p])$ is stable under φ_{inf}. This observation led Fontaine to consider the ring A_{cris} defined in Definition 1.1. The only issue with A_{cris} is that it is not $(\mathbb{Q}_p, \Gamma_K)$-regular, which turns out to be resolved by considering the ring $B_{cris} = A_{cris}[1/t]$.

Proposition 1.5. *The ring* B_{cris} *is naturally a filtered subalgebra of* B_{dR} *over* K_0 *which is stable under the action of* Γ_K.

Proof. By construction we have

$$A_{inf}[1/p] \subseteq A_{cris}[1/p] = B_{cris}^+ \subseteq B_{cris} \subseteq B_{dR}.$$

In addition, the proof of Proposition 2.2.15 yields a unique homomorphism $K \to B_{dR}$ which extends a natural homomorphism $K_0 \to A_{inf}[1/p]$. Hence by Example 2.3.2 we naturally identify Bcris as a filtered subalgebra of B_{dR} over K_0 with $Fil^n(B_{cris}) := B_{cris} \cap t^n B_{dR}^+$.

It remains to show that $B_{cris} = A_{cris}[1/t]$ is stable under the action of Γ_K. Since Γ_K acts on t by the cyclotomic character as noted in Theorem 2.2.21, we only need to show that A_{cris} is stable under the action of Γ_K. Consider an arbitrary element $\gamma \in \Gamma_K$ and an arbitrary sequence (a_n) in A_{inf} with $\lim_{n\to\infty} a_n = 0$. Since $\ker(\theta)$ is stable under the Γ_K-action as noted in Theorem 2.2.21, we may write $\gamma(\xi) = c_\gamma \xi$ for some $c_\gamma \in A_{inf}$ by Proposition 2.2.6. We then have $\lim_{n\to\infty} \gamma(a_n)c_\gamma^n = 0$ as the Γ_K-action on A_{inf} is evidently continuous with respect to the p-adic topology. Hence we find

$$\gamma\left(\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\right) = \sum_{n=0}^{\infty} \gamma(a_n) c_\gamma^n \frac{\xi^n}{n!} \in A_{cris}$$

as desired.

□

Remark. We provide a functorial perspective for the Γ_K-actions on B_{cris} and B_{dR} which can be useful in many occasions. Since the definitions of B_{cris} and B_{dR} only depend on the valued field $\mathbb{C}_K$, we may regard B_{cris} and B_{dR} as functors which associate topological rings to each complete and algebraically closed valued field. Then by functoriality the action of Γ_K on $\mathbb{C}_K$ induces the actions of Γ_K on B_{cris} and B_{dR}. In particular, since B_{cris} is a subfunctor of B_{dR} we deduce that the Γ_K-action on B_{cris} is given by the restriction of the Γ_K-action on B_{dR} as asserted in Proposition 1.5.

We also warn that $Fil^\circ(B_{cris}) = B_{cris} \cap B_{dR}^+$ is not equal to B_{cris}^+. For example, the element

$$\alpha + \frac{[\varepsilon^{1/p^2}] - 1}{[\varepsilon^{1/p}] - 1}$$

lies in $B_{cris} \cap B_{dR}^+$ but not in B_{cris}^+.

In order to study the Γ_K-action on B_{cris} we invoke the following crucial (and surprisingly technical) result without proof.

Proposition 1.6. *The natural Γ_K-equivariant map $B_{cris} \otimes_{K_0} K \to B_{dR}$ is injective.*

Remark. The original proof by Fontaine in [Fon94] is incorrect. A complete proof involving the semistable period ring can be found in Fontaine and Ouyang's notes [FO, Theorem 6.14]. Note however that the assertion is obvious if we have $K = K_0$, which amounts to the condition that K is unramified over $\mathbb{Q}_p$.

Proposition 1.7. *There exists a natural isomorphism of graded K-algebras*

$$\mathrm{gr}(B_{cris} \otimes_{K_0} K) \cong \mathrm{gr}(B_{dR}) \cong B_{HT}.$$

Proof. We only need to establish the first identification as the second identification immediately follows from Theorem 2.2.21 as noted in Example 2.3.2. By Proposition 1.6 the natural map $B_{cris} \otimes_{K_0} K \to B_{dR}$ induces an injective morphism of graded K-algebras

$$\mathrm{gr}(B_{cris} \otimes_{K_0} K) \hookrightarrow \mathrm{gr}(B_{dR}). \tag{3}$$

In particular, we have an injective map

$$\mathrm{gr}^\circ(B_{cris} \otimes_{K_0} K) \hookrightarrow \mathrm{gr}^\circ(B_{dR}) \cong \mathbb{C}_K$$

where the isomorphism is induced by θ_{dR}^+. Moreover, this map is surjective since the image of $B_{cris} \otimes_{K_0} K$ in B_{dR} contains $A_{inf}[1/p]$ and consequently maps onto $\mathbb{C}_K$ by θ_{dR}^+. Therefore we obtain an isomorphism

$$\mathrm{gr}^\circ(B_{cris} \otimes_{K_0} K) \cong \mathrm{gr}^\circ(B_{dR}) \cong \mathbb{C}_K$$

