Prop Let $\mathcal{D}$ be a presentation of the Cartier divisor $D = D(s_D)$. Then there is a unique locally bounded metric on $\mathcal{O}(D) = L \otimes M^{-1}$ given on a local section s by

$$\|s(x)\| = \min_k \max_l \left| \frac{s t_l}{s_k}(x) \right|$$

The local height $\lambda_{\mathcal{D}}$ is equal to the local height $\lambda(D, \|\cdot\|)$ with respect to the Néron divisor $(D, \|\cdot\|)$.

Def An M-metric on a line bundle L is a family $\{\|\cdot\|_u\}_{u \in M}$ such that $\|\cdot\|_u$ is a metric with respect to the absolute value M_u satisfying the following compatibility condition:
For every $x \in X$ and $u_1, u_2 \in M$, with the same restriction to $K(x)$, the equation $\|\cdot\|_{u_1} = \|\cdot\|_{u_2}$ holds on $L_x(K(x))$.

Def M-metric locally bounded if for $\forall$ nowhere vanishing section of L on an open set U of X the function $(x, u) \longmapsto \log\|s(x)\|_u$ on $U \times M$ is locally M-bounded.

Hence Néron divisor $\hat{D}$ is a Cartier divisor with a locally-bounded M-metric on $\mathcal{O}(D)$

Thm Let D be a Cartier divisor on a complete variety X over K and let $(\|\cdot\|_u)_{u \in M}$, $(\|\cdot\|'_u)_{u \in M}$ be locally M-bounded metrics on

$\mathcal{O}(D)$ giving rise to local heights

$\lambda_{\hat{D}}(P, u) := -\log \| s_D(P) \|_u$

$\lambda_{\hat{D}'}(P, u) := -\log \| s_D(P) \|'_u$

For $v \in M_K$, there are constants $\gamma_v \in \mathbb{R}$, with $\gamma_v \neq 0$ only for finitely many v, such that $|\lambda_{\hat{D}'}(P, u) - \lambda_{\hat{D}}(P, u)| \leq \gamma_v$, for all $P \in X \setminus \operatorname{Supp}(D)$

Thm (Weil's Theorem of decomposition).
f - non-zero rational function on irreducible regular projective variety X/k.

$\operatorname{div}(f) = \sum_{j=1}^{n} m_j Y_j$, Y_j prime divisors.

so $\exists$ non-negative local ht λ_j relative to Y_j. For $\forall$ u on $\bar{K}$, $\exp(-\lambda_j(P, u))$ measures the u-distance from $P \in X(\bar{K})$ to Y_j. By the generalization of Prop 2.7.6 and Thm 2.7.14, $\exists$ family $(\gamma_v)_{v \in M_K}$ of ≥ 0 real number, $\gamma_v = 0$ for $\forall$ but finitely many v, such that

$$\sum_{j=1}^{m} m_j \lambda_j(P, u) - \gamma_v \leq -\log |f(P)|_u \leq \sum_{j=1}^{n} m_j \lambda_j(P, u) + \gamma_v$$

for any $u \in M$, $u | v \in M_K$, and $P \in X(\bar{K})$ and $P \notin \operatorname{supp}(\operatorname{div}(f))$.

Next goal: Let $\bar{L}$ be locally bounded M-metrized line bundle on K-variety X. Goal: define associated global height function $h_{\bar{L}}$.

Let $P \in X(\bar{K})$, choose a finite subextension F/K of $\bar{K}/K$ with $P \in X(F)$. For $\forall w \in M_F$ choose $\forall u \in M$ with $u|w$. consider

$$\|\cdot\|_w := \|\cdot\|_u^{[T_w : K_u] / [F:K]}$$

T_w: completion F_w if F/K separable. on $L_P(F)$. By the compatibility condition in the definition of an M-metric, the norm is independent of the choice of u.

