DIOPHANTINE GEOMETRY WEEK 11-12 NOTES

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Abstract. This is the final topic of our seminar on Diophantine geometry. In the last two weeks, we survey the proofs of the three most important results: Faltings's theorem, Nevanlinna theory, and a little favor of Vojta's conjecture, again following the approach as outlined in [1]. Through this, our goal is to introduce the readers some of the most important pre-modern treatments in Arithmetic geometry.

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1. Faltings's theorem: Introduction

The first goal is to prove Faltings's theorem:

Theorem 1.1 (Faltings)**.** *Let C be a geometrically irreducible smooth projective curve of genus $g \geq 2$, defined over a number field K. Then the number of K-rational points of C is finite.*

Remark 1.1. The assumption $g \geq 2$ cannot be further improved since the theorem would fail for $C = \mathbb{P}^1_L$ or elliptic curves of possible rank. Where we've shown from Mordell–Weil's theorem that $E(\mathbb{Q})$ is a finitely generated abelian group, thus $E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \times \mathbb{Z}^r$ where $E(\mathbb{Q})_{tors}$ is the finite torsion subgroup and we define r to be the rank of the elliptic curve.

Remark 1.2. The second assumption may be easily dropped, though. To see this we first note two facts: any curve over K is birational to a projective curve over K, and the fact that for any curve C over K, $C_{\overline{K}}$ is birational to a smooth projective for finite-dimensional subextension $\overline{K}/K$.

From those two facts, if Cf is any curve over the number field K, then there exist a finite extension L/K such that C_L is birational to a finite disjoint union of geometrically irreducible smooth projective curves C_j over L and we may apply Faltings's theorem to $C_j(L)$ assuming that the genus of $C_j \geq 2$.

1.1. **Overview.** Before presenting the actual proof, we will first present a brief historical overview on the subject. The above theorem was conjectured by Mordell (for the rational field $K = \mathbb{Q}$) at the end of his paper [3] proving the finite generation of the group of rational points of an elliptic curve defined over $\mathbb{Q}$.

Igor Shafarevich conjectured that there are only finitely many isomorphism classes of abelian varieties of fixed dimension and fixed polarization degree over a fixed number field with good reduction outside a fixed finite set of places. Aleksei Parshin showed that Shafarevich's finiteness conjecture would imply the Mordell conjecture, using what is now called Parshin's trick.

The Mordell conjecture was at last proved by G. Faltings [4] in 1983, as a consequence of his proofs of the Tate conjecture and the Shafarevich conjecture. Faltings proved Shafarevich's finiteness conjecture using a known reduction to a case of the Tate conjecture, together with tools from algebraic geometry, including the theory of Néron models. The main idea of Faltings's proof is the comparison of Faltings heights and naive heights via Siegel modular varieties. However, this proof is significantly more involved, with the idea of Faltings's height difficult to understand.

A completely new proof was then given by P. Vojta, first in the function field case [5] and then in the arithmetic case [6]. In this chapter, following Bombieri's approach in [1] and in his paper [2] we shall give a even simplified version of Vojta's proof.

Again, even with the simplification the proof is very involved. We will omit the proof of some of the more important result, in the hope that the reader would be able to fill in the details. The goal is to survey some of the important techniques used to prove Faltings' theorem.

To begin with, we first introduce some basic definitions:

1.2. **Vojta divisor.** Let C be an irreducible smooth projective curve of genus g over the field K with a K-rational point. Now fix a point $P_0 \in C(K)$.

In fact, the Vojta divisor is devoted to purely geometric properties of certain divisors on $C \times C$, and we never use the assumption that K is a number field. But we will soon see this as an important ingradient in the proof of Falting's theorem.

By Δ we denote the diagonal of $C \times C$ and for simplicity of notation we shall also write

$$\Delta' := \Delta - \{P_0\} \times C - C \times \{P_0\}.$$

We study here properties of divisors on $C \times C$, which are expressed as linear combinations of the divisors $\{P_0\} \times C$, $C \times \{P_0\}$, and Δ. It is worth noting that, since we are in characteristic 0, for a general curve C, these three divisors generate the full group of divisors of $C \times C$ up to algebraic equivalence, hence this situation considered may easily be applied to the general case .

Hence we have the following lemma about hte intersection numbers:

Lemma 1.2. *The following table gives the intersection numbers:*

	$\{P_0\} \times C$	$C \times \{P_0\}$	Δ'
$\{P_0\} \times C$	*0*	*1*	*0*
$C \times \{P_0\}$	*1*	*0*	*0*
Δ'	*0*	*0*	$-2g$

The proof of this theorem would involve more algbraic geometry, the reader are welcome to reference [1], A.9.25. for the first two rows. For the last row, the result should be a direct consequence of Riemann-Roch Theorem on C, where the proof is described in A.13.6 of [1].

Definition 1.3. *For $d_1, d_2, d \in \mathbb{N}$, the divisor*

$$V := d_1\{P_0\} \times C + d_2 C \times \{P_0\} + d\delta'$$

*is called a **Vojta divisor**.*

Now our main goal is to explicitly calculate the height associated with V. To do this, our approach is express V as the difference of two well-chosen, in the sense that they are very ample divisors on $C \times C$.

Recall that the definition of very ampleness for a divisor is the following (we will try to supply a sheaf-free definition here):

Definition 1.4 (Very ampleness)**.** *If X is a smooth projective variety, a divisor D on X is called very ample if it is the section of an immersion of X in a projective space $\mathbb{P}^r$ with a hyperplane of $\mathbb{P}^r$ not containing X.*

Fix $N \geq 2g + 1$ for the rest of our discussion. The standard algebraic geometry result implieds that $N[P_0]$ as a divisor is very ample on C. Let

$$\varphi_{N[P_0]} : C \to \mathbb{P}^n_K,$$

the one can easily show that the product $\varphi_{N[P_0]} \times \varphi_{N[P_0]}$ gives a closed embedding

$$\psi : C \times C \to \mathbb{P}^n_K \times \mathbb{P}^n_K,$$

and we have

$$O_{C\times C}(\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\}) \cong \psi * O_{\mathbb{P}^n\times\mathbb{P}^n}(\delta_1, \delta_2).$$

And we follow [1] to make the notational abbreviation $O(d) := O_{\mathbb{P}^m}(d)$, and $O(\delta_1, \delta_2) := O_{\mathbb{P}^n\times\mathbb{P}^n}(\delta_1, \delta_2)$.

