Now $\text{ind}(W) = \text{ind}(U) + \text{ind}(V)$.
we obtain upper bound by using Roth's lemma inductively on # variables to estimate $\text{ind}(U)$ and $\text{ind}(V)$.

Suppose true for $l < m$ variables. Now apply inductive assumption to $U$ & $V$ but with $(s+1)d_j$ in place of $d_j$. Now may verify assumptions are satisfied. Hence

$$\text{ind}(U) \leq 2(m-1)(s+1)\sigma^{1/2^{m-2}}, \quad \text{ind}(V) \leq (s+1)\sigma^{1/2^{m-2}}$$

Now $\min(\text{ind}(P), \text{ind}(P)^2) \leq 4(m-1)\sigma^{1/2^{m-2}} + 4\delta$.
$\Rightarrow \text{ind}(P) \leq m$. Hence preceding bound may be simplified to

$$\text{ind}(P)^2 \leq 4m(m-1)\sigma^{1/2^{m-2}} + 4m\varepsilon \leq 4m^2\sigma^{1/2^{m-2}}$$

this proves Roth's lemma.

## Now

## Roth's theorem

**Thm** $K$ #-field. $S$: finite set of places. $F$: f.d. extension of $K$. $v \in S$. Let $\alpha_v \in F$. $|\cdot|_v$ extends to $|\cdot|_{v,K}$ of $F$. Then for $\kappa > 2$ there are only finitely many $\beta \in K$ s.t.

$$\prod_{v \in S} \min(1, |\beta - \alpha_v|_{v,K}) \leq H(\beta)^{-\kappa}$$

The statement implies Roth's theorem by

embedding $\overline{K}$ into $\overline{K_v}$ for each $v \in S$, extending $|\cdot|_v$ to $\overline{K_v}$ using the reduction $|\cdot|_{v,K}$ so $F = K(\{\alpha_v | v \in S\}) \subset \overline{K}$.

<u>Pf</u> By contradiction, assume infinitely-many $\beta \in K$ would satisfy.

<u>Step 0</u>: Approximation classes.

Define $\Delta(\beta) = \prod_{v \in S} \min(1, |\beta - \alpha_v|_{v,K})$.

<u>Def</u> if $\beta \in K$ $\Delta(\beta) < 1$ $\Rightarrow$ we say non-trivial approximation

Now consider for $\beta$ non-trivial
$(\log \min(1, |\beta - \alpha_v|_{v,K}) / \log \Delta(\beta))_{v \in S}$.

this is a vector, and a point in $|S|$-dimensional unit cube lying on the hyperplane where the sum of coordinate is 1

Partition this cube by means of a grid of semi-open subcubes of side $1/N$, where $N$ is a positive integer, and classify $\beta$ according to subcube containing the corresponding vector.

Set of $\beta$ gives the same subcube is called an approximation class

$1/N$ is called size of approximation class.

$\vec{\lambda} := (\lambda_v)_{v \in S}$ be the south-west ~~center~~ corner of a subcube, namely

$$\vec{\lambda} = (\lfloor N x_v \rfloor / N)_{v \in S}$$

with $\vec{x}$ any point in the subcube.

$Q(\vec{\lambda})$ : corresponding sub-cube

$C(\vec{\lambda}; N)$ approximation class of size $1/N$ determined by $Q(\vec{\lambda})$.

For every $\beta \in C(\vec{\lambda}, N)$, $v \in S$ we have

$$\Delta(\beta)^{\lambda_v + \frac{1}{N}} < \min(1, |\beta - d_v|_{v,K}) \le \Delta(\beta)^{\lambda_v}$$

Note $\quad 1 - \frac{|S|}{N} \le \sum_{v \in S} \lambda_v \le 1$.

because $Q(\vec{\lambda})$ always contain a point $\vec{x}$ with $\sum x_v = 1$ if $C(\vec{\lambda}, N) \neq \phi$.

