p-adic Hodge Theory (Spring 2023): Week 10

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This week: de Rham Representations

1 Filtered Vector Spaces

In this subsection we set up a categorical framework for our discussion of B_{dR}-admissible representations in the next subsection.

Definition 1.1. *Let L be an arbitrary field.*

1. *A filtered vector space over L is a vector space V over L along with a collection of subspaces $\{Fil^n(V)\}_{n\in\mathbb{Z}}$ that satisfies the following properties:*
   (a) *$Fil^n(V) \supset Fil^{n+1}(V)$ for every $n \in \mathbb{Z}$.*
   (b) *$\bigcap_{n\in\mathbb{Z}} Fil^n(V) = 0$ and $\bigcup_{n\in\mathbb{Z}} Fil^n(V) = V$.*
2. *A graded vector space over L is a vector space V over L along with a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$.*
3. *A L-linear map between two filtered vector spaces V and W over L is called a morphism of filtered vector spaces if it maps each $Fil^n(V)$ into $Fil^n(W)$.*
4. *A L-linear map between two graded vector spaces $V = \bigoplus_{n\in\mathbb{Z}} V_n$ and $W = \bigoplus_{n\in\mathbb{Z}} W_n$ over L is called a morphism of graded vector spaces if it maps each V_n into W_n.*
5. *For a filtered vector space V over L, we define its associated graded vector space by*
$$gr(V) := \bigoplus_{n\in\mathbb{Z}} Fil^n(V)/Fil^{n+1}(V)$$
   and write $gr^n(V) := Fil^n(V)/Fil^{n+1}(V)$ for every $n \in \mathbb{Z}$.
6. *We denote by Fil_L the category of finite dimensional filtered vector spaces over L.*

Example 1.2. *We present some motivating examples for our discussion*

1. *The ring B_{dR} is a filtered K-algebra with $Fil^n(B_dR) := t^n B_{dR}^+$ and $gr(B_{dR}) \cong B_{HT}$*
2. *For a proper smooth variety X over K, the de Rham cohomology $H^n_{dR}(X/K)$ with the Hodge filtration is a filtered vector space over K whose associated graded vector space recovers the Hodge cohomology.*
3. *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, we may regard $D_{B_{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K}$ as a filtered vector space over K with*
$$Fil^n(D_{B_{dR}}(V)) := (V \otimes_{\mathbb{Q}_p} t^n B_{dR}^+)^{\Gamma_K}.$$

For an arbitrary proper smooth variety X over K, we have a canonical Γ_K equivariant isomorphism of filtered vector spaces
$$D_{B_{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{dR}(X/K)$$

Lemma 1.3. *Let V be a finite dimensional filtered vector space over a field L. There exists a basis $(v_{i,j})$ for V such that for every $n \in \mathbb{Z}$ the vectors $v_{i,j}$ with $i \geq n$ form a basis for $Fil^n(V)$.*

Definition 1.4. *Let L be an arbitrary field.*

1. *Given two filtered vector spaces V and W over L, we define the convolution filtration on $V \otimes_L W$ by*

$$Fil^n(V \otimes_L W) := \sum_{i+j=n} Fil^i(V) \otimes_L Fil^j(W).$$

2. *For every filtered vector space V over L, we define the dual filtration on the dual space $V^\vee = \mathrm{Hom}_L(V, L)$ by*

$$Fil^n(V^\vee) := \{f \in V^\vee : Fil^{1-n}(V) \subseteq \ker(f)\}$$

3. *We define the unit object $L[0]$ in Fil_L to be the vector space L with the filtration*

$$Fil^n(L[0]) \begin{cases} L & \text{if } n \leq 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Remark. The use of $Fil^{1-n}(V)$ rather than $Fil^{-n}(V)$ in the (2) above is to ensure that $L[0]$ is self-dual.