With this we in fact would be able to construct the following two very ample divisor:

Lemma 1.5. *For integers $\delta_1, \delta_2 \geq 1$, the divisor $\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\}$ is very ample.*

Lemma 1.6. *If M is a sufficiently large integer, then*

$$B := M(\{P_0\} \times C + C \times \{P_0\}) - \Delta'$$

is a very ample divisor on $C \times C$.

With this, we now have the decomposition of Vojta divisor into two very ample divisor, which permits a canonical way of calculating the height: We fix such an M once for all. Let

$$\phi_B : C \times C \to \mathbb{P}^m_K$$

be a corresponding closed embedding such that $\phi *_B O_{\mathbb{P}^m}(1) \cong O(B)$. The coordinates of $\mathbb{P}_K^m$ and $\mathbb{P}_K^n \times \mathbb{P}_K^n$ will be denoted by $\mathbf{y}$ and $(\mathbf{x}, \mathbf{x}')$. We consider $C \times C$ as a closed subvariety of $\mathbb{P}_K^m$ or $P_K^n \times P_K^n$, as the case may be, without mentioning the closed embeddings $\phi_B or \psi$. Because they are defined with a basis of global sections, no coordinate x_j, x_j' , or y_i vanishes identically on $C \times C$. With the smoothness Let us consider the condition:

(1) $$\delta_1 := (d_1 + Md)/N \text{ and } \delta_2 := (d_2 + Md)/N$$

are positive integers.

By adding to d_1 and d_2 integers bounded by N , (1) will be satisfied. Hence we get a decomposition

$$V = (\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\}) - dB$$

of the Vojta divisor V into the difference of two very ample divisors, as in the two lemmas above. Now from 10.38 of appendix A in [1], for sufficiently large δ_1, δ_2, d the following condition is satisfied:

(2) The first cohomology groups of $J_{\psi(C\times C)}(\delta_1, \delta_2)$ and $J_{\phi_B(C\times C)}(d)$ vanish.

With these two conditions, we note the following lemma:

Lemma 1.7. *Suppose that V is a Vojta divisor satisfying (1) and (2). For any global section s of $O(V)$, there are polynomials $F_i(\boldsymbol{x}, \boldsymbol{x}')$, $i = 0, \cdots, m$, bihomogeneous of bidegree (δ_1, δ_2), such that*

(3) $$s = F_i(\boldsymbol{x}, \boldsymbol{x}')/y_i^d|_{C\times C}.$$

for $i = 0, \cdots, m$. Conversely, assume that $F_i(\boldsymbol{x}, \boldsymbol{x}')$, $i = 0, \cdots, m$, are bihomogeneous polynomials of bidegree (δ_1, δ_2), satisfying

$$F_i(\boldsymbol{x}, \boldsymbol{x}')/y_i^d = F_j(\boldsymbol{x}, \boldsymbol{x}')/y_j^d.$$

on $C \times C$ for every i, j. Then there is a unique global section s on $O(V)$ such that 3 is valid for every i.

With this initial discussion on Vojta's divisor, let us now consider a generalization of Mumford's formula on to Vojta's divisor, which effectively gives a upper bound for the height.

1.3. Mumford's method and an upper bound for the height.

Proposition 1.1. *Let $P, Q \in C(K)$ and $z = j(P), w = j(Q)$. Then*

(4) $$h_V(P,Q) = \frac{d_1}{2g}|z|^2 + \frac{d_2}{2g}|w|^2 - d\langle z, w\rangle + d_1 O(|z|) + d_2 O(|w|) + (d_1 + d_2 + d + 1)O(1).$$

The proof of the proposition is a direct result of Mumford's formula on the heights of Δ' and $\{P_0\} \times C$, $C \times \{P_0\}$:

$$h_{\Delta'}(P,Q) = -\langle z, w\rangle + O(1),$$

and

$$h_{\{P_0\}\times C}(P,Q) = \frac{1}{2g}|z|^2 - \frac{1}{2g}\widehat{h}_{\theta-\theta^-}(z) + O(1),$$

and a similarly for $h_{C\times\{P_0\}}$.

Remark 1.3. By the proposition, there is a natural quadratic form on $J(\overline{K}) \times J(\overline{K})$ associated to $h_V(P,Q)$, namely

$$\frac{d_1}{2g}|z|^2 + \frac{d_2}{2g}|w|^2 - d\langle z, w\rangle.$$

Using the Cauchy–Schwarz inequality, this form is indefinite if and only if $d_1 d_2 < g_2 d_2$.

We next state without proving this important result, known as the local Eisenstein theorem, which allows us to bond the coefficient of the polynomial.

Theorem 1.8. *Let K be a field of characteristic 0 complete with respect to an absolute value $|\ |$ and let $p(x,t) \in K[x,t]$ be a polynomial in two variables with partial degrees at most d. Let ξ be an algebraic function of x such that $p(x,\xi(x)) = 0$. We suppose that $\xi(0) \in K$, $|\xi(0)| \leq 1$ and*

$$\frac{\partial p}{\partial t}(0, \xi(0)) \neq 0.$$

Then the Taylor series expansion $\xi(x) = \sum_{k=0}^{\infty} a_k x^k$ has coefficients in K and the following bound holds for $l \geq 1$

$$|a_l| \leq C^l (|p| / |\frac{\partial p}{\partial t}(0, \xi(0))|)^{2l-1},$$

where $|p| = \max |\text{coefficients of } p|$ is the Gauss norm of p, and

$$C = \begin{cases} 1 & \text{in the non-archimedean case} \\ |8(d+1)^7| & \text{in the archimedean case.} \end{cases}$$

With this, we are ready to supply a lower bound for the height.

1.4. **A lower bound for the height.** Let C be an irreducible projective smooth curve over a number field K and let us fix $P_0 \in C(K)$. Let V be a Vojta divisor satisfying equations (1) and (2). We are interested in getting a lower bound for the height $h_V(P,Q)$, where $P,Q \in C(K)$. This is obtained by means of two lemmas which we will soon see. The first lemma gives an explicit lower bound in term of the Taylor coefficients of local coordinates for C viewed as algebraic functions of a uniformizing parameter of C at P or Q. The second lemma applies the local Eisenstein theorem to bound these Taylor coefficients.