$\exists$ an invertible meromorphic section s of L with $P \notin \text{supp}(D(s))$, since $\exists$ open dense trivialization of L in a neighborhood of P. Then the M-metric on L yields a Néron divisor $\hat{D}(s)$ and we set

$$\lambda_{\hat{D}(s)}(P, w) := -\log \|s(P)\|_w,$$

$$h_{\bar{L}}(P) := \sum_{w \in M_F} \lambda_{\hat{D}(s)}(P, w).$$

and Lemma 1.3.7, $h_{\bar{L}}(P)$ independent of choice of F. and 2.7.10 (c), clearly $h_{\bar{L}}(P)$ not depend on choice of s.

Prop For global height function:

(a) $h_{\bar{L}}$ depends only on isometry class of $\bar{L}$.

(b) $\bar{L}_1, \bar{L}_2$ locally-bounded M-metrized line bundles on X, then

$$h_{\bar{L}_1 \otimes \bar{L}_2} = h_{\bar{L}_1} + h_{\bar{L}_2}$$

(c) Let $\varphi: X' \to X$ morphism of k-varieties. Then $h_{\varphi^* \bar{L}} = h_{\bar{L}} \circ \varphi$.

(d) Let $\mathcal{D}$ be a presentation of Cartier divisor D, and let $\hat{D}$ be the associated Néron divisor. Then the global height w.r.t. $\mathcal{D}$ is equal to $h_{\mathcal{O}(\hat{D})}$.

(e) If X is complete (or more generally M-bounded) then $h_{\bar{L}}$ does not depend on the choice of the locally bounded M-metric up to bounded functions.

Rmk Every line bundle admits a locally-bounded M-metric. On complete varieties, the corresponding global height depends only on the isomorphism class of L up to bounded functions. Moreover, theorem 2.3.8 holds for arbitrary complete varieties X, Y over k.

Abelian Varieties

Goal: Fundamental properties of Abelian Varieties
Jacobians of Algebraic curves.
+ Group Varieties.
~~[illegible]~~ + Theorem of cubes / theorem of square.
+ isogeny multiplication by n.

Group Varieties

K field $\bar{K}$ algebraic closure of K.
Assume all varieties & morphisms are defined over K.

Def (Group Variety) Let G be a variety. If G is endowed with
$m: G \times G \to G, \quad (x,y) \mapsto xy$ (multiplication)
$\iota: G \to G \quad x \mapsto x^{-1}$ (inverse).
with an element $\varepsilon \in G(K)$ ~~[illegible]~~ ~~[illegible]~~ such that $G(\bar{K})$ is a group with multiplication, inverse, identity induced by m, ι, ε then G is a group variety.

If G_1, G_2 are group varieties with multiplications m_1, m_2 then a morphism $\varphi: G_1 \to G_2$ with $\varphi \circ m_1 = m_2 \circ (\varphi \times \varphi)$ is called a homomorphism of group varieties.

If in addition $\exists \psi: G_2 \to G_1$ $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity then φ (resp ψ) is an isomorphism of algebraic varieties. If $G_1 = G_2$ then a homo (iso) is called an endomorphism (automorphism) of group varieties.

Closed subvariety of G whose $\bar{K}$-rational points form a subgroup of $G(\bar{K})$ is a group variety. We say that it is a closed subgroup of G.

Def Abelian variety. Group Variety that is geometrically irreducible & reduced.

Homo (iso) / auto/endo of Abelian variety is a ______ of group varieties.

Abelian subvariety B of an abelian variety A is a geometrically reduced and irreducible closed subgroup of A since B is closed $\Rightarrow$ again B is an abelian variety.

Ex M_n : $n \times n$ matrices with entries in $\bar{K}$. $\mathbb{A}^{n^2}(\bar{K})$ with $M_n \Rightarrow$ irreducible affine group variety over K.
det: $M_n \to \mathbb{A}^1_K$ so affine open irreducible subvariety $GL(n)_K$, defined as the complement of vanishing locus of determinant

$GL(n)_K$ and matrix multiplication forms a group variety. All closed subgroups, as for example the special linear group

$$SL(n)_K := \{a \in GL(n)_K \mid \det(a) = 1\}$$

of upper triangular matrices, are affine group varieties.