Proposition 1.5. *Let V be a filtered vector space over a field L. Then we have canonical isomorphisms of filtered vector spaces*

$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V \quad \text{and} \quad (V^\vee)^\vee \cong V$$

Proof. For every $n \in \mathbb{Z}$ we find

$$Fil^n(V \otimes_L L[0]) = \sum_{i+j=n} Fil^i(V) \otimes_L Fil^j(L[0]) \cong \sum_{i \geq n} Fil^i(V) = Fil^n(V),$$

and consequently obtain an identification of filtered vector spaces

$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V.$$

Moreover, the natural evaluation isomorphism $\epsilon : V \cong (V^\vee)^\vee$ yields an isomorphism of filtered vector spaces since for every $n \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Fil}^n\left((V^\vee)^\vee\right) &\cong \left\{\, v \in V : \mathrm{Fil}^{1-n}(V^\vee) \subseteq \ker(\epsilon(v)) \,\right\} \\ &= \left\{\, v \in V : f(v) = 0 \text{ for all } f \in \mathrm{Fil}^{1-n}(V^\vee) \,\right\} \\ &= \left\{\, v \in V : f(v) = 0 \text{ for all } f \in V^\vee \text{ with } \mathrm{Fil}^n(V) \subseteq \ker(f) \,\right\} \\ &= \mathrm{Fil}^n(V). \end{aligned}$$

Therefore we complete the proof. ☐

Proposition 1.6. *Let V and W be finite dimensional filtered vector spaces over a field L. Then we have a natural identification of filtered vector space*

$$(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee.$$

Proof. By Lemma 1.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i \geq n}$ and $(w_{j,l})_{j \geq n}$ respectively form bases for $Fil^n(V)$ and $Fil^n(W)$. Let $(f_{i,k})$ and $(g_{j,l})$ be the dual bases for $V^\vee$ and $W^\vee$. Then the vectors $(f_{i,k} \otimes g_{j,l})$ form a basis for the vector space $(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee$. Moreover, for every $n \in \mathbb{Z}$ the vectors $(f_{i,k})_{i \leq -n}$ and $(g_{j,l})_{j \leq -n}$ respectively form bases for $Fil^n(V^\vee)$ and $Fil^n(W^\vee)$. Hence we find that for every $n \in \mathbb{Z}$ both $Fil^n((V \otimes_L W)^\vee)$ and $Fil^n(V^\vee \otimes_L W^\vee)$ are spanned by the vectors $(f_{i,k} \otimes g_{j,l})_{i+j \leq -n}$, thereby deducing the desired assertion. ☐

Lemma 1.7. *Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $W = \bigoplus_{n \in \mathbb{Z}} W_n$ be graded vector spaces over a field L. A morphism $f : V \to W$ of graded vector spaces is an isomorphism if and only if it is bijective.*

Proof. The assertion immediately follows by observing that f is the direct sum of the induced morphisms $f_n : V_n \to W_n$. ☐

Proposition 1.8. *Let L be an arbitrary field. A bijective morphism $f : V \to W$ in Fil_L is an isomorphism in Fil_L if and only if the induced map $gr(f) : gr(V) \to gr(W)$ is bijective.*

Proof. If f is an isomorphism of filtered vector spaces, then $gr(f)$ is clearly an isomorphism. Let us now assume that $gr(f)$ is an isomorphism. We wish to show that for every $n \in \mathbb{Z}$ the induced map $Fil^n(f) : Fil^n(V) \to Fil^n(W)$ is an isomorphism. Since each $Fil^n(f)$ is injective by the bijectivity of f, it suffices to show

$$\dim_L Fil^n(V) = \dim_L Fil^n(W) \text{ for every } n \in \mathbb{Z}.$$

The map $gr(f)$ is an isomorphism of graded vector spaces by Lemma 1.7, and consequently induces an isomorphism

$$gr^n(V) \cong gr^n(W) \text{ for every } n \in \mathbb{Z}.$$

for every $n \in \mathbb{Z}$ we find

$$\dim_L Fil^n(V) = \sum_{i \geq n} \dim_L gr^i(V) = \sum_{i \geq n} \dim_L gr^i(W) = \dim_L Fil^n(W)$$

as desired. □

Example 1.9. *Let us define $L[1]$ to be the vector space L with the filtration*

$$Fil^n(L[1]) := \begin{cases} L & \text{if } n \leq 1, \\ 0 & \text{if } n > 1. \end{cases}$$

The bijective morphism $L[0] \to L[1]$ given by the identity map on L is not an isomorphism in Fil_L since $Fil^1(L[0]) = 0$ and $Fil^1(L[1]) = L$ are not isomorphic. Moreover, the induced map $gr(L[0]) \to gr(L[1])$ is a zero map.