This implies that each $\mathrm{gr}^n(B_{cris} \otimes_{K_0} K)$ is a vector space over $\mathbb{C}_K$. Moreover, each $\mathrm{gr}^n(B_{cris} \otimes_{K_0} K)$ contains a nonzero element given by $t^n \otimes 1$. Hence the injective map (3) must be an isomorphism since each $\mathrm{gr}^n(B_{dR})$ has dimension 1 over $\mathbb{C}_K$. □

Theorem 1.8 (Fontaine (Fon94))**.** *The ring B_{cris} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{cris}^{\Gamma_K} \cong K_0$.*

Proof. Let C_{cris} denote the fraction field of B_{cris}. Proposition 1.5 implies that C_{cris} is a subfield of B_{dR} which is stable under the action of Γ_K. Hence we have $K_0 \subseteq B_{cris}^{\Gamma_K} \subseteq C_{cris}^{\Gamma_K}$.

Then Proposition 3.1.6 and Theorem 2.2.21 together yield injective maps

$$B_{cris}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{dR}^{\Gamma_K} \cong K$$

and

$$C_{cris}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{dR}^{\Gamma_K} \cong K$$

thereby implying $K_0 = B_{cris}^{\Gamma_K} = C_{cris}^{\Gamma_K}$.

It remains to check the condition (ii) in Definition 1.1.1. Consider an arbitrary nonzero element $b \in B_{cris}$ on which Γ_K acts via a character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$. We wish to show that b is a unit in B_{cris}.

By Proposition 2.2.19 we may write $b = t^i b'$ for some $b' \in (B_{dR}^+)^\times$ and $i \in \mathbb{Z}$. Since t is a unit in B_{cris} by construction, the element b is a unit in B_{cris} if and only if b' is a unit in B_{cris}. Moreover, Theorem

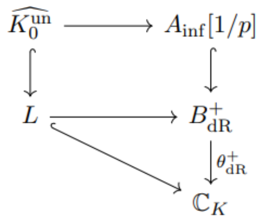
2.2.21 implies that Γ_K acts on $b' = b \cdot t^{-i}$ via the character $\eta\chi^{-i}$. Hence we may replace b by b' to assume that b is a unit in B_{dR}^+.

Since θ_{dR}^+ is Γ_K-equivariant as noted in Theorem 2.2.21, the action of Γ_K on $\theta_{dR}^+(b) \in \mathbb{C}_K$ is given by the character η. Then by the continuity of the Γ_K-action on $\mathbb{C}_K$ we find that η is continuous. Therefore we may consider η as a character with values in $\mathbb{Z}_p^\times$. Moreover, we have $\theta_{dR}^+(b) \neq 0$ as b is assumed to be a unit in B_{dR}^+. Hence Theorem 1.1.8 implies that $\eta^{-1}(I_K)$ is finite.

Let us denote by K^{un} the maximal unramified extension of K in $\overline{K}$, and by $\widehat{K^{un}}$ the p-adic completion of K^{un}. By definition $\widehat{K^{un}}$ is a p-adic field with I_K as the absolute Galois group.

Therefore by our discussion in the preceding paragraph there exists a finite extension L of $\widehat{K^{un}}$ with the absolute Galois group Γ_L such that η^{-1} becomes trivial on $\Gamma_L \subseteq I_K$. Since Γ_K acts on $\theta_{dR}^+(b)$ via η, we find $\theta_{dR}^+(b) \in C_K^{\Gamma_L} = C_L^{\Gamma_L} = L$.

Let us write $W(k)$ for the ring of Witt vectors over k, and $\widehat{K_0^{un}}$ for the fraction field of $W(k)$. Proposition 2.2.15 yields a commutative diagram



where all maps are Γ_K-equivariant. Moreover, both horizontal maps are injective as $\widehat{K_0^{un}}$ and L are fields. We henceforth identify $\widehat{K_0^{un}}$ and L with their images in B_{dR}. Then we have

$$\widehat{K_0^{un}} \subseteq A_{inf}[1/p] \subseteq B_{cris} \tag{4}$$

We assert that b lies in (the image of) L. Let us write $\widehat{b} := \theta_{dR}^+(b)$. If suffices to show $b = \hat{b}$. Suppose for contradiction that b and $\hat{b}$ are distinct. Since we have $\theta_{dR}^+(\hat{b}) = \hat{b} = \theta_{dR}^+(b)$ by the commutativity of the diagram above, we may write $b - \hat{b} = t^j u$ for some $j > 0$ and $u \in (B_{dR}^+)^\times$. Moreover, we find

$$\gamma(b - \hat{b}) = \gamma(b) - \gamma(\hat{b}) = \eta(\gamma)(b - \hat{b})$$

for every $\gamma \in \Gamma_K$.