Lem # of approximation class of size $1/N$ determined by non-trivial approximations does not exceed

$$\binom{N + |S|}{|S|} < 2^{N + |S|}.$$

A Choose independent solutions

By hypothesis $\Delta(\beta) \le H(\beta)^{-K} S$ has infinitely many solutions, the preceding lemma shows for $\forall N$ $\exists$ approximation class $C(\vec{\lambda}, N)$

containing infinitely-many such $\beta$s.

Let $\beta_1, \ldots, \beta_m$ be elements of $K$ and let $M > 2$.
def $(\beta_j)$ to be $(L, M)$ independent if $h(\beta_1) \geq L$ and $h(\beta_{j+1}) \geq M h(\beta_j)$ for $j = 1, \ldots, n-1$.

Now Northcott's theorem $\Rightarrow$ $\forall$ infinite sequence $K$ contains an infinite sequence of $(L, M)$ independent elements. so for $\forall$ $N$, $(L, M)$ can find infinite subsequence of $(L, M)$ elements belonging to a fixed approximation class $C(\vec{t}, N)$.

## Step I: The auxilliary poly

$D$ be a large real #, which will be tending to $\infty$. choose

$$d_j = \lfloor D / h(\beta_j) \rfloor, \quad j = 1, \ldots, m.$$

Let $\vec{t} = (t_v)_{v \in S}$, with $t_v = (\frac{1}{2} - \varepsilon) m$, and let $\vec{\alpha_v} := (\alpha_v, \ldots, \alpha_v) \in F^m$, $\beta = (\beta_1, \ldots, \beta_m) \in K^m$

Also abbreviate $r := [F : K]$.
Now by $\left( V_m\left( (\frac{1}{2} - \varepsilon) m \right) \leq e^{-6m\varepsilon^2} \right)$ lem

we have $r V_m(\vec{t}) = r|S| V_m\left( (\frac{1}{2} - \varepsilon) m \right)$

$$< r|S| e^{-6m\varepsilon^2} \leq \frac{1}{2}$$

provided $m > \frac{\log(2r|S|)}{6\varepsilon^2}$

which we will assume for the rest of this section

Now hyp of $\boxed{\text{Lem } 6.3.4}$ is verified, we estimate (b) of Lem 6.3.4 by noting that

$$r V_m(\vec{t}) / (1 - r V_m(\vec{t})) \le 1.$$

So we obtained non-trivial poly $P$ with coefficients in $K$, partial degrees $d_1, \dots, d_m$ such that

$\operatorname{ind}(P; \vec{d}; \alpha_v) \ge (\frac{1}{2} - \varepsilon) m$ for $v \in S$, and

$$h(P) \le \Big\{ \sum_{v \in S} \sum_{j=1}^{m} (h(\alpha_v) + \log 2)/h(\beta_j) \Big\} D + o(D)$$

as $D \to \infty$. If we define

$$C_1 := |S| (\max_{v \in S} h(\alpha_v) + \log 2).$$

we obtain for large $D$ the bound

$$h(P) \le 2 C_1 D / L.$$

**Step II: application of Roth's lemma**

Would like $P$ to have additional property that $P(\vec{\beta}) \neq 0$. In order to do this, apply Roth's lemma to show $P$ does not vanish too much at $\vec{\beta}$ if $\vec{\beta}$ is $(L, M)$-indep and $L, M$ are large, then work with derivative of $P$.

Let $0 < \sigma \leq \frac{1}{2}$, by Roth's Lemma, we have $\operatorname{ind}(P; \vec{d}; \vec{\beta}) \leq 2m\,\sigma^{1/2^{m-1}}$ if $d_{j+1}/d_j \leq \sigma$ and $d_j h(\beta_j) \geq \frac{1}{\sigma}(h(P) + 4md_1)$

In case $d_j = \lfloor D/h(\beta_j) \rfloor \sim D/h(\beta_j)$ and $h(\beta_{j+1}) \geq M h(\beta_j)$ hence $d_{j+1}/d_j$ is verified if $M \geq 2\sigma^{-1}$.

and $D$ is large enough.