First we make a few notational definitions: Let ∂, ∂' be non-zero vectors in the tangent space of C at P,Q. we abbreviate

$$\partial_i := \frac{1}{i!}\partial^i, \partial'_i := \frac{1}{i!}\partial'^i.$$

Definition 1.9 (Admissibility)**.** *Let $s \in \Gamma(C \times C, O(V)) \setminus \{0\}$. A pair $(i_1^*, i_2^*) \in \mathbb{N}^2$ is called admissible if and only if*

$$\partial_{i_1^*}\partial'_{i_2^*} s(P,Q) \neq 0,$$

and

$$\partial_{i_1}\partial'_{i_2} s(P,Q) = 0,$$

whenever $i_1 \leq i_1^*$, $i_2 \leq i_2^*$ and $(i_1, i_2) \neq (i_1^*, i_2^*)$. In order to make sense of the above formulas, we should choose a trivialization of $O(V)$ in (P,Q). It is also clear that admissibility is independent of the choice of the trivialization and the choice of ∂, ∂'.

With this, we are now going to choose an explicit height function relative to $O(V)$. Choose a finite extension L/K such that $P, Q \in C(L)$. We use the decomposition

$$V = (\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\}) - dB$$

of V as a difference of two very ample divisors. Now we have generating sections $\mathbf{x}^{\mathbf{h}}\mathbf{x}'^{\mathbf{h}'}$ ($|\mathbf{h}| = \delta_1, |\mathbf{h'}| = \delta_2$) of $O(\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\})$ and $y_i (|i| = d)$ of $O(dB)$. With respect to the presentation

$$\left(sV; O(\delta_1 N\{P_0\} \times C + \delta_2 NC \times \{P_0\}), \mathbf{x}^{\mathbf{h}}\mathbf{x}'^{\mathbf{h}'}; O(dB), \mathbf{y}^{\mathbf{i}}\right),$$

we have the global height function

$$\begin{aligned} hV(P,Q) &:= \sum_{v \in M_L} \max_{\mathbf{h},\mathbf{h'}} \min_{\mathbf{i}} \log \left| \frac{\mathbf{x}^{\mathbf{h}}\mathbf{x}'^{\mathbf{h}'}}{\mathbf{y}^{\mathbf{i}}}(P,Q) \right|_v \\ &= \sum_{v \in M_L} \max_{j,j'} \min_{i} \log \left| \frac{x_j^{\delta_1} x'^{\delta_2}_{j'}}{y_i^d}(P,Q) \right|_v \end{aligned}$$

Note that the vectors $\mathbf{x}(P)$, $\mathbf{x'}(Q)$ and $y(P,Q)$ are only defined up to a multiple. By the product formula, $h_V(P,Q)$ is well-defined; it is the difference of two Weil heights, the first given by the closed embedding

$$(P,Q) \mapsto \left(\mathbf{x}^{\mathbf{h}}(P)\mathbf{x}'^{\mathbf{h}}(Q)\right)_{|\mathbf{h}|=\delta_1, |\mathbf{h'}|=\delta_2}$$

and the second given by the closed embedding

$$(P,Q) \mapsto \left(\mathbf{y}^{\mathbf{i}}(P,Q)\right)_{|\mathbf{i}|=d}.$$

Now we will state the two important bounding lemmas which will be used later:

Lemma 1.10. *Let s be a non-zero global section of $O(V)$ and let (i_1^*, i_2^*) be admissible for s at (P,Q). With the notation introduced above, we have*

$$\begin{aligned} h_V(P,Q) \geq &-h(\boldsymbol{F}) - n \log\left((\delta_1 + n)(\delta_2 + n)\right) \\ &- \sum_{v \in M_L} \max_{i_\lambda} \left(\sum_{\lambda} \max_{\nu} \log |\partial_{i_\lambda} \xi_{\nu j_v}(P)|_v \right) \\ &- \sum_{v \in M_L} \max_{i'_\lambda} \left(\sum_{\lambda} \max_{\nu} \log |\partial'_{i'_\lambda} \xi_{\nu j_v}(P)|_v \right) \\ &- (\delta_1 + \delta_2 + i_1^* + i_2^*), \end{aligned}$$

where $\{i_\lambda\}$ and $\{i'_\lambda\}$ run over all partitions of i_1^* and i_2^*.

On the other hand, our next task consists in majorizing the sums appearing in Lemma 1.10: to achieve this, we do by an application of the local Eisenstein theorem. Fix a nonconstant $f \in K(C)$. For any $P \in C(K)$ which is neither a pole of f nor a zero of df , the function $\zeta = f - f(P)$ is a local uniformizer at P (i.e. a local parameter in the local ring). Therefore the completion of $\mathcal{O}_{C,P}$ with respect to its maximal ideal is isomorphic to $K(P)[[\zeta]]$ (by the Cohen structure theorem). Moreover, we may differentiate with respect to ζ.

By assumption, $K(C)$ is a finite extension of $K(f)$. Choose $g_{ij}(x,t) \in K[x,t]$ such that $g_{ij}(f,t) \in K(f)[t]$ is a minimal polynomial of ξ_{ij} over $K(f)$. Since char$(K) = 0$, we have $\frac{\partial}{\partial t} g_{ij}(f, \xi_{ij}) \neq 0$ in $K(C)$ (irreducible polynomials are separable). Let $\deg(g_{ij})$ be the total degree of $g_{ij}(x,t)$.

Let us denote by Z the finite subset of $C(K)$ consisting of:

(1) all zeros of x_j , for $j = 0, \cdots, n$;
(2) all poles of f ;
(3) the support of div(df);
(4) the zeros of $\frac{\partial}{\partial t} g_{ij}(f, \xi_{ij})$, for $i, j = 0. \cdots, n$.

We are going to apply the local Eisenstein theorem to the polynomials

$$p_{ij}(x,t) := g_{ij}(x + f(P), t + \xi_{ij}(P))$$

for any $P \notin Z$. Note that $p_{ij}(0,0) = 0$ and $\frac{\partial}{\partial t} p_{ij}(0,0) \neq 0$.

Now since ξ_{ij} is regular at P , we get from the local Eisenstein theorem in 11.4.1

$$|\partial k \xi_{ij}(P)|_v \leq |C_2|_v^{k\eta_v} \left(\frac{|p_{ij}|_v}{|\frac{\partial}{\partial t} p_{ij}(0,0)|_v} \right)^{2k-1},$$

for $k \geq 1$ and $\partial_k = \frac{1}{k!}(\partial/\partial\zeta)^k$. Now we may relate the sum in the previous lemma with the canonical form on the Jacobian.