<u>Lemma</u> (Constancy) X, Y, Z be varieties s.t. X is complete & X and Y are geometrically irredu. If $f: X \times Y \to Z$ is a morphism such that $f(X \times \{y_0\}) = \{z_0\}$

for some $y_0 \in Y$, $z_0 \in Z$ then $f(X \times \{y\})$ is a point for every $y \in Y$.

<u>Rmk</u> lem only holds when X is a complete variety. Take, e.g. $X = \mathbb{A}_K^1$, this is not complete, because $xy = 1$ is a closed subvariety of $\mathbb{A}_K^1 \times \mathbb{A}_K^1$, while its projection on the second factor is $\mathbb{A}_K^1 \setminus \{0\}$, which is not closed in $\mathbb{A}_K^1$.

Now consider $f: \mathbb{A}_K^1 \times \mathbb{A}_K^1 \to \mathbb{A}_K^1$, given by $(x, y) \mapsto xy$

so $f(\mathbb{A}_K^1 \times \{0\}) = \{0\}$, and clearly constancy lemma does not hold in this case.

<u>Cor</u> X, Y geometrically irreducible varieties with

at least 1 K-rational point. Assume that X is complete. Morphism $f: X \times Y \to G$ of a product into a group variety G factorizes as $f(x,y) = g(x)\, h(y)$ for suitable morphisms $g: X \to G$ and $h: Y \to G$.

Cor Let $\varphi: A \to G$ be a morphism of the abelian variety A into the group variety G. Then $\psi: A \to G \quad a \mapsto \varphi(a)\, \varphi(\varepsilon_A)^{-1}$ is a homomorphism of group varieties.

Pf Apply constancy lemma with $f: A \times A \to G$ given by $(x,y) \mapsto \psi(x)\psi(y)\psi(xy)^{-1}$

with y_0 and z_0 the identity elements ε_A, ε_G of A and G. We conclude that the restriction of f to $A \times \{y\}$ is a constant map for every y. Since $f(\{\varepsilon_A\} \times A) = \{\varepsilon_G\}$
$\Rightarrow$ f is constant, w/ image the identity of G.

Cor An abelian variety is commutative.

Pf By Cor, inverse map ι is a homomorphism
$\Rightarrow$ equivalent to commutativity

$\boxed{\text{So just for convention:}}$ $m(x,y) = x+y$

$\iota(x) = -x$

identity is denoted 0.
For $a \in A$. $T_a(x) := x+a$ is called a translation by a. (A abelian variety)

For $n \in \mathbb{Z}$ denote $[n]$ the endomorphism of A, which is multiplication by n. The kernel of $[n]$ is denoted by $A[n]$. This is the torsion subgroup of A. Also use this for any abelian group.

Prop Geometrically reduced group variety is smooth.

Pf Base change if necessary $\Rightarrow K$ algebraically closed. Now $\exists$ open dense smooth subset U. $\Rightarrow$ can define left & right-translation by a point of group variety. They are automorphisms and so left-translation of U is also smooth. If we vary left-translations, get an open cover of the group variety, proving the claim.

Prop For a group variety G/K TFAE:
(a) G is connected
(b) G is geometrically connected ($G_{\bar{K}}$ connected)
(c) G is irreducible
(d) G is geometrically irreducible.

Rmk Let $\varphi: G \to H$ be a homomorphism of group of varieties

Then the image $\mathrm{im}(\varphi)$ is a closed subgroup of H.

(Dimension Theorem) Theorem Let $\varphi: G \to H$ be a surjective homomorphism of irreducible group varieties. Then
$\dim(G) = \dim(H) + \dim(\ker(\varphi))$

Pf Let $\varphi: G \to H$ be a surjective homomorphism of irreducible group varieties. Then φ is flat. Moreover, if $\dim(G) = \dim(H)$, then φ is finite and $|\ker(\varphi)|$ is equal to the separable degree of the field $k(G)$ over $k(H)$.

Pf By generic flatness, there is an open dense subset U of G s.t. $\varphi|_U$ is flat. Assume first k algebraically closed $\Rightarrow$ may cover G by translates of U. $\Rightarrow \varphi$ is flat.