Proposition 1.10. *Let L be an arbitrary field. For any $V, W \in Fil_L$ there exists a natural isomorphism of graded vector spaces*

$$gr(V \otimes_L W) \cong gr(V) \otimes_L gr(W).$$

Proof. Since we have a direct sum decomposition

$$gr(V) \otimes_L gr(W) = \bigoplus \left(\bigoplus_{i+j=n} gr^i(V) \otimes_L gr^j(W) \right),$$

it suffices to find a natural isomorphism

$$gr^n(V \otimes_L W) \cong \bigoplus_{i+j=n} gr^i(V) \otimes_L gr^j(W) \text{ for every } n \in \mathbb{Z}.$$

By Lemma 1.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i \geq n}$ and $(w_{j,l})_{j \geq n}$ respectively span $Fil^n(V)$ and $Fil^n(W)$. Let $v_{i,k}$ denote the image of $v_{i,k}$ under the map $Fil^i(V) \twoheadrightarrow gr^i(V)$, and let $w_{j,l}$ denote the image of $w_{j,l}$ under the map $Fil^j(W) \twoheadrightarrow gr^j(W)$. Since each $Fil^n(V \otimes_L W)$ is spanned by the vectors $(v_{i,k} \otimes w_{j,l})_{i+j \geq n}$, we obtain the identification (2.7) by observing that both sides are spanned by the vectors $(v_{i,k} \otimes w_{j,l})_{i+j=n}$. □

2 Properties of de Rham representations

Definition 2.1. *We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is de Rham if it is B_{dR}-admissible. We write $\mathrm{Rep}^{dR}_{\mathbb{Q}_p}(\Gamma_K) := \mathrm{Rep}^{B_{dR}}_{\mathbb{Q}_p}(\Gamma_K)$ for the category of de Rham p-adic Γ_K-representations. In addition, we write D_{HT} and D_{dR} respectively for the functors $D_{B_{HT}}$ and $D_{B_{dR}}$.*

Example 2.2. *Below are some important examples of de Rham representations.*

1. *For every $n \in \mathbb{Z}$ the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is de Rham; indeed, the inequality*

$$\dim_K D_{dR}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$

is an equality, as $D_{dR}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$.

2. *Every $\mathbb{C}_K$-admissible representation is de Rham by a result of Sen.*

3. *For every proper smooth variety X over K, the étale cohomology $H^n_{et}(XK, Qp)$ is de Rham by a theorem of Faltings as briefly discussed in Chapter I, Theorem 1.2.2.*

The general formalism discussed in §1 readily yields a number of nice properties for de Rham representations and the functor D_{dR}. Our main goal in this subsection is to extend these properties in order to incorporate the additional structures induced by the filtration $\{t^n B_{dR}^+\}_{n\in\mathbb{Z}}$ on B_{dR}.

Lemma 2.3. *Given any $n \in \mathbb{Z}$, every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is de Rham if and only if $V(n)$ is de Rham.*

Proof. Since we have identifications

$$V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n) \quad \text{and} \quad V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n),$$

the assertion follows from Proposition 1.2.4 and the fact that every Tate twist of $\mathbb{Q}_p$ is de Rham as noted in Example 2.4.2. □

