Pf $m^*(\vec{c}) - p_1^*(\vec{c}) - p_2^*(\vec{c})\big|_{A\times\{a\}} = T_a^*(\vec{c}) - \vec{c}$, and by seesaw principle. Now pull-back equation (a) by the morphism

$$A \longrightarrow A\times A \qquad a \longmapsto (a,-a)$$

we get $[-1]^*(\vec{c}) = -\vec{c}$.

$X(K)$ assume non-empty fix P_0 on X. Let T be an irred. variety. We say that $\vec{c} \in Pic(X\times T)$ is a subfamily of $Pic^0(X)$ parametrized by T if

(a) $\vec{c}_t \in Pic^0(X_{K(t)})$ for $\forall\, t\in T$.

(b) $\vec{c}_{p_0} = \vec{0} \in Pic(T)$

So by seesaw, $\vec{c}$ is uniquely determined by family $(\vec{c}_t)_{t\in T}$ and by condition (b).

Thm There is a subfamily $\vec{p}$ of $Pic^0(X)$, parametrized by an irreducible smooth complete variety B, with the following universal property: For $\forall$ subfamily $\vec{c}$ of $Pic^0(X)$, parametrized by an irredu. variety T, there is a unique morphism $\varphi: T \longrightarrow B$ with $(id_X\times\varphi)^*(\vec{p}) = \vec{c}$.

B is called a <u>Picard Variety</u> of X and $\vec{p}$ is Poincaré class. If $(B',\vec{p}')$ is another such pair, then $\exists\ \varphi: B'\to B$, $\varphi': B\to B'$ such that $\vec{p}' = (id_X\times\varphi)^*(\vec{p})$ and $\vec{p} = (id_X\times\varphi')^*(\vec{p}')$.

Since $(id_X\times(\varphi\circ\varphi'))^*(\vec{p}) = \vec{p}$, we conclude that $\varphi\circ\varphi' = id_B$ by uniqueness. Interchanging the role of $(B,\vec{p})$ and $(B',\vec{p}')$ we notice that φ is an isomorphism. In this sense the pair $(B,\vec{p})$ is uniquely determined.

In fact, we can endow a scheme structure, but that's beyond scope of our discussion.

Rmk: F-rational points of the Picard Variety may be identified with $Pic^0(X_F)$ for any extension F/K. In particular, the set of points of the Picard variety corresponds to $Pic^0(X_{\bar{K}})$. We denote Picard variety of X cursively by $\mathcal{P}ic^0(X)$ to distinguish from $Pic^0(X)$ of $Pic(X)$.

Cor Let F be an extension field of K.

(a) By base change, we have $Pic(X) \subset Pic(X_F)$

(b) $\mathcal{P}ic^0(X_F) = \mathcal{P}ic^0(X)_F$ and its Poincaré class is obtained from $\vec{p}$ by base-change to F.

(c) $\mathcal{P}ic^0(X)(F) = Pic^0(X_F)$ by identifying b with $\vec{p}_b$.

Rmk By seesaw Principle, the Poincaré class $\vec{p}$ is uniquely characterized by the conditions

(a) $\vec{p}_c = \vec{c}$ for $\forall\, \vec{c} \in \mathcal{P}ic^0(X)$

(b) $\vec{p}_{P_0} = \vec{0}$

Now the morphism is given by
$$\varphi(t) = \vec{c}_t = \breve{p}_{\varphi(t)}$$

This is clear by the restriction of $(id \times \varphi)^*(\vec{p}) = \vec{c}$ to the fibre $X \times \{t\}$ and then using the rule $(f \circ g)^* = g^* \circ f^*$ to show that
$$\vec{c}_t = (id_X \times \varphi)^*(\vec{p})|_{X \times \{t\}}\ (id_X \times \varphi(t))^*(\breve{p}) = \vec{p}_{\varphi(t)} = \varphi(t)$$

Thm Together with the canonical group structures induced by tensor product of line bundles, $Pic^0(X)$ is an abelian variety of K.