Similarly, using $d_j h(\beta_j) \sim D$, $d_1 \leq D/h(\beta_1) \leq D/L$ we see that $d_j h(\beta_j) \geq \sigma^{-1}(h(P) + c_1 m d_1)$ verified if

$$D \geq \sigma^{-1} \frac{2c_1 + 5m}{L} D.$$

which is so if $L \geq (2c_1 + 5m)\sigma^{-1}$

Thus if $M \geq 2\sigma^{-1}$, $L \geq (2c_1 + 5m)\sigma^{-1}$ and $D$ large enough,

$$\operatorname{ind}(P; \vec{d}; \vec{\beta}) \leq 2m\,\sigma^{1/2^{m-1}}$$

We choose $\sigma = \varepsilon^{2^{m-1}}$, and deduce that there is $\mu$ such that $\partial_{\vec{\mu}} P(\vec{\beta}) \neq 0$ and

$$\sum_{j=1}^{m} \frac{\mu_j}{d_j} \leq 2m\varepsilon.$$

Now by construction $\operatorname{ind}(P; \vec{d}; \vec{\alpha_v}) \geq (\frac{1}{2} - \varepsilon)m$. for $v \in S$. Let $Q = \partial_\mu P$, we have, with $C_1$ given by $(\ast\ast)$

**Lem** Suppose $\beta_1, \dots, \beta_m$ are $(L, M)$ indep with

$m > \log(2r|S|)/(6\varepsilon^2)$ and

$L \geq (2C_1 + Sm)\,\varepsilon^{-2^{m-1}}$, $\quad M \geq 2\varepsilon^{-2^{m-1}}$.

Then for $D$ suffi. large, $\exists$ polynomial $Q$ with $Q \in k[x_1, \dots, x_m]$, with partial degrees at most $d_j = LD/h(\beta_j)$, such that

(a) $\mathrm{ind}(Q; \vec{d}, \vec{\alpha}_v) \geq (\frac{1}{2} - 3\varepsilon)m$ for $v \in S$.

(b) $Q(\vec{\beta}) \neq 0$

(c) $h(Q) \leq 4C_1 D/L$.

## Step III: the upper bound

First an upper bound for $\log|Q(\vec{\beta})|_v$, for each place $v$. $\log^+ t = \max(0, \log t)$ for $t \geq 0$, define

$$\varepsilon_v := \begin{cases} [k_v : \mathbb{Q}_v]/[k:\mathbb{Q}] & \text{if } v \text{ archimedean} \\ 0 & \text{if } v \text{ non-archimedean} \end{cases}$$

Here $o(1) \to 0$ as $d_j \to \infty$.

If instead $v \in S$, expand $Q$ in Taylor series with center $\vec{\alpha}_v$, obtaining

$$Q(\vec{\beta}) = \sum \partial_{\vec{j}}\, Q(\vec{\alpha}_v)(\beta_1 - \alpha_v)^{j_1} \cdots (\beta_m - \alpha_v)^{j_m}$$

and estimate each term in Taylor expansion.

Now note by lemma 6.4.7.

$$\partial_{\vec{j}}\, Q(\vec{\alpha}_v) = 0 \quad \text{if } \frac{j_1}{d_1} + \dots + \frac{j_m}{d_m} < (\tfrac{1}{2} - 3\varepsilon)m$$

and also direct estimate yields

$$\log |\partial_{\vec{k}} Q(\vec{\alpha}_v)|_{v,k} \le \log |Q|_v + \sum_{j=1}^{m} \log^+ |\alpha_v|_{v,k} (d_j - k_j) + \varepsilon_v (\log 2 + o(1)) d_j$$

Now from the inequality

$$\log |a-b|_{v,k} \le -\log^+ \frac{1}{|a-b|_{v,k}} + \log^+ |a|_{v,k} + \log^+ |b|_{v,k} + \varepsilon_v \log 2$$