Proposition 2.4. *Let V be a de Rham representation of Γ_K. Then V is Hodge-Tate with a natural K-linear isomorphism of graded vector spaces*

$$gr(D_{dR}(V)) \cong D_{HT}(V).$$

Proof. For every $n \in \mathbb{Z}$ we have a short exact sequence

$$0 \to t^{n+1}B_{dR}^+ \to t^n B_{dR}^+ \to t^n B_{dR}^+/t^{n+1}B_{dR}^+ \to 0,$$

which induces an exact sequence

$$0 \to (V \otimes_{\mathbb{Q}_p} t^{n+1}B_{dR}^+)^{\Gamma_K} \to (V \otimes_{\mathbb{Q}_p} t^n B_{dR}^+)^{\Gamma_K} \to (V \otimes_{\mathbb{Q}_p} (t^n B_{dR}^+/t^{n+1}B_{dR}^+))^{\Gamma_K}$$

Therefore we obtain an injective K-linear map of graded vector spaces

$$gr^n(D_{dR}(V)) = Fil^n(D_{dR}(V))/Fil^{n+1}(D_{dR}(V)) \hookrightarrow (V \otimes_{\mathbb{Q}_p} (t^n B_{dR}^+/t^{n+1}B_{dR}^+))^{\Gamma_K}.$$

and consequently yields an injective K-linear map

$$gr(D_{dR}(V)) \hookrightarrow \bigoplus_{n\in\mathbb{Z}} (V \otimes_{\mathbb{Q}_p} (t^n B_{dR}^+/t^{n+1}B_{dR}^+))^{\Gamma_K} \cong (V \otimes_{\mathbb{Q}_p} B_{HT})^{\Gamma_K} = D_{HT}(V)$$

where the middle isomorphism follows from Theorem 2.2.21. We then find

$$\dim_K D_{dR}(V) = \dim_K gr(D_{dR}(V)) \le \dim_K D_{HT}(V) \le \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1. Since V is de Rham, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 2.5. *Let V be an extension of $\mathbb{Q}_p(m)$ by $\mathbb{Q}_p(n)$ with $m < n$. We assert that V is de Rham. By Lemma 2.3 we may assume $m = 0$. Then we have a short exact sequence*

$$0 \to \mathbb{Q}_p(n) \to V \to \mathbb{Q}_p \to 0. \tag{1}$$

Since the functor D_{dR} is left exact by construction, we obtain a left exact sequence

$$0 \to D_{dR}(\mathbb{Q}_p(n)) \to D_{dR}(V) \to D_{dR}(\mathbb{Q}_p).$$

We wish to show $\dim_K D_{dR}(V) = \dim_{\mathbb{Q}_p} V = 2$. Since we have

$$\dim_K D_{dR}(\mathbb{Q}_p(n)) = \dim_K D_{dR}(\mathbb{Q}_p) = 1$$

by Example 2.2, it suffices to show the surjectivity of the map $D_{dR}(V) \to D_{dR}(\mathbb{Q}_p) \cong K$. As B_{dR}^+ is faithfully flat over $\mathbb{Q}_p$, the sequence (1) yields a short exact sequence

$$0 \to \mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{dR}^+ \to V \otimes_{\mathbb{Q}_p} B_{dR}^+ \to \mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{dR}^+ \to 0.$$

In addition, we have identifications

$$(\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{dR}^+)^{\Gamma_K} \cong (t^n B_{dR}^+)^{\Gamma_K} = 0,$$

$$(\mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{dR}^+) \cong (B_{dR}^+)^{\Gamma_K} \cong K.$$

We thus obtain a long exact sequence

$$0 \to 0 \to (V \otimes_{\mathbb{Q}_p} B_{dR}^+)^{\Gamma_K} \to K \to H^1(\Gamma_K, t^n B_{dR}^+).$$

Since we have $(V \otimes_{\mathbb{Q}_p} B_{dR}^+)^{\Gamma_K} \subseteq (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K} = D_{dR}(V)$, it suffice to prove

$$H^1(\Gamma_K, t^n B_{dR}^+) = 0. \tag{2}$$

We have a short exact sequence

$$0 \to t^{n+1} B_{dR}^+ \to t^n B_{dR}^+ \to \mathbb{C}_K(n) \to 0,$$

which in turn yields a long exact sequence

$$\mathbb{C}_K(n)^{\Gamma_K} \to H^1(\Gamma_K, t^{n+1} B_{dR}^+) \to H^1(\Gamma_K, t^n B_{dR}^+) \to H^1(\Gamma_K, \mathbb{C}_K(n)).$$