Lemma 1.11. *If $P \notin Z$, then*

$$\sum_{v \in M_L} \max_{\{i_\lambda\}} \left(\sum_{\lambda} \max_{\nu} \log |\partial_{i_\lambda} \xi_{\nu j_\nu}(P)|_v \right) \ll i_1^*(|j(P)|^2 + 1),$$

where the maximum runs over all partitions $\{i_\lambda\}$ of i_1^*. The constant implied in the symbol $\ll$ is independent of P and i_1^*.

The proof of these two results can be purely algebraic in nature, which we would omit here.

Now we are ready to develop one the mostly important tool used to prove Faltings theorem: since we already have an upper bound on the height of the Vojta divisor. Now our goal is to construct a Vojta divisor of very small height.

In the last part of this subsection, we prove the existence of a section of $O(V)$, with V a Vojta divisor, of small height. The argument is fairly standard. The space of sections of $O(V)$ is presented as a subspace, given by linear relations with small height, of a vector space with a standard basis. The Riemann–Roch theorem shows that this subspace has large dimension, and the existence of a small section follows by Siegel's lemma or, equivalently, by geometry of numbers.

Let C be an irreducible projective smooth curve of genus g over a number field K and again fix $P_0 \in C(K)$. We shall use the notation of Section 11.2, in particular V will be a Vojta divisor satisfying (1) and (2).

Lemma 1.12. *The following holds*

$$\dim \Gamma(C \times C, \psi^* O(\delta_1, \delta_2)) = (N\delta_1 + 1 - g)(N\delta_2 + 1 - g)$$

and, for $d_1 + d_2 > 4g - 4$

$$\dim \Gamma(C \times C, O(V)) \geq d_1 d_2 - g d_2 + O(d_1 + d_2).$$

The proof of this result would involve the use of Riemann Roch.

We need now another assumption for the parameters of the Vojta divisor.

(5) $$d_1 + d_2 > 4g - 4 \text{ and } d_1 d_2 - g d_2 > \gamma d_1 d_2 \text{ for some } \gamma > 0.$$

Here, γ is independent of d_1, d_2, d. As we have seen at the beginning of the chapter, we may map $C \times C$ by a birational morphism onto a hypersurface of degree D in $\mathbb{P}^3_K$. We denote the projective coordinates in $\mathbb{P}^3_K$ by z. All presentations (see 2.5.4) will refer to this set up. Now we are ready to construct a Vojta divisor with small height.

Lemma 1.13. *There are two positive constants C_4 , C_5 independent of d_1 , d_2 , d and γ with the following property. Let V be a Vojta divisor satisfying (1), (2), (3), and $d_1, d_2 \geq C_4/\gamma$. Then there is a non-zero global section s of $O(V)$ such that the polynomials $F_0, \cdots, F_m$ in Lemma 1.7 may be chosen with*

$$h(F) \leq C_5(d_1 + d_2)/\gamma.$$

Sketch of proof: The idea is to apply Siegel's lemma to get a section of small height. Thus we have to transfer the equations in Lemma 1.7 into a linear system of equations with coefficients in K. Thereby, we must be careful not to increase the height of the linear system too much.

1.5. **Roth's lemma.** Finally, we need one last important ingredient to prove Falting's theorem, which is the Roth's lemma. We will present a out line of lemmas that lead to the main lemma, presented at the end of this section. Again, the proof of this is slightly involved, which we will omit here.

Let C be an irreducible projective smooth curve over the number field K and let $P_0 \in C(K)$. With the notation introduced in Section 1.2, let V be a Vojta divisor satisfying (1) and (2). Let $F_i(\mathbf{x}, \mathbf{x'})$, $i = 0, \cdots, m$ denote bihomogeneous polynomials of bidegree (δ_1, δ_2), describing a non-trivial global section s of $O(V)$ as in Lemma 1.7, hence

$$s = F_i(\mathbf{x}, \mathbf{x'})/y_i^d|_{C \times C}$$

for $i = 0, \cdots, m$. We are looking for an upper bound of the admissible pair (i_1^*, i_2^*) in the point $(P, Q) \in (C \times C)(K)$. The idea is to project down to $\mathbb{P}^1_K \times \mathbb{P}^1_K$ to get a bihomogeneous polynomial instead of s and then apply Roth's lemma to that polynomial. In the first four lemmas of this subsection, we describe the push-down of sy_i^d and show that it has similar properties as s. The final lemma is the application of Roth's lemma, which we would need to prove Falting's Theorem.

We will first state the sequence of four lemmas that lead to the application of Roth's lemma:

Lemma 1.14. *There is a bihomogeneous polynomial $G_i \in K[x_0, x_1, x_0^{'}, x_1^{'}]$ of bidegree $(N^2\delta_1, N^2\delta_2)$ such that*

$$(\pi \times \pi)_* div(F_i|C \times C) = div(G_i).$$

Here $\pi \times \pi$ denotes the natural morphism $C \times C \rightarrow\ \mathbb{P}^1_K \times P^1_K$ induced by π.

Remark 1.4. Let Norm denote the norm with respect to the field extension $K(C \times C)/K(P^1_K \times P^1_K)$. Then we may choose

$$Gi(\xi_1, \xi_1^{'}) = \text{Norm}(F_i(\xi, \xi_1^{'})). \tag{6}$$

In order to see this, note that

$$\begin{aligned} \text{div}(\text{Norm}(Fi(\xi, \xi^{'})) &= (\pi \times \pi)_* \text{div}(Fi(\xi, \xi'))) \\ &= \text{div}(G_i(\mathbf{x}, \mathbf{x}')) - (\pi \times \pi)_* \text{div}(x_0^{\delta_1} x_0'^{\delta_2}|_{C \times C}) \end{aligned}$$

by A.9.11. Since π has degree N and $div(x_0|C) = \pi^*([\infty])$ on C, the projection formula for proper intersections yields $\pi^*(\mathrm{div}(x_0|C)) = N[\infty]$. This is equal to $\mathrm{div}(x_0^N)$ on $\mathbb{P}^1_K$ and a similar identity holds for x'_0 . This easily proves (6).