If k not alg. closed, then perform base change to $\bar{k}$. In this case work in category of schemes to ensure that a morphism over k is flat if and only if its base change to $\bar{k}$ is flat.

If $\dim(G) = \dim(H)$, then there is an open dense set U' of H such that φ induces a finite map $V := \varphi^{-1}(U') \to U'$ whose fibers have cardinality equal to the separable degree of $k(G)$ over $k(H)$. Also, this cardinality equals $|\ker(\varphi)|$. Again, we assume k algebraically closed to cover G by translates of U proves finiteness of φ. If k not alg. closed, then we use base-change to $\bar{k}$ and the fact that a morphism over k is finite iff its base change to $\bar{k}$ is finite.

Def Rational curve: Curve birational to $\mathbb{P}^1_k$. A variety X is called rationally-connected if any 2 points in $X(\bar{k})$ may be connected by a rational curve over $\bar{k}$. It follows from the constancy lemma that abelian varieties do not contain rational curves

In particular, a morphism $X \to A$ of X into abelian variety A contracts the rational curves of X into points. It follows that any morphism of a rationally connected variety, such as projective space $\mathbb{P}^n$, into an abelian variety is constant.

Prop Any $f: \mathbb{P}^1_k \longrightarrow G$ of projective line into a group variety is constant

Pf see 8.2.19 of [Bom].

Cor Let U be an open set of the projective line $\mathbb{P}^1_k$. Then any morphism $f: U \longrightarrow A$ of U into an abelian variety is constant.

Thm Let $\varphi: X \dashrightarrow G$ be a rational map of a smooth variety X to a group variety G and let U_{max} be the domain of φ. Then every irreducible component of $X \setminus U_{max}$ is of codimension 1.

Cor A rational map from a smooth variety to an abelian variety is a morphism.

Next, we prove that the differential of multiplication on a group variety is given by addition.

Prop Let $m: G \times G \to G$ be multiplication of a smooth group variety. Then the differential of m at ε is the map $T_{G,\varepsilon} \oplus T_{G,\varepsilon} \to T_{G,\varepsilon}$ given by addition of tangent vectors.

Cor Let G be a smooth variety and for $n \in \mathbb{Z}$, let $[n]: G \to G$ be the morphism $x \mapsto x^n$. Then the differential of $[n]$ at ε is the endomorphism of $T_{G,\varepsilon}$ given by multiplying tangent vectors with n.

Prop Let G be an irreducible smooth group variety. Then the tangent bundle T_G on G is a trivial vector bundle of rank equal to $\dim(G)$.

Elliptic Curves

From prop the contangent bundle of an abelian variety over the field K is trivial. Thus the abelian variety of dim 1 has genus 1 $\Rightarrow$ it is an elliptic curve.

Now: show E-curve has group structure and is an abelian variety.

Def An elliptic curve $/K$ is a geometrically irreducible smooth projective curve E of genus $g(E)=1$. defined over K, equipped with a rational point $P_0 \in E(K)$.

Rmk Geometrically irredu. $\Leftrightarrow$ irreducible.
Let E be an elliptic curve, over k and let D be a divisor on $E_{\bar{k}}$ of degree $\deg(D)>0$.

Space of global ~~function~~ sections $\Gamma(E_{\bar{k}}, \mathcal{O}(D))$ may be realized as subspace

$$L(D) := \{ f \in \bar{k}(E_{\bar{k}})^{\times} \mid \operatorname{div}(f) \geq -D\} \cup \{0\}$$

in $\bar{k}(E_{\bar{k}})$, using the homomorphism $s \mapsto s/s_D$

By Riemann-Roch theorem, we have

$$\dim_{\bar{k}} L(D) = \deg(D).$$

hence the corresponding linear system $|D_{\bar{k}}|$ has dimension $\deg(D)-1$. It follows that two distinct points on E are <u>never</u> rationally equivalent over $\bar{k}$.