Then by Theorem 3.1.12 in Chapter II we obtain an identification

$$H^1(\Gamma_K, t^{n+1} B_{dR}^+) \cong H^1(\Gamma_K, t^n B_{dR}^+). \tag{3}$$

Hence by induction we only need to prove (2) for $n = 1$.
Take an arbitrary element $\alpha_1 \in H^1(\Gamma_K, tB_{dR}^+)$. We wish to show $\alpha_1 = 0$. Regarding α_1 as a cocycle, we use to inductively construct sequences (α_m) and (y_m) with the following properties:

1. *$\alpha_m \in H^1(\Gamma_K, t^m B_{dR}^+)$ and $y_m \in t^m B_{dR}^+$ for all $m \geq 1$.*
2. *$\alpha_{m+1}(\gamma) = \alpha_m(\gamma) + \gamma(y_m) y_m$ for all $\gamma \in \Gamma_K$ and $m \geq 1$.*

Now, since t is a uniformizer in B_{dR}^+, we may take an element $y = \sum y_m \in B_{dR}^+$. Then we have

$$\alpha_1(\gamma) + \gamma(y) - y \in H^1(\Gamma_K, t^m B_{dR}^+) \quad \text{for all } \gamma \in \Gamma_K \text{ and } m \geq 0,$$

and consequently find $\alpha_1(\gamma) + \gamma(y) - y = 0$ for all $\gamma \in \Gamma_K$. We thus deduce $\alpha_1 = 0$ as desired.

Remark. Highly nontrivial fact that every non-splitting extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge-Tate but not de Rham. The existence of such an extension follows from the identification

$$\mathrm{Ext}^1_{\mathbb{Q}_p[\Gamma_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p) \cong H^1(\Gamma_K, \mathbb{Q}_p(-1)) \cong K$$

where the second isomorphism is a consequence of the Tate local duality for p-adic representations. The difficult part is to prove that such an extension is not de Rham. For this part we need a very deep result that every de Rham representation is potentially semistable.

Proposition 2.6. *Let V be a de Rham representation of Γ_K. For every $n \in \mathbb{Z}$ we have $gr^n(D_{dR}(V)) \neq 0$ if and only if n is a Hodge-Tate weight of V.*

Remark. Proposition 2.6 provides the main reason for Serin's choice of the sign convention in the definition of Hodge-Tate weights. In fact, under our convention the Hodge-Tate weights of a de Rham representation V indicate where the filtration of $D_{dR}(V)$ has a jump. In particular, for a proper smooth variety X over K, the Hodge-Tate weights of the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ give the positions of "jumps" for the Hodge filtration on the de Rham cohomology $H^n_{dR}(X/K)$ by the isomorphism of filtered vector spaces

$$D_{dR}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{dR}(X/K).$$

Example 2.7. *The Tate twist $\mathbb{Q}_p(m)$ of $\mathbb{Q}_p$ is a 1-dimensional de Rham representation with the Hodge-Tate weight $-m$ as noted in Example 2.2. Hence by Proposition 2.6 we find*

$$Fil^n(D_{dR}(\mathbb{Q}_p(m))) \cong \begin{cases} K & \text{for } n \leq -m, \\ 0 & \text{for } n > -m. \end{cases}$$

In particular, for $m = 0$ we obtain an identification $D_{dR}(\mathbb{Q}_p) \cong K[0]$.