Besides these, we will need another lemma for the proof of our final lemma:

Lemma 1.15. *There is a bihomogeneous polynomial $E(x_0, x_1, x'_0, x'_1)$, of bidegree (Nd_1, Nd_2), with the following properties*

1. $(\pi \times \pi)_*(div(s)) = div(E)$.
2. *If (j_1^*, j_2^*) is an admissible pair of E in $(\pi(P), \pi(Q))$, then there is an admissible pair (i_1^*, i_2^*) of s in (P, Q) such that $i_1^* \leq j^*1$, $i^*2 \leq j^*2$.*
3. $h(E) \leq N^2 h(F) + O(d_1 + d_2 + d)$.

Remark 1.5. We note the similarity of this result with the one we presented in the proof of Mordell-Weil Theorem. In fact, this bond gives a similar treatment as we did in the proof of Mordell-Weil Theorem.

With this, we are now ready to present one of the most lemma of this section:

Lemma 1.16 (Roth's lemma, application)**.** *There is a constant $C_6 > 0$, independent of d_1, d_2, d, and γ , such that for $0 < \epsilon \leq \frac{1}{\sqrt{2}}$, for any Vojta divisor satisfying (1), (2), (3), with*

$$d_2 \geq C_4/\gamma, d_2/d_1 \leq \epsilon^2$$

and for any $P, Q \in C(K)$ with

$$\min\left(d_1 h_{N[P_0]}(P), d_2 h_{N[P_0]}(Q)\right) \geq C_6 \frac{d_1}{\gamma\epsilon^2} \tag{7}$$

there exists a global section s of $O(V)$ with an admissible pair (i_1^*, i_2^*) in (P, Q) such that

$$h(F) \leq C_5 \frac{d_1 + d_2}{\gamma}, \quad \frac{i_1^*}{d_1} + \frac{i_2^*}{d_2} \leq 4N\epsilon.$$

With all these, we are now ready to prove the Flatings's theorem.

2. Proof of Faltings's Theorem

Let C be an irreducible projective smooth curve of genus $g \geq 2$, defined over a number field K, with a point P_0 defined over K. The proof of Faltings's Theorem would be a direct consequence of the following theorem, which is known as the **Vojta's theorem**.

Theorem 2.1. *There are constants C_7 , C_8 , depending only on C and P_0 , with the following property: Let $P, Q \in C(K)$ and $z = j(P), w = j(Q)$. Then one of*

$$|z| \leq C_7, |w| \leq C_8|z|, \langle z, w\rangle \leq \frac{3}{4}|z||w|$$

holds.

Remark 2.1. In particular, we note that the constant $\frac{3}{4}$ has no special significance and can be replaced by any constant in $(\frac{1}{\sqrt{g}}, 1]$: what we need is that strictly less than 1. We will see this follows from the proof.

Proof. We consider $P, Q \in C(K)$ with $|z| \geq C_7, |w| \geq C_8|z|$ for large constants C_7, C_8 to be determined later. Since the set Z defined in 11.6.6 is finite and effectively determinable, we may assume that the constants C_7 and C_8 are so large that $P, Q \notin Z$. Suppose that V is a Vojta divisor satisfying equations (1), (2), (3), and $d_1, d_2 \geq C_4/\gamma$ (cf. lemmas 1.6 and equation (5)). Then by Proposition 1.1, Lemma 1.10, Lemma 1.11 and Lemma 1.13 show that, for a positive constant C_9 depending only on C and P_0, we have

$$\begin{aligned} - C_9 \left(\frac{d_1 + d_2}{\gamma} + i_1^*|z|^2 + i_2^*|w|^2 + i_1^* + i_2^* \right) \\ \leq \frac{d_1}{2g}|z|^2 + \frac{d_2}{2g}|w|^2 - d\langle z, w \rangle + O(d_1|z| + d_2|w| + d_1 + d_2). \end{aligned}$$

Now we also assume that there is an ϵ, with $0 < \epsilon \leq 1/\sqrt{2}$, with

$$d_2/d_1 \leq \epsilon^2 \tag{8}$$

and

$$\min(d_1 h_{N[P_0]}(P), d_2 h_{N[P0]}(Q)) \geq C_6 \frac{d-1}{\gamma\epsilon^2} \tag{9}$$

as in Lemma 1.15. Now with lemma 1.15, we get

$$\begin{aligned} -C_9 \cdot \left(2\frac{2d_1}{\gamma} + 4N\epsilon d_1|z|^2 + 4N\epsilon d_2|w|^2 \right) \\ \leq \frac{d_1}{2g}|z|^2 + \frac{d_2}{2g}|w|^2 - d\langle z, w \rangle + O(d_1|z| + d_2|w| + d_1). \end{aligned} \tag{10}$$

For a small positive number $\gamma_0 < 1$ and $D \in N$, we choose

$$d1 = \sqrt{g + \gamma_0}\frac{D}{|z|^2} + O(1),$$

$$d2 = \sqrt{g + \gamma_0}\frac{D}{|w|^2} + O(1),$$

and

$$d = \frac{D}{|z| \cdot |w|} + O(1).$$

The $O(1)$ terms are for small adjustments so that $d_1, d_2, d, \delta_1, \delta_2$ are all nonzero natural numbers. This is a choice which makes above relatively sharp and fulfills (1), (2), (3), for D sufficiently large as a function of $|z|, |w|, \gamma_0$. It is immaterial here how this notion of D being large depends on $|z|, |w|, or \gamma_0$, since in the end we shall let $D \to \infty$. Note that we have

$$d_1 d - 2 - g d_2 \geq \gamma d_1 d_2$$

for

$$\gamma = \frac{\gamma_0}{g + \gamma_0} + o(1),$$

where the term implicit in $o(1)$ tends to 0 as $D \to \infty$. Using

$$\frac{d_2}{d_1} = \frac{|z|^2}{|w|^2} + o(1),$$

eqiatopm (8) becomes

$$\frac{|z|}{|w|} \leq \epsilon + o(1). \tag{11}$$

As remarked in the proof of Lemma 1.11, we have

$$h_{N[P0]}(P) = \frac{N}{2g}|z|^2 + O(|z|) + O(1),$$

with a similar equation for Q and $|w|$. Thus the condition

$$|z| \geq C_{10}/(\epsilon\sqrt{\gamma}), \tag{12}$$

with $C_{10} \geq 1$ a positive constant depending on C and P_0, implies equation (9) for sufficiently large D.