Fix $P_0 \in E(k)$. , for $P_1, P_2 \in E(\bar{k})$
$D := [P_1]+[P_2]-[P_0] \Rightarrow \deg(D)=1$, $L(D)$ is one-dim, generated by some f, unique up to mult by scalar.
By construction f has pole divisor majorized by $[P_1]+[P_2]$. If $P_0 \notin \{P_1, P_2\}$
then f has pole divisor $[P_1]+[P_2]$ and vanishes at one other point P_3, which is the unique point rationally equiv to $[P_1]+[P_2]-[P_0]$.

Make sense even P_1/P_2 equals to P_0
Hence well-defined composition law on E by $(P_1, P_2) \longmapsto P_1+P_2 := P_3$

Now for addition:

Prop) If the group structure on an elliptic curve E/k with base pt $P_0 \in E(k)$ is given by the bijective map.
$E \longrightarrow \mathrm{Pic}^0(E_{\bar{k}})$, $P \mapsto \mathrm{cl}([P]-[P_0])$
then E is an abelian variety def'ed over k.

where $\mathrm{Pic}^0(E_{\bar{k}})$ is the group of rational equivalence classes of divisors of degree 0.

Prop) (Addition law) Let E be the elliptic curve in normal form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The origin O of the group E is the unique point at infinity and the group law $+$ is defined as follows. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ be two finite points on E and set

$$a = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{if } x_1 \neq x_2$$

$$a = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3} \quad \text{if } x_1 = x_2, \quad b = y_1 - ax_1$$

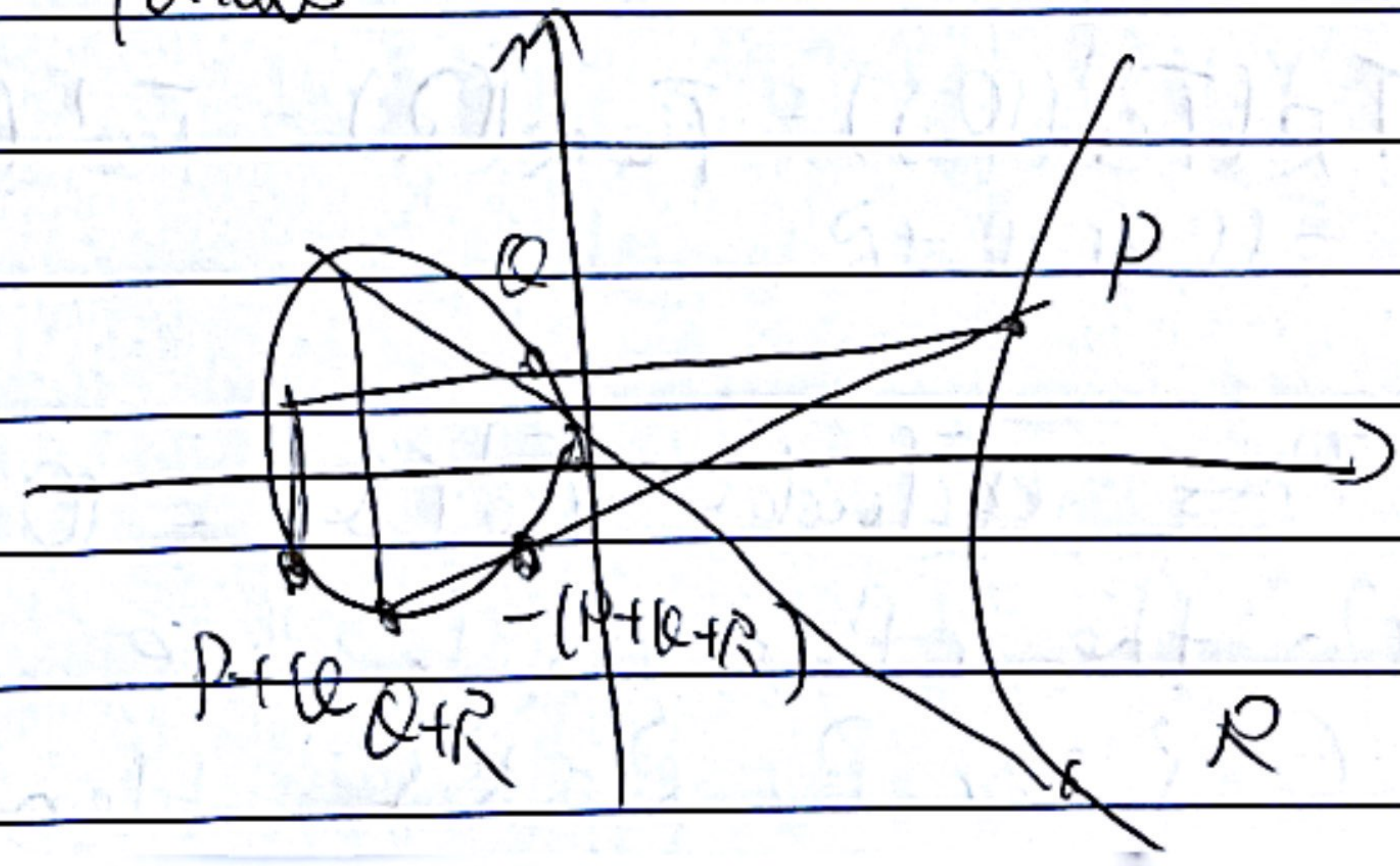
Then: (a) The inverse of P_1 is given by $-P_1 = (x_1, -a_1x_1 - a_3 - y_1)$.