Then under the Γ_K-equivariant isomorphism

$$t^j B_{dR}^+ / t^{j+1} B_{dR}^+ \cong \mathbb{C}_K(j)$$

given by Theorem 2.2.21, the element $b - \hat{b} \in t^j B_{dR}^+$ yields a nonzero element in $\mathbb{C}_K(j)$ on which Γ_K acts via the character η. Therefore Theorem 1.1.8 implies that $(\chi^j \eta^{-1})(I_K)$ is finite. Since $\eta^{-1}(I_K)$ is also finite as noted above, we deduce that $\chi^j(I_K)$ is finite as well, thereby obtaining a desired contradiction by Lemma 1.1.7.

Let us now regard b as an element in L. Proposition 2.2.15 implies that L is a finite extension of $\widehat{K_0^{un}}$. Hence we can choose a minimal polynomial equation

$$b^d + a_1 b^{d-1} + \cdots + a_{d-1} b + a_d = 0$$

with $a_n \in \widehat{K_0^{un}}$.

Since the minimality of the equation implies $a_d \neq 0$, we obtain an expression

$$b^{-1} = -a_d^{-1}(b^{d-1} + a_1 b^{d-2} + \cdots + a_{d-1}).$$

We then find $b^{-1} \in B_{cris}$ by (4), thereby completing the proof.

□

Proposition 1.9. *Let A^0_{cris} be the A_{inf}-subalgebra in $A_{inf}[1/p]$ generated by the elements of the form $\xi^n/n!$ with $n \geq 0$.*

1. *The ring A_{cris} is naturally identified with the p-adic completion of A^0_{cris}.*
2. *The action of Γ_K on A_{cris} is continuous.*

Lemma 1.10. *The Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B^+_{cris}.*

Proof. The Frobenius automorphism of A_{inf} uniquely extends to an automorphism on $A_{inf}[1/p]$, which we denote by φ_{inf}. By construction we have

$$\varphi_{inf}(\xi) = [(p^\flat)^p] - p = [p^\flat]^p - p = (\xi + p)^p - p. \tag{5}$$

Hence we may write $\varphi_{inf}(\xi) = \xi^p + pc$ for some $c \in A_{inf}$. Let us define A^0_{cris} as in Proposition 1.9. Then we have

$$\varphi_{inf}(\xi) = p \cdot (c + (p-1)! \cdot (\xi^p/p!)),$$

and consequently find

$$\varphi_{inf}(\xi^n/n!) = (p^n/n!) \cdot (c + (p-1)! \cdot (\xi^p/p!))^n \in A^0_{cris} \quad \text{for all } n \geq 1$$

by observing that $p^n/n!$ is an element of $\mathbb{Z}_p$. Hence A^0_{cris} is stable under φ_{inf}. Moreover, by construction φ_{inf} is Γ_K-equivariant and continuous on $A_{inf}[1/p]$ with respect to the p-adic topology. We thus deduce by Proposition 1.9 that the endomorphism φ_{inf} on A^0_{cris} uniquely extends to a continuous Γ_K-equivariant endomorphism φ^+ on $B^+_{cris} = A_{cris}[1/p]$. □

Remark. The identity (5) shows that $\varphi_{inf}(\xi)$ is not divisible by ξ, which implies that $\ker(\theta)$ is not stable under φ_{inf}. Hence the endomorphism φ^+ on B^+_{cris} (or the Frobenius endomorphism on B_{cris} that we are about to construct) is not compatible with the filtration on B_{dR}.

Proposition 1.11. *The Frobenius automorphism of A_{inf} naturally extends to a Γ_K-equivariant endomorphism φ on B_{cris} with $\varphi(t) = pt$.*

Proof. As noted in Lemma 1.10, the Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B^+_{cris}. In addition, the proof of Proposition 1.2 shows that the power series expression

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n}$$

converges with respect to the p-adic topology in A_{cris}. Hence we use the continuity of φ^+ on A_{cris} to find that

$$\varphi^+(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon) = pt.$$

Since Γ_K acts on t via χ, it follows that φ^+ uniquely extends to a Γ_K-equivariant endomorphism φ on $B_{cris} = B^+_{cris}[1/t]$. □

Remark. The endomorphism φ is not continuous on B_{cris}, even though it is a unique extension of the continuous endomorphism φ^+ on B^+_{cris}. The issue is that the natural topology on B^+_{cris} induced by the p-adic topology on A_{cris} does not agree with the subspace topology inherited from B_{cris}.

Definition 1.12. *We refer to the endomorphism φ in Proposition 1.11 as the Frobenius endomorphism of B_{cris}. We also write*

$$B_e := \{b \in B_{cris} : \varphi(b) = b\}$$

for the ring of Frobenius-invariant elements in B_{cris}.

We close this subsection by stating two fundamental results about φ without proof.

Theorem 1.13. *The Frobenius endomorphism φ of B_{cris} is injective.*

Theorem 1.14. *The natural sequence*

$$0 \to \mathbb{Q}_p \to B_e \to B_{dR}/B^+_{dR} \to 0$$

is exact.

Definition 1.15. *We refer to the exact sequence in Theorem 1.14 as the fundamental exact sequence of p-adic Hodge theory.*