Hence $\log \left| \partial_{\vec{k}} Q(\vec{\alpha}_v) \prod_{j=1}^{m} (\beta_j - \alpha_v)^{k_j} \right|_{v,k}$

$$\le - \sum_{j=1}^{m} k_j \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,k}} + \log |Q|_v$$

$$+ \sum_{j=1}^{m} (\log^+ |\beta_j|_v + \log^+ |\alpha_v|_{v,k} + (\log 4 + o(1)) \varepsilon_v) d_j .$$

Now can estimate $\log |Q(\vec{\beta})|_v$

$$\log |Q(\vec{\beta})|_v = \log \left| \sum \partial_{\vec{k}} Q(\vec{\alpha}_v) \prod_{j=1}^{m} (\beta_j - \alpha_v)^{k_j} \right|_{v,k}$$

$$\le -\min' \left( \sum_{j=1}^{m} k_j \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,k}} \right) + \log |Q|_v$$

$$+ \sum_{j=1}^{m} (\log^+ |\beta_j|_v + \log^+ |\alpha_v|_{v,k} + (\log 4 + o(1)) \varepsilon_v) d_j ,$$

where min' taken over $(k_1, \dots, k_m)$ with $\frac{k_1}{d_1} + \dots + \frac{k_m}{d_m} \ge (\frac{1}{2} - 3\varepsilon) m$.

Now putting everything together, we have

$$\sum_{v \in M_K} \log |Q(\vec{\beta})|_v \le -\sum_{v \in S} \min{}' \left( \sum_{j=1}^{m} k_j \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,K}} \right) + h(Q)$$
$$+ \sum_{j=1}^{m} \left( h(\beta_j) + \sum_{v \in S} \log^+ |\alpha_v|_{v,K} + \log 4 + o(1) \right) d_j .$$

And $\sum d_j \le 2 d_1 \le 2 D/L + o(D/L)$.

$h(\beta_j) d_j \sim D$ $\quad h(\beta_j) \ge L$, hence

$$\sum_{v \in M_K} \log |Q(\vec{\beta})|_v \le -\sum_{v \in S} \min{}' \left( \sum_{j=1}^{m} k_j \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,K}} \right)$$
$$+ \left(m + \frac{C_2}{L}\right) D + o(D)$$

say $C_2 = 4C_1 + 4\log 2 + 2|S| \max_{v \in S} \log^+ |\alpha_v|_{v,K}$.

Now remains to estimate the minimum.
We use the fact that $\beta_j$ are of similar type.
Let $C(\vec{\lambda}; N)$ be the approximation class of approximations $\beta_j$, recall

$$\Delta(\beta_j) := \prod_{v \in S} \min(1, |\beta_j - \alpha_v|_{v,K})$$

Now $\lambda_v \kappa h(\beta_j) \le \lambda_v \log \frac{1}{\Delta(\beta_j)} \le \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,K}}$

$h(\beta_j) d_j \sim D$.

$$\Rightarrow \sum_{v \in S} \min{}' \left( \sum_{j=1}^{m} k_j \log^+ \frac{1}{|\beta_j - \alpha_v|_{v,K}} \right)$$
$$\ge \sum_{v \in S} \min{}' \left( \sum_{j=1}^{m} \lambda_v \kappa\, h(\beta_j) k_j \right) \sim D \kappa \left( \sum_{v \in S} \lambda_v \right) \min{}' \left( \sum_{j=1}^{m} \frac{k_j}{d_j} \right)$$