(b) if $x_2 = x_1$ and $y_2 = -a_1x_1 - a_3 - y_1$, then $P_1 + P_2 = O$

(c) Otherwise, we have $P_1 + P_2 = (a^2 + a_1a - a_2 - x_1 - x_2, -(a + a_1)(a^2 + a_1a - a_2 - x_1 - x_2) - a_3 - b)$.

The addition & associative law for an elliptic curve in Weierstrass form can be seen visually as follows:

$\boxed{y^2 = x^3 - x}$



We may show that addition is a morphism. Since by construction it is a rational map.

WLOG (A-11.9) we may assume K alg. closed.
$\boxed{\text{First}}$: show T_Q by $Q \in E$ is a morphism (translat
May assume $Q \neq 0$

By formulas, T_Q is a rational map which restricts to a morphism $E \setminus \{0, Q, -Q\} \longrightarrow E \setminus \{Q, 0, Q+Q\}$.
Since every rational map between smooth projective curve extends to a morphism (A-11.10) get a morphism $T_Q' : E \to E$ which agrees with T_Q on $E \setminus \{0, Q, -Q\}$.

Remains to prove $T_Q = T_Q'$. For $R \in E$,
$T_Q' \circ T_R' = T_{Q+R}'$. In particular, every T_Q' is an isomorphism with the inverse T_{-Q}'.
Conclude that T_Q' maps $\{0, Q, -Q\}$ onto $\{Q, Q+Q, 0\}$. For any
$R \notin \{0, Q, -Q, Q+Q, -Q-Q\}$, we have

$T_R'(T_Q'(Q)) = T_{Q+R}'(Q) = T_Q'(T_R'(Q)) = T_Q'(Q+R)$
$= Q+Q+R$.

This excludes $T_Q'(Q) = Q$ immediately.
On the other hand, we know $T_Q'(Q)$
$\in \{0, R, R+R\}$. Hence $T_Q'(Q) = 0$

is only possible if $Q + Q = O$. This proves
$T_Q'(Q) = Q + Q = T_Q(Q)$

The equation $T_Q'(-Q) = O = T_Q(-Q)$

is proved in a similar fashion. Hence T_Q' is a bijection, hence $T_Q'(O) = Q = T_Q(O)$

All exceptions handled, thus $T_Q = T_Q'$

$\boxed{\text{Finally}}$: Show addition is a morphism. addition is a rational map, which is morphism outside of

$Z := \{(P,P) \mid P \in E\} \cup \{(P,-P) \mid P \in E\}$
$\cup (E \times \{O\}) \cup (\{O\} \times E)$.

for $(P,Q) \in Z$. There are $R, S \in E$ such that $(P+R, Q+S) \notin Z$. Since translations are morphisms by our above considerations, we see that

$$T_{-P-Q} \circ m \circ (T_R \times T_S) \text{ is a morphism in}$$

a neighborhood of (P,Q) and agrees with "+" everywhere. This proves + is morphism.