We substitute the values for d_1, d_2, d, γ in (10) to derive

$$-C_{11} \cdot \left(\frac{1}{gamma0|z|^2} + \epsilon \right) D \leq \frac{\sqrt{g+\gamma_0}}{g} D - \frac{\langle z, w \rangle}{|z| \cdot |w|} D + O\left(\left(\frac{1}{|z|} + \frac{1}{|w|} D \right) \right) + o(D),$$

for a certain constant C_{11} depending only on C and P_0 . Assuming (12), we divide by D, let D tend to ∞, simplify, and find after rearranging terms

$$\frac{\langle z, w \rangle}{|z| \cdot |w|} - \frac{\sqrt{g+\gamma_0}}{g} \leq C_{12}\epsilon \tag{13}$$

with C_{12} depending only on C, P_0 . We still need conditions in equation (12) and $|z| < \epsilon|w|$, the limit of (11) for $D \to \infty$. To this end, we choose first γ_0 so small that

$$\frac{3}{4} - \frac{\sqrt{g+\gamma_0}}{g} > 0,$$

and then ϵ so small that

$$\frac{3}{4} - \frac{\sqrt{g+\gamma_0}}{g} > C_{12}\epsilon. \tag{14}$$

Here we have used $\frac{3}{4} > \frac{1}{\sqrt{2}}$ and $g \geq 2$. Again, we stress that the choice of $\frac{3}{4}$ can be replced with any number greater than $\frac{1}{\sqrt{2}}$ and less than or equal to 1. Let

$$C_7 > C_{10} / \left(\epsilon \sqrt{\frac{\gamma_0}{g+\gamma_0}} \right)$$

and

$$C_8 > \frac{1}{\epsilon}.$$

For $P, Q \in C(K)$ satisfying

$$|z| \geq C_7, |w| \geq C_8|z|,$$

(11) and (12) are both satisfied and inequality (13) holds. Finally, because of (14) and $C_{10} \geq 1$, we see that (13) implies

$$\frac{\langle z, w \rangle}{|z| \cdot |w|} \leq \frac{3}{4},$$

which completes the proof. □

Finally, to prove Falting's theorem, we recall the following result of Néron Tate Height, which were used in the proof of Mordell Weil theorem as well:

Theorem 2.2. *Let K be a number field and let c be ample and even. Then $\widehat{h}_{\boldsymbol{c}}$ vanishes exactly on the torsion subgroup of $A(K)$. Moreover, there is a unique scalar product $\langle , \rangle$ on the abelian group $A(K) \otimes_{\mathbb{Z}} \mathbb{R}$ such that*

$$\widehat{h}_{\boldsymbol{c}}(x) = \langle x \otimes 1, x \otimes 1 \rangle$$

for every $x \in A(K)$.

and also an application from **gap principle**, a standard result in analysis:

Lemma 2.3. *Let $\|\|$ be a norm on $\mathbb{R}^r$. Let E be a subset of the ball*

$$B_t := \{x \in \mathbb{R}^r | \|x\| \leq t\}$$

of radius t . Then for any $\epsilon > 0$, we can cover E with $(1 + 2t/\epsilon)^r$ translates, all centred on the set E, of the ball B_ϵ.

With this, we are now ready to prove Faltings's theorem:

Proof of Faltings's Theorem. We may assume that C has a base point $P_0 \in C(K)$. From Theorem2.2 above and the Mordell–Weil theorem we proved in part II of the write-up, we know that $J(K) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite-dimensional euclidean space. By Lemma above, we may cover it by finitely many cones T centered at 0 with angle $\alpha/2$ from the axis to the ending, where $\cos\alpha > \frac{3}{4}$ (or any number $> \frac{1}{\sqrt{2}}$). Let C_7, C_8 be the constants in Theorem 11.9.1.

By Mumford's gap principle with $\epsilon \leq 2gcos\alpha - 3$, there is a constant C_{13} , depending only on C and P_0 , such that for any pair of distinct points P, Q in a same cone T , with $C_{13} \leq |j(P)| \leq |j(Q)|$, we have

$$|j(Q)| \geq 2|j(P)|. \tag{15}$$

Let $C_{14} = \max(C_7, C_{13})$. The set of K-rational points in the ball with center 0 and radius C_{14} is finite by Northcott's theorem we proved in week 08, so it remains to see that

$$S := T \cap \{P \in C(K) | |j(P)| > C_{14}\}$$

is finite. Suppose $P_0, P_1, \cdots, P_k$ are different points of S such that

$$|j(P_i)| \leq |j(P_{i+1})|, i = 0, \cdots, k-1.$$

By (15), we have

$$|j(P_{i+1})| \geq 2|j(P_i)|,$$

yielding

$$|j(P_k)| \geq 2k|j(P_0)|.$$

On the other hand, Vojta's theorem (2.1) shows that

$$|j(P_k)| \leq C_8|j(P_0)|.$$

This proves

$$k \leq \log(C)_8)/\log 2$$

and the Faltings's theorem.

□

This completes the overview of the Falting's Theorem

3. Nevanlinna theory: an overview

In this next section, we will give a quick overview on Nevanlinna theory, which we feel important as it bridges the concepts of complex analysis with Arithmetic Geometry. Again, the goal is to survey some of the important results, and hence we will only present proofs of the important theorems in detail, with the hope that the reader would be able to fill in the details.

3.1. **Introduction.** In the mathematical field of complex analysis, Nevanlinna theory is part of the theory of meromorphic functions. It was devised in 1925, by Rolf Nevanlinna. In 1987 Vojta formulated a sweeping set of precise conjectures about the structure of the set of rational points on algebraic varieties. The rationale about these conjectures was a rather precise analogy between the Nevanlinna theory of the distribution of values of meromorphic functions and diophantine approximation. In this way, Vojta motivated, clarified, and unified results and conjectures in diophantine approximation and diophantine equations. The analogy between Nevanlinna theory and diophantine approximation had also been noticed earlier by Ch. Osgood, in a somewhat different setting.

In this chapter, we will be introducing the two main theorems of Navanlinna, with the ultimate goal to present the analogus *abc* conjecture.