$\boxed{\text{In Summary:}}$

Thm Let X be an irreducible smooth complete variety over K and let $P_0 \in X(K)$ be a base point of X. Then the group $Pic^0(X_{\bar{K}})$ has a unique structure as an abelian variety over K, called the Picard variety and denoted by $Pic^0(X)$, with the properties:

(a) There is $\vec{p} \in Pic(X \times Pic^0(X))$ such that $\breve{p}_{\vec{b}} = \vec{b}$ for $\vec{b} \in Pic^0(X)$ and $\breve{p}_{p_0}$ is trivial.

(b) For any subfamily $\vec{c}$ of $Pic^0(X)$ parametrized by an irreducible variety T over K, the set theoric map $T \to Pic^0(X)$, $t \mapsto \breve{c}_t$ is actually a morphism over K.
The uniquely determined class $\breve{p}$ is called

the Poincaré class.

$\boxed{\text{Theorem of Square}}$ (Mostly just state, without proof).

For details, check §5 of Bombieri.

Prop Let $\vec{c} \in \mathrm{Pic}(A)$, and $a \in A$. Then $\varphi_{\vec{c}}(a) := T_a^*(\vec{c}) - \vec{c}$ $\in \mathrm{Pic}^0(A)(k(a))$ and $\varphi_{\vec{c}} : A \to \mathrm{Pic}^0(A)$ is a homomorphism of abelian varieties over k.

Thm (theorem of square). For $a, b \in A$, we have
$$T_{a+b}^*(\vec{c}) + \vec{c} = T_a^*(\vec{c}) + T_b^*(\vec{c})$$

Prop A class $\vec{c} \in \mathrm{Pic}(A)$ is ample iff $\ker(\varphi_{\vec{c}})$ is finite and in addition $H^0(A, n\vec{c}) \neq 0$ for some positive integer n.

Most importantly, the following statements:

Prop For $\vec{b} \in \mathrm{Pic}(A)$, the following statements are equivalent:

(a) $\vec{b} \in \mathrm{Pic}^0(A)$

(b) $\ker(\varphi_{\vec{b}}) = A$.

(c) For every ample $\vec{c} \in \mathrm{Pic}(A)$, there is $a \in A$ such that $\vec{b} = T_a^*(\vec{c}) - \vec{c}$.

(d) There is an ample $\vec{c} \in \mathrm{Pic}(A)$ such that $\vec{b} = T_a^*(\vec{c}) - \vec{c}$ for some $a \in A$.

Def The Picard variety $\mathrm{Pic}^0(A)$ is called the dual abelian variety of A and will be denoted by $\hat{A}$.

Cor The dual abelian variety $\hat{A}$ has the same dimension as A.

Theorem of Cube

This section discuss elementary facts for involutions and quadratic functions on an abelian group then prove the theorem of cubes.

Def M abelian group. with "$*$" involution, defined as a linear map $M \to M$ $x \mapsto x^*$. with $(x^*)^* = x$ for $\forall x \in M$.
$x \in M$ even if $x^* = x$
$x \in M$ odd if $x^* = -x$

Lem $x \in M$, then $2x$ has a decomposition into an even and an odd part. If the subgroup of odd elements is divisible by 2, then x also has such a decomposition.

Pf $2x = (x + x^*) + (x - x^*)$.
divisibility by $2 \Rightarrow \exists z$ odd, $2z = x - x^*$.
let $x_+ := x - z$ $x_- := z$.
$(x_+)^* = x^* - z^* = (x - 2z) + z = x_+$
so x_+ is even, done.

Ex A abelian variety over k. Consider the involution $\vec{c} \mapsto [-1]^* \vec{c}$ on the abelian group $\mathrm{Pic}(A)$.
Hence line bundle is even / odd if
$[-1]^* L \cong L$ / $[-1]^* L \cong L^{\otimes(-1)}$.

Def $q: M \to N$ set map on abelian groups
if $b: M \times M \to N$ $(x,y) \mapsto q(x+y) - q(x) - q(y)$
is bilinear, the q is quadratic function
with b associated bilinear form

Quadratic form: quadratic function which is homogeneous of degree 2.