Picard Variety

If $\varphi : X \to Y$ is a morphism of varieties over K and $y \in Y$, then the fibre of φ over y is denoted by X_y. The pull-back of $\vec{c} \in \mathrm{Pic}(X)$ to the fibre X_y is denoted by $\vec{c}_y$. It is an element of $\mathrm{Pic}(X_y)$. Note that X_y and $\vec{c}_y$ are only defined over $k(y)$. Often, we identify X with X_y using $x \mapsto (x, y)$. of course this is only defined over $k(y)$.

Consider $\vec{c} \in \mathrm{Pic}(X \times Y)$, and fibres w.r.t. the projections p_1, p_2 onto the factors. For $x \in X$, $y \in Y$, we have

$$\vec{c}_y = \vec{c}\,|_{X \times \{y\}} \in \mathrm{Pic}(X_{k(y)})$$

$$\vec{c}_x = \vec{c}\,|_{\{x\} \times Y} \in \mathrm{Pic}(Y_{k(x)}).$$

<u>Thm</u> (See saw Principle). X geometrically irreducible smooth complete variety over K and Y an irreducible smooth variety over K. Let $\vec{c} \in \mathrm{Pic}(X \times Y)$ and suppose that there is a dense open subset U of Y such that $\vec{c}_y = 0$ for all $y \in U$. Then $\vec{c}$ is equal to the pull-back of an element of $\mathrm{Pic}(Y)$ by p_2.

<u>Pf</u> Let s be meromorphic section of $\vec{c}$. Then there is a rational function $f \in K(X \times Y)^{\times}$ such that $\mathrm{div}(f)$ and $\mathrm{div}(s)$ are equal on $X \times V$. Therefore, their difference $\mathrm{div}(f) - \mathrm{div}(s)$ is supported in $X \times Z$ where Z is a closed sub-variety of co-dimension 1 in Y. If $Z_1, \ldots, Z_r$ are irreducible components of Z, then $X \times Z_1, \ldots, X \times Z_r$ are irredu.

components of $X \times Z$. Therefore the divisor $\mathrm{div}(f) - \mathrm{div}(s)$ is a linear combination of the $X \times Z_i$ $(i = 1, \cdots, r)$, i.e. a pull-back of a divisor on Y. This proves the claim.

Cor Let X, Y be smooth varieties over k and assume that Y is irreducible and that X is complete and geometrically irreducible. Let $\vec{c} \in \mathrm{Pic}(X \times Y)$ with $\vec{c}_y = 0$ for all y in an open dense subset of Y and with $\vec{c}_x = \vec{0}$ for some $x \in X(k)$. Then $\vec{c} = \vec{0}$.

Pf By thm, have $\vec{c} = p_2^* \vec{c}'$ for some $\vec{c}' \in \mathrm{Pic}(Y)$. Now consider the closed embedding $\iota_x : Y \to X \times Y$, $y \mapsto (x, y)$, mapping Y isomorphically onto the fibre over x. Since $p_2 \circ \iota_x$ is the identity map on Y, we get

$$\vec{c}' = \iota_x^* p_2^* \vec{c}' = \vec{c}_x = \vec{0}.$$

This also proves that $\vec{c} = \vec{0}$.

Cor Let A be an abelian variety over k, let p_i be the ith projection $A \times A$ onto A, and let m be addition as usual. TFAE.

(a) $m^*(\vec{c}) = p_1^*(\vec{c}) + p_2^*(\vec{c})$ $\quad \vec{c} \in \mathrm{Pic}(A)$.

(b) $T_a^*(\vec{c}) = \vec{c}$ for $\forall a \in A$

If (a) and (b) are satisfied $[-1]^*(\vec{c}) = -\vec{c}$.