3.2. **Nevanlinna theory in one variable.** In plain terms, the Nevallina theory in one variable analyzes the distributions/density of values of a non-constant meromorphic functions that maps to $\mathbb{P}^1_{an}$. In this section, we will present some of the important fundamental results leading towards Nevallina's two main theory. Throughout this section, we shall suppose that the meromorphic function f is not a constant, unless specified otherwise, and that $f : \mathbb{C} \to \mathbb{P}^1_{an}$.

The usual way to study the distribution of values of a meromorphic function $f(z)$ is to consider the number of solutions, counted with multiplicity, of the equation $f(z) = a$ in a disk $\{|z| < r\}$, as r varies. Recall that $ord_z(f)$ denotes the order of f at $z \in \mathbb{C}$. This classical result in complex analysis had appeared in many places, common known as the **Jensen Formula** (or Poisson-Jensen Formula):

Theorem 3.1. *Let f be meromorphic in the closed disk $|z| \le R$ and assume that $f(z) \ne 0, \infty$. Then for $|z| < R$ it follows that*

$$\begin{aligned}\log|f(z)| = &-\sum_{|a|<R, a\neq z} ord_a(f) \log\left|\frac{R^2-\overline{a}z}{R(z-a)}\right| \\ &+\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \cdot Re\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\end{aligned} \tag{16}$$

The case in which there are no zeros or poles is called Poisson's formula and the case $z = 0$ is called Jensen's formula.

The proof of this result have appeared in many literature, for example check Stein and Sharkarchi's [9].

The special case of the Poisson–Jensen formula in which $z = 0$ is particularly important to us and we proceed to rewrite it as follows. First, we introduce some notation. For a real-valued function $F(r)$, $r > 0$, quantitative estimates such as $F(r) = O(\log r)$ are always meant with respect to $r \to \infty$. Also we define

$$F^+(r) = \max\{F(r), 0\}, \quad F^-(r) = -\min\{F(r), 0\},$$

so that $F(r) = F^+(r) - F^-(r)$. This is the standard approach where we used to prove Schwarz's reflection principle.

Definition 3.2. *][Enumerating Function] For $a \in \mathbb{C}$ and $r > 0$, we define enumeration function as*

$$n(r, a, f) := \sum_{|z|<r} ord_z^+(f - a),$$

which is the number of solutions of $f(z) = a$ in the disk $|z| < r$ counted with multiplicity. For $a = \infty$, we replace f by $\frac{1}{f}$ to obtain

$$n(r, \infty, f) := \sum_{|z|<r} ord_z^-(f),$$

But then the problem with this function is that it is too irregular, in a sense that it was not smooth at many point. So a standard logarithmic average transformation (similar to Mellin Transform) would allow us to have much better better behavior:

Definition 3.3. *For $a \in \mathbb{C}$ and $r > 0$, we define the counting function as the smoothed average of enumerating function, namely*

$$\begin{aligned} N(r, a, f) &:= \int_0^r \frac{n(t, a, f) - ord_0^+(f - a)}{t} dt + ord_0^+(f - a) \log r \\ &= ord_0^+(f - a) \log r + \sum_{0<|z|<r} ord_z^+(f - a) \log |\frac{r}{z}|. \end{aligned}$$

Where for $a = \infty$, we replace f by $\frac{1}{f}$ and that ∞ by 0 to obtain

$$\begin{aligned} N(r, \infty, f) &:= \int_0^r \frac{n(t, \infty, f) - ord_0^-(f)}{t} dt + ord_0^-(f) \log r \\ &= ord_0^-(f) \log r + \sum_{0<|z|<r} ord_z^-(f) \log |\frac{r}{z}|. \end{aligned}$$

On the other hand, the function $N(r, a, f)$ so defined is perfectly suited for a compact reformulation of the Poisson–Jensen formula at $z = 0$, which is Jensen's formula:

Proposition 3.1. *Let*

$$c(f, 0) := \lim_{z \to 0} f(z) z^{-ord_0(f)}$$

be the leading coefficient in the Laurent series of f at 0. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta + N(r, \infty, f) - N(r, 0, f) = \log |c(f, 0)|.$$

Finally, we made the following definition of **proximity function**:

Definition 3.4. *The* ***proximity function*** *is:*

$$m(r, a, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta.$$

For $a = \infty$, we replace f by $1/f$ then

$$m(r, \infty, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

Remark 3.1. This is the version we see on Stein and Sharkarchi [9], and this is a special case of theorem 3.1.

Also we make the following two remarks, which will gear towards Nevanlinna's first theorem:

Remark 3.2. The counting function is a logarithmically weighted degree of the zero divisor of $f - a$ on the open disk $D(r) := \{z \in \mathbb{C} | |z| < r\}$. The proximity function is a logarithmic average on the boundary $\partial D(r)$ measuring how close $f(z)$ is to a. Note that $m(r, a, f) < \infty$ because values $f(re(i\theta) = a$ lead only to integrable logarithmic singularities on $\partial D(r)$.

Remark 3.3. We also note the following upper and lower bound of the proximity function, which will be extremly useful in Nevanlinna's first main theorem.

Suppose that f is an entire function. Then the proximity function and $\log \|f\|_r$, where $\|f\|_r := \max_{|z| \leq r} |f(z)|$, are comparable. In one direction, we have

$$m(r, \infty, f) \leq \log^+ \|f\|_r.$$

On the other direction, by Poisson-Jensen inequality we may easily show that

$$\log^+ \|f\|_r \leq 3mm(2r, \infty, f)$$

In the case of entire function, as we see from the above observation that the proximity function at $a = \infty$ plays the same role as the logarithm of maximum modulus. The case of meromoprhic function, however, does not have the same attribute. Hence the first Nevanlinna's theorem:

Theorem 3.5 (Nevanlinna, first). *For $a \in \mathbb{C}$ the following formula holds:*

$$m(r, a, f) + N(r, a, f) = m(r, \infty, f) + N(r, \infty, f) - \log |c(f - a, 0)| + \epsilon(r, a, f),$$

with $|\epsilon(r, a, f)| \leq \log^+ |a| + \log 2$.

Proof. We see that this result is immediate from proposition 3.1, when applied to $f - a$, noting first that

$$\log |f| = \log^+ |f| - \log^- |f|, \quad ord_a(f) = ord_a^+(f) - ord_a^-(f)$$

, and then that

$$\left| \log + |f - a| - \log^+ |f| \right| \leq \log^+ |a| + \log 2.$$

□

Now we note that $m(r, a, f) + N(r, a, f)$ is in fact independent of a up to a bounded function, we can therefore making a new definition:

Definition 3.6. *The characteristic function of f is*

$$T(r, f) := m(r, \infty, f) + N(r, \infty, f).$$

The significance of this function is that, it is well behaved as a function of r.