Involution: $q^*(x) := q(-x)$
By lemma 8.6.2, decompose $2q$ into Q even, L odd, given by
$Q(x) := q(x) + q(-x)$ and $L(x) := q(x) - q(-x)$
Q: associated quadratic form
L: associated linear form

Lem $q: M \to N$ be a quadratic function, $n \in \mathbb{Z}$.
Then $q(nx) = \frac{n^2+n}{2} q(x) + \frac{n^2-n}{2} q(-x)$

for all $x \in M$.

Cor A quadratic function is even if and only if it is a quadratic form.

Ex: $M = (\mathbb{Z}/2\mathbb{Z})^2$, $N = \mathbb{Z}/2\mathbb{Z}$.
$q : M \to N$ $\quad q(\vec{x}) = 0$ iff $\vec{x} = \vec{0}$
then q is odd quadratic function not linear

Def $k \in \mathbb{N}^+$, $I \subset \{1, \dots, k\}$, define $S_I : M^k \to M$

$$S_I(x_1, \dots, x_k) = \sum_{i \in I} x_i$$

set $S_\phi = \vec{0}$

Lem let $q : M \to N$ be a quadratic function and let k be an integer. If $k \geq 3$, then we have for $\vec{x} \in M^k$:

$$\sum_{I \subset \{1, \dots, k\}} (-1)^{|I|} q(S_I(\vec{x})) = 0.$$

Apply the lemma to the abelian group $M = \mathrm{Mor}(X, A)$ of morphisms from X to A and to the abelian group $N = \mathrm{Pic}(X)$.

Thm Let X be a variety over the field K and let A be an abelian variety over K with $\vec{c} \in \mathrm{Pic}(A)$. Then the map of $\mathrm{Mor}(X, A)$ into $\mathrm{Pic}(X)$ given by $\varphi \mapsto \varphi^*(\vec{c})$ is quadratic.

Let $k \geq 3$ and $X = A^k$ with ith projection p_i onto A. For $I \subset \{1, \dots, k\}$, we have

$$S_I(p_1, \dots, p_k) = \sum_{i \in I} p_i.$$

Lemma 8.6.10 shows that the theorem implies

$$\sum_{I \subset \{1, \dots, k\}} (-1)^{I} s_I(p_1, \dots, p_k)^* (\vec{c}) = 0$$

For $k=3$, the equation

$$\sum_{I \subset \{1,2,3\}} (-1)^{|I|} \left(\sum_{i \in I} p_i\right)^\wedge (\vec{c}) = \vec{0}$$

is called the theorem of cubes.

For more information about theorem of cubes for varieties, and a version of Riemann Roch for abelian varieties:

$$\dim (\Gamma(A, L)) = \sqrt{|\det E(\lambda_j, \lambda_k)|}$$

The isogeny multiplication by n

A abelian variety over K field.

this section: study $[n]: A \longrightarrow A$ given by multiplication by $n \in \mathbb{Z}$ of the abelian variety A.

Significance: • Construction of Néron-Tate Height, needed for proof of Mordell-Weil.

over $\mathbb{C}$. ($k = \mathbb{C}$) we have

$$A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim(A)}.$$

Prop Let $\vec{c} \in \mathrm{Pic}(A)$ and $n \in \mathbb{Z}$, then
$$[n]^*(\vec{c}) = \frac{n^2+n}{2}\vec{c} + \frac{n^2-n}{2}[-1]^*\vec{c}.$$

In particular, we have $[n]^*\vec{c} = n^2\vec{c}$ if $\vec{c}$ is even and $[n]^*(\vec{c}) = -\vec{c}$ if $\vec{c}$ is odd.

Prop 8.7.2 Let $n \in \mathbb{Z}\setminus\{0\}$. Then $[n]$ is a finite flat surjective morphism of degree $n^{2\dim(A)}$. The separable degree of $[n]$ equals the number of points of any fibre. If $\mathrm{Char}(k) \nmid n$, then $[n]$ is an étale morphism and
$$A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2\dim(A)}$$

If $p = \mathrm{Char}(k) \mid n$, then $[n]$ is not separable.

Finally, We include some materials about curves & Jacobians.

Def The Picard variety of C is called the Jacobian Variety of C.

where C is an irreducible smooth projective curve over a field k of genus $g \geq 1$ with base-point $P_0 \in C(k)$.