Now $\sum \lambda_v \geq 1 - |S|/N$ gives

$$\min'\left(\sum k_j \log |\alpha_j|\right) \geq \left(\frac{1}{2} - 3\varepsilon\right) m.$$

$$\sum_{v \in S} \min'\left(\sum_{j=1}^{m} k_j \log^{+} \frac{1}{|\beta_j - \alpha_j h_{v,k}|}\right) \geq \kappa\left(1 - \frac{|S|}{N}\right)\left(\frac{1}{2} - 3\varepsilon\right) m D$$
$$+ o(D)$$

$$\sum_{v \in M_K} \log |Q(\vec{\beta})|_v \leq -\kappa\left(1 - \frac{|S|}{N}\right)\left(\frac{1}{2} - 3\varepsilon\right) m D + \left(m + \frac{c_2}{L}\right) D$$
$$+ o(D)$$

Step IV lower bound.

since $Q(\vec{\beta}) \neq 0$ is in $K$, by product formula.

$$\sum_{v \in M_K} \log |Q(\vec{\beta})|_v = 0.$$

Step V Comparison of upper & lower bounds.

$$-\kappa\left(1 - \frac{|S|}{N}\right)\left(\frac{1}{2} - 3\varepsilon\right) + 1 + \frac{c_2}{mL} \geq 0.$$

Assuming $\frac{1}{2} > 3\varepsilon$

$$\kappa \leq \left(1 + \frac{c_2}{L}\right)\left(1 - \frac{|S|}{N}\right)^{-1}\left(\frac{1}{2} - 3\varepsilon\right)^{-1}$$

The right-hand side of this inequality tends to 2 as $\varepsilon \to 0$, $N \to \infty$ and $L \to \infty$. contradicting $\kappa > 2$.

# Metrized line bundles and local hts

$K$ field, $\overline{K}$ algebraic closure
fix $| \, |$ on $\overline{K}$.

**Def** $L$: line bundle on $K$-variety $X$. Metric on $L$ is a norm $\|\cdot\|$ on every fibre $L_x$, $x \in X$.
i.e. a real function $\neq 0$ s.t.

$$\|\lambda v\| = |\lambda| \cdot \|v\| \quad (\lambda \in \overline{K}, v \in L_x).$$

$(L, \|\cdot\|)$ metrized line bundle $\overline{L}$.
metric said locally bounded if $\log \|s\|$ is locally bounded on $U$ for every $U \subset X$ open and every nowhere-vanishing section $s \in L(U)$.

**Ex** $f$ regular func on $X$. $\log |f|$ is locally bounded on $\{x \in X \mid f(x) \neq 0\}$ trivial metric on $\mathcal{O}_X$ is characterized by $\|1\| = |1|$.

Let $\overline{L} = (L, \|\cdot\|)$, $\overline{M} = (M, \|\cdot\|)$ be metrized bundle on $K$-variety $X$.

$\overline{L} \otimes \overline{M}$ is the metrized line bundle $(L \otimes M, \|\cdot\|)$, where the metric on $L \otimes M$ is given

$$\|v \otimes w\| := \|v\| \cdot \|w\| \quad (v \in L_x, w \in M_x).$$

for $\forall x \in X$

Two metrized line bundles on $X$ are called isometric if there is an isomorphism which is fiberwise an isometry.

Isometry classes of the bundles on $X$ form a group $\widehat{\mathrm{Pic}}(X)$

$\mathcal{O}_X$ is the identity (with trivial metric).

and $(\overline{L})^{-1} = (L^{-1}, \|\cdot\|^{-1})$

**Def** $\varphi : X' \longrightarrow X$ morphism, pull-back $\varphi^*(\overline{L}) = (\varphi^* L, \|\cdot\|)$ as metrized line bundle on $X'$ with metric on $\varphi^*(L)$ characterized by $\|\varphi^*(s)\| = \|s\| \circ \varphi$ for any

Ex $\mathcal{O}_{\mathbb{P}^n_K}(1)$ standard metric: , $\ell$ - linear fun.

$$\|\ell(\vec{x})\| = \frac{|\ell(\vec{x})|}{\max_{j=0,\dots,n} |x_j|}$$

$|\cdot|$ archimedean $\Rightarrow$ Fubini-Study metric

$$\|\ell(\vec{x})\|_2 = \frac{|\ell(x)|}{\left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}}$$

May check both metrics are locally bounded.