In fact, when $f(z)$ is a polynomial of degree d. Then the fundamental theorem of algebra shows

$$N(r, a, f) = d \log r + O(1), m(r, a, f) = O(1)$$

for $a \neq \infty$ and $r \to \infty$, the implied constant in the $O(1)$ symbol depends on f and a. For $a = \infty$, we have

$$N(r, \infty, f) = 0, m(r, \infty, f) = d \log r + O(1),$$

hence

$$T(r,f) = d\log r + O(1).$$

We also note the following result, known as Cartan's formula, which is a direct consequence Jensen's formula.

Proposition 3.2. *Let* $C := log^+|f(0)|$ *if* $f(0) \neq \infty$ *and* $C := \log|c(f,0)|$ *if* $f(0) = \infty$. *Then*

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta}, f)d\theta + C.$$

We also note from this that we have the following two important observations:

Observation 3.1. *The Nevanlinna characteristic* $T(r,f)$ *is an increasing convex function of* $\log r$. *Moreover,* $T(r,f)$ *is increasing but it has not to be strictly increasing. For example, if* $f\{|z| \leq r\} \subset \{|w| < 1\}$, *then* $T(r,f) = 0$.

We now present the following proposition, and its converse as the next proposition:

Proposition 3.3. *If* f *is not a constant, then* $T(r,f)$ *is unbounded. If*

$$\lim\inf_{r\to\infty} \frac{T(r,f)}{\log r} < +\infty,$$

then f *is a rational function.*

The proof of this result would use Cauchy facility and the upper bound we derived in Remark 3.3. The converse of this proposition is given as the following:

Proposition 3.4. *Let* f *be a non-constant rational function. Then there are coprime polynomials* P, Q *with* $f(z) = P(z)/Q(z)$. *Recall that the degree of* f, *considered as a finite morphism, is given by* $\deg(f) = \max\{\deg(P), \deg(Q)\}$.

We may compute directly

$$N(r,a,f) = \deg(P - aQ)\log r + O(1),$$

$$m(r,a,f) = (\deg(f) - \deg(P - aQ))\log r + O(1)$$

for $a \neq \infty$ *and*

$$N(r,\infty,f) = \deg(Q)\log r + O(1),$$

$$m(r,\infty,f) = (\deg(f) - deg(Q))logr + O(1).$$

This illustrates the first main theorem and shows that, if f *is a non-constant rational function, then* $T(r,f) = deg(f)\log r + O(1)$.

Finally, we make the following definitions, which would allow us present the second main theorem of Nevanlinna:

Definition 3.7. *Let* f_1 *and* f_2 *be two entire functions, then we define the Wronskian* $W(f_0, f_1)$ *of* $(f_0 : f_1)$ *as*

$$W(f_0, f_1) := \det\begin{pmatrix} f_0 & f_1 \\ f_0' & f_1' \end{pmatrix}$$

and hence we define

$$N_{ram}(r,f) := N(W(f_0,f_1), 0, r).$$

Where we note that this construction would actually allow the definition of N_{ram} to every meromorphic function f, since by Weierstrass factorization theorem, we know that every meromorphic function f may be written as f_1/f_0 with f_0, f_1 entire and shares no common zero.

With this, we are now ready to present Nevanlinna's theory. The most important aspect in Nevanlinna second theory is the following lemma:

Lemma 3.8. *The estimate*

$$m(r, \infty, f'/f) = O(\log T(r, f)) + O(\log r)$$

holds for r outside a set E of finite Lebesgue measure.

The proof of this is elementary, and the readers may check [14] chapter 3 for a complete proof.

We have seen that the average of the proximity function $m(r, a, f)$ on a circle of radius R is at most $\log 2 + \log^+(1/R)$. So that we expect the measure of teh values of a for which $m(r, a, f)$ is large, to the extent that almost as large as $T(r, f)$ in order must be close to 0, and we expect $N(r, a, f)/T(r, f)$ to be near 1 almost surely. And this is the statment of the Second Navenlinna's theorem:

Theorem 3.9. *Let $a_1, \cdots, a_q$ be different elements of $\mathbb{P}^1_{an} = \mathbb{C} \cup \{\infty\}$. Then*

$$\sum_{j=1}^{q} m(r, a_j, f) + N_{ram}(r, f) \leq 2T(r, f) + O(\log T(r, f)) + O(\log r)$$

outside of a set E of finite Lebesgue measure. Moreover, if f has finite order the result holds for all large r surely. (Not almost surely)

To end this chapter, we present the an analgoue of an *abc*-theorem for meromorphic functions, which we can see as a closely related consequence of Navenlinna's second theorem.

We first supply the following two definitions:

Definition 3.10. *Let f be a non-constant meromorphic function and let $a \in \mathbb{C}$. The truncated counting function $N^{(1)}(r, a, f)$ is*

$$N^{(1)}(r, a, f) := \min\{1, ord_0^+(f-a)\} \log r + \sum_{0<|z|<r} \min\{1, ord_0^+(f-a)\} \log |\frac{r}{z}|,$$

and for $a = \infty$ we define

$$N^{(1)}(r, \infty, f) := \min\{1, ord_0^-(f)\} \log r + \sum_{0<|z|<r} \min\{1, ord_0^-(f)\} \log |\frac{r}{z}|,$$

and hence

Definition 3.11. *Let f be a non-constant meromorphic function. The conductor of f in $|z| \leq r$ is by definition*

$$cond(r, f) := N^{(1)}(r, 0, f) + N^{(1)}(r, \infty, f) = N(r, \infty, f'/f).$$

Theorem 3.12. *Let f, g be non-constant meromorphic functions such that $f + g = 1$. Then*

$$T(r, f) \leq cond(r, fg) + O(\log T(r, f)) + O(\log r)$$

for all r outside of a set E of finite Lebesgue measure. Moreover, if f has finite order, then we may choose E bounded.

With this we concluded the section on Nevanlinna's theorem. We hope that more could be updated as a next step, to explain to greater detail its relationship with *abc* conjecture, as well as a potential next step in Vojta conjecture.

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