Cor The Jacobian Variety of C has dimension g.

Lem Assume that the ground field K is algebraically closed. Let L be a line bundle on C. Then for every $P \in C$, we have

$$\dim \Gamma(C, L(-[P])) \geq \dim \Gamma(C, L) - 1.$$

Equality holds iff P is not a base-point of L.

Lem Let L be a line bundle on C and let $r \in \{1, \ldots, \dim \Gamma(C, L)\}$. Then there is a dense open subset U of C^r such that

$$\dim \Gamma\left(C_{\bar{K}}, L_{\bar{K}}\left(-\sum_{j=1}^{r} [P_j]\right)\right) = \dim \Gamma(C, L) - r$$

for all $(P_1, \ldots, P_r) \in U$.

Lem Let $r \in \{1, \ldots, g\}$. Then

$$U_r := \left\{(P_1, \ldots, P_r) \in C^r \,/\, \forall i \neq j \Rightarrow P_i \neq P_j,\ \dim \Gamma\left(C_{\bar{K}}, \mathcal{O}\left(\sum_{j=1}^{r} [P_j]\right)\right) = 1\right\}.$$

is an open dense subset of C^r.

Rmk For any $r \in \mathbb{N}$, we have a map

$$j_r : C^r \to J, \quad (P_1, \ldots, P_r) \mapsto cl\left(\sum_{j=1}^{r} [P_j] - r[P_0]\right)$$

since $j = j_1$ and addition on J are both morphisms, we easily deduce that j_r

is a morphism. Note that its image is closed because C^r is complete.

Let $a := j_r(P_1, \dots, P_r)$, then the fiber over a is

$$j_r^{-1}(a) = \{(Q_1, \dots, Q_r) \in C^r \mid \sum_{j=1}^{r} [Q_j] \sim \sum_{j=1}^{r} [P_j]\}$$

Let $r \in \{1, \dots, g\}$ and $(P_1, \dots, P_r) \in U_r$.

Then the fibre over a is obtained by permuting the entries, namely

$$j_r^{-1}(j_r(P_1, \dots, P_r)) = \{(P_{\pi(1)}, \dots, P_{\pi(r)}) \mid \pi \in S_r\}$$

By the dimension theorem, we conclude that $\dim j_r(C^r) = r$. In particular, Corollary 8.10.7 of [Bom] imply that morphism j_g is surjective. Moreover, $\Theta = j_{g-1}(C^{g-1})$ is indeed a divisor.

<u>Prop</u> The map $j : C \to J$, $P \mapsto cl([P] - [P_0])$ is a closed embedding.

Aside: An important role Jacobian Variety played is the theta divisor.

$$\Theta := \underbrace{j(C) + \dots + j(C)}_{g-1 \text{ times}}$$

The three lemmas presented just now in particular verified that Θ is indeed a divisor.

Rmk As a divisor on J, we also consider
$$\Theta^- := [-1]^* \Theta = \underbrace{\jmath(C) - \cdots - \jmath(C)}_{g-1}.$$

In $\mathrm{Pic}(J)$, we use $\theta := \mathrm{cl}(\mathcal{O}(\Theta))$ and $\theta^- := [-1]^* \theta$. For $a \in J$, we set $\jmath_a := T_a \circ \jmath$, i.e. $\jmath_a$ is the map $C \to J$ given by $\jmath_a(P) = \jmath(P) - a$.

Prop Assume k is algebraically closed. For all $(P_1, \ldots, P_g) \in C^g$, we have the rational equivalence relation
$$\sum_{i=1}^{g} [P_i] \sim \jmath_a^*(\Theta^-) \text{ of divisors on } C,$$
where $a := \jmath_g(P_1, \ldots, P_g)$.

If k is not algebraically closed, then Corollary 8.4.10 of [Bom] shows that rational equivalence holds over any field of definition for $P_1, \ldots, P_g$

Rmk The idea is to show that the intersection of $\jmath(C)$ and $\Theta^- + a$ is transverse for generic $(P_1, \ldots, P_g)$. Then the proposition will follow in the generic case from our first step below. An application of the theorem of square will lead to the general case.