Proposition 2.8. *For every $V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(\Gamma_K)$, we have a natural Γ_K-equivariant isomorphism of filtered vector spaces*

$$D_{dR}(V) \otimes_K B_{dR} \cong V \otimes_{\mathbb{Q}_p} B_{dR}.$$

Proposition 2.9. *The functor D_{dR} with values in Fil_K is faithful and exact on $\mathrm{Rep}^{dR}_{\mathbb{Q}_p}(\Gamma^K)$.*

Proof. Let Vec_K denote the category of finite dimensional vector spaces over K. The faithfulness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ is an immediate consequence of Proposition 1.2.2 since the forgetful functor $Fil_K \to Vec_K$ is faithful. Hence it remains to verify the exactness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$. Consider an exact sequence of de Rham representations

$$0 \to U \to V \to W \to 0. \tag{4}$$

The functor D_{dR} with values in Fil_K is left exact by construction. In other words, for every $n \in \mathbb{Z}$ we have a left exact sequence

$$0 \to Fil^n(D_{dR}(U)) \to Fil^n(D_{dR}(V)) \to Fil^n(D_{dR}(W)). \tag{5}$$

We wish to show that this sequence extends to a short exact sequence. By Proposition 1.2.2 the sequence (4) induces a short exact sequence of vector spaces

$$0 \to D_{HT}(U) \to D_{HT}(V) \to D_{HT}(W) \to 0.$$

Moreover, by the definition of D_{HT} we find that this sequence is indeed a short exact sequence of graded vector spaces. Then by Proposition 2.4 we may rewrite this sequence as

$$0 \to gr(D_{dR}(U)) \to gr(D_{dr}(V)) \to gr(D_{dR}(W)) \to 0.$$

by Proposition 2.4. Hence for every $n \in \mathbb{Z}$ we have

$$\begin{aligned}\dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) &= \sum_{i\geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V)) \\ &= \sum_{i\geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(U)) + \sum_{i\geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(W)) \\ &= \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) + \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(W)),\end{aligned}$$

thereby deducing that the sequence (5) extends to a short exact sequence as desired. □

Corollary 2.10. *Let V be a de Rham representation. Every subquotient W of V is a de Rham representation with $D_{dR}(W)$ naturally identified as a subquotient of $D_{dR}(V)$ in Fil_K.*

Proof. This is an immediate consequence of Proposition 1.2.3 and Proposition 2.9. □

Proposition 2.11. *Given any $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ with a natural isomorphism of filtered vector spaces*

$$D_{dR}(V) \otimes_K D_{dR}(W) \cong D_{dR}(V \otimes_{\mathbb{Q}_p} W). \tag{6}$$

Proposition 2.12. *For every de Rham representation V, we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ and $\mathrm{Sym}^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ with natural isomorphisms of filtered vector spaces*

$$\wedge^n(D_{dR}(V)) \cong D_{dR} \quad \text{and} \quad \mathrm{Sym}^n(D_{dR}(V)) \cong D_{dR}(\mathrm{Sym}^n(V)).$$

Proposition 2.13. *For every de Rham representation V, the dual representation $V^\vee$ is de Rham with a natural perfect paring of filtered vector spaces*

$$D_{dR}(V) \otimes_K D_{dR}(V^\vee) \cong D_{dR}(V \otimes_{Q_p} V^\vee) \to D_{dR}(\mathbb{Q}_p) \cong K[0].$$

Proof. We find $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ and obtain the desired perfect pairing as a map of vector spaces. Moreover, Proposition 2.11 implies that this pairing is a morphism in Fil_K. We thus obtain a bijective morphism of filtered vector spaces $D_{dR}(V)^\vee \to D_{dR}(V^\vee)$. Therefore it suffices to show that the induced map

$$gr(D_{dR}(V)) \to gr(D_{dR}(V^\vee)) \tag{7}$$

is an isomorphism. Since V is Hodge-Tate by Proposition 2.4, we have a natural isomorphism

$$D_{HT}(V)^\vee \cong D_{HT}(V^\vee) \tag{8}$$

by Proposition 1.2.6. We thus deduce the desired assertion by identifying the maps (7) and (8) using Proposition 2.4. □

Let us now discuss some additional facts about de Rham representations and the functor D_{dR}.

Proposition 2.14. *Let V be a p-adic representation of Γ_K. Let L be a finite extension of K with absolute Galois group Γ_L.*

1. *There exists a natural isomorphism of filtered vector spaces*

$$D_{dR,K}(V) \otimes_K L \cong D_{dR,L}(V)$$

where we set $D_{dR,K}(V) := (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_K}$ and $D_{dR,L}(V) := (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_L}$.

2. *V is de Rham if and only if it is de Rham as a representation of Γ_L.*

Proof. We only need to prove the first statement, as the second statement immediately follows from the first statement. Let L' be the Galois closure of L over K with the absolute Galois group $\Gamma_{L'}$ and set $D_{dR,L'}(V) := (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_{L'}}$. Then we have identifications

$$D_{dR,K}(V) \otimes_K L = (D_{dR,K}(V) \otimes_K L')^{Gal(L'/L)} \quad \text{and} \quad D_{dR,L}(V) = D_{dR,L'}(V)^{Gal(L'/L)}.$$

Hence we may replace L by L' to assume that L is Galois over K. Moreover, since the construction of B_{dR} depends only on $\mathbb{C}_K$, we get a natural L-linear map

$$D_{dR,K}(V) \otimes_K L \to D_{dR,L}(V).$$

It is evident that this map induces a morphism of filtered vector spaces over L where the filtrations on the source and the target are given as in Example 2.2. We then have

$$Fil^n(D_{dR,K}(V)) = Fil^n(D_{dR,L}(V))^{Gal(L/K)} \quad \text{for all } n \in \mathbb{Z},$$

thereby deducing the desired assertion by the Galois descent for vector spaces. □

Remark. Proposition 2.14 extends to any complete discrete-valued extension L of K inside $\mathbb{C}_K$, based on the "completed unramified descent argument" as explained in [BC, Proposition 6.3.8]. This fact has the following immediate consequences:

1. Every potentially unramified p-adic representation is de Rham; indeed, we have already mentioned this in Example 2.2 since being $\mathbb{C}_K$-admissible is the same as being potentially unramified as noted in Example 1.1.4.

2. For one-dimensional p-adic representations, being de Rham is the same as being Hodge-Tate.

Example 2.15. *Let $\eta : \Gamma_K \to \mathbb{Z}_p^\times$ be a continuous character with finite image. Then there exists a finite extension L of K with absolute Galois group Γ_L such that $\mathbb{Q}_p(\eta)$ is trivial as a representation of Γ_L. Hence by Example 2.7 and Proposition 2.14 we find that $\mathbb{Q}_p(\eta)$ is de Rham with an isomorphism of filtered vector spaces*

$$D_{dR}(\mathbb{Q}_p(\eta)) \otimes_K L \cong L[0],$$

and consequently obtain an identification

$$D_{dR}(\mathbb{Q}_p(\eta)) \cong K[0] \cong D_{dR}(\mathbb{Q}_p).$$

In particular, we deduce that the functor D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{dR}(\Gamma_K)$ with values in Fil_K is not full.

We close this section by introducing a very important conjecture, known as the FontaineMazur conjecture, which predicts a criterion for the "geometricity" of global p-adic representations.

Conjecture 2.16 (Fontaine-Mazur (FM95))**.** . *Fix a number field E, and denote by $\mathcal{O}_E$ the ring of integers in E. Let V be a finite dimensional representation of $Gal(\overline{\mathbb{Q}}/E)$ over $\mathbb{Q}_p$ with the following properties:*

1. *For all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_E$, the representation V is unramified at $\mathfrak{p}$ in the sense that the action of the inertia group at $\mathfrak{p}$ is trivial.*

2. *For all prime ideals of $\mathcal{O}_E$ lying over p, the restriction of V to $Gal(\overline{\mathbb{Q}}_p/E_\mathfrak{p})$ is de Rham.*

Then there exist a proper smooth variety X over E such that V appears as a subquotient of the étale cohomology $H^n_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))$ for some $m, n \in \mathbb{Z}$.