Pf outline Step I: For $(P_1, \ldots, P_g) \in C^g$ with $\dim P(C, \mathcal{O}(\sum_{i=1}^{g} [P_i])) = 1$, we have

$$(\Theta^- + a) \cap j(C) = \{j(P_1), \ldots, j(P_g)\}$$

Step II: For $1 \le r \le g$, and $(P_1, \ldots, P_r) \in U_r$, the differential $dj_r : T_{C^r, (P_1, \ldots, P_r)} \to T_{J, j_r(P_1, \ldots, P_r)}$

is injective.

Step III: For generic $(P_1, \ldots, P_g) \in C^g$, the intersection of $\Theta^- + a$ and $j(C)$ is transverse

Step IV: proof of proposition for generic $(P_1, \ldots, P_g) \in C^g$

Step V: proof of proposition for all

$(P_1, \ldots, P_g) \in C^g$.

Cor: For all $(P_1, \ldots, P_g) \in U_g$ and $a := j_g(P_1, \ldots)$ we have $\sum_{i=1}^{g} [P_i] = j_a^*(\Theta^-)$ as an identity of divisors

Cor For $\vec{a} \in J = Pic^0(C)$, we have
$$j_{\vec{a}}^{*}(\Theta^-) - j^{*}(\Theta^-) = \vec{a}.$$

Finally, note there are two Poincaré classes in the context of Jacobians.

(1) $\vec{p}_C \in Pic(C \times J)$

(2) $\vec{p}_J \in Pic(J \times \hat{J})$, where $\hat{J}$ is the dual abelian variety of J.

Prop Let Δ denote the diagonal in $C \times C$, then
$$(id_C \times j)^{*}(\vec{p}_C) = cl(\mathcal{O}(\Delta - C \times \{P_0\} - \{P_0\} \times C))$$

Prop Let $m: J \times J \to J$ be addition and let p_1, p_2 be the projections of $J \times J$ onto the corresponding factor. For $\vec{c} := m^{*}\Theta^- - p_1^{*}\Theta^- - p_2^{*}\Theta^- \in Pic(J \times J)$ we have
$$(j \times id_J)^{*}(\vec{c}) = -\vec{p}_C$$

Prop Let $\varphi_{\vec{\Theta}}, \varphi_{\vec{\Theta}^-}$ be the morphism $J \to \hat{J}$ introduced in 8.5.1 let $\vec{c} := m^{*}\Theta^- - p_1^{*}\Theta^- - p_2^{*}\Theta^- \in Pic(J \times J)$ as in the previous proposition, then
$$(id_J \times \varphi_{\Theta^-})^{*}(\vec{p}_J) = (id_J \times \varphi_{\vec{\Theta}})^{*}(\vec{p}_J) = \vec{c}.$$

We summarize here our findings.

Given a curve C, $g \geq 1$ genus, base point $P_0 \in C(k)$, $\exists$ natural embedding j of C into the Jacobian variety. By § 5-1, we have a dual homomorphism $\hat{j} : \hat{J} \to J$. The theta divisor is defined by

$$\Theta := \underbrace{j(C) + \cdots + j(C)}_{g-1}$$

and the corresponding class in $\mathrm{Pic}(J)$ is denoted by θ. Let $\theta^{-} := [-1]^{*}\theta$ and $\varphi_\theta : J \to \hat{J}$ be the natural morphism introduced in § 5-1 of [Bom].

There are 3 canonical morphisms from $J \times J$ to J, namely:

① addition m ② first projection p_1
③ second projection p_2.

The pull-back of the Poincaré class $\wp_J \in \mathrm{Pic}(J \times \hat{J})$ by $id_J \times \varphi_\theta$ is equal to the class

$$c := m^{*}\theta^{-} - p_1^{*}\theta^{-} - p_2^{*}\theta^{-}$$

and it follows from previous

proposition that

$$\vec{c} = m^* \theta - p_1^* \theta - p_2^* \theta.$$

Finally, we note that:

Prop The map φ_θ is an isomorphism of J onto $\hat{J}$ whose inverse is $-\hat{j}$. Moreover, Θ is ample.