**Prop** Every line bundle on an arbitrary variety over $K$ admits a locally bounded metric.

(For proof see Page 58-60 of Bombieri)

**Rmk** Sometimes useful to impose additional requirements on the metrics.

In the archimedean case, often convenient to require the functions $\|s\|$ to be $C^\infty$ for every open subset $U$ and non-vanishing $s \in \mathcal{L}(U)$. In this case, work with $\|\cdot\|_2$ on $\mathcal{O}_{\mathbb{P}^n_k}(1)$, because standard metric is non-differentiable.

Non archimedean: $|\cdot| = |\cdot|$ on $\bar{k}$ with place $u$, natural to assume $\|s(x)\| \in |k^\times|$ for every $x \in X$. May also assume $\|s\|$ are continuous w.r.t $u$-topology on $U(\bar{k})$ defined in 2.6.4. (See [Bom] for more info).

**Ex** Now go back to previous Ex, consider the presentation

$$D = (\ell(\vec{x}); \mathcal{O}_{\mathbb{P}^n_k}(1), x_0, \ldots, x_n; \mathcal{O}_{\mathbb{P}^n_k}(1)).$$

of hyperplane $D = \mathrm{div}(\ell(\vec{x}))$. Local height $\lambda_D$ is given by

$$\lambda_D(P) = \max_k \log \left| \frac{x_k}{\ell(\vec{x})} \right| \quad \text{for}$$

any $P = \vec{x} \notin \mathrm{supp}(D)$,

we conclude that $\lambda_D(P) = -\log \|\ell(\vec{x})\|$ depends only on $D$ and the standard metric $\|\cdot\|$ on $\mathcal{O}(D) = \mathcal{O}_{\mathbb{P}^n}(1)$.

This is used in what follows to generalize the concept of local heights, replacing the presentation by a suitably locally bounded metric on $\mathcal{O}(D)$.

Néron-divisor $\hat{D}$ on the $K$-variety $X$ is Cartier divisor $D$ on $X$ with a locally bounded metric on the line bundle $\mathcal{O}(D)$, corresponding locally-metrized line bundle is denoted by $\mathcal{O}(\hat{D})$. Note $D$ induces a canonical meromorphic section $s_D$ of $\mathcal{O}(D)$ and we have $D = D(s_D)$

Néron divisor on $X$ form a group with composition law

$$\hat{D} + \hat{E} = (D+E, \mathcal{O}(\hat{D}) \otimes \mathcal{O}(\hat{E})).$$

Let $\varphi: X' \to X$ be a morphism of $K$-varieties s.t. no irreducible component of $X'$ is mapped into $\mathrm{supp}(D)$. Then the pull-back $\varphi^A(D)$ is a well-defined Cartier divisor on $X'$.

and we define the pull-back $\varphi^A(\hat{D})$ as the Néron divisor

$$\varphi^A(\hat{D}) = (\varphi^A(D), \varphi^A \mathcal{O}(\hat{D})).$$

Local height associated to a Néron divisor $\hat{D} = (D, \|\cdot\|)$ on a variety $X$ is given by.

$$\lambda_{\hat{D}}(P) := -\log\|s_D(P)\|, \quad P \in X \setminus \mathrm{Supp}(D)$$

Prop. Let $\hat{D} = (D, \|\cdot\|)$ be a Néron divisor on the $K$-variety $X$.

(a) If $\hat{E}$ is a Néron divisor on $X$, then

$$\lambda_{\hat{D}+\hat{E}}(P) = \lambda_{\hat{D}}(P) + \lambda_{\hat{E}}(P), \quad P \notin \mathrm{Supp}(D) \cup \mathrm{Supp}(E)$$

(b) If $\varphi: X' \to X$ is a $K$-morphism such that no irreducible component of the $K$-variety $X'$ is mapped into $\mathrm{Supp}(D)$, then

$$\lambda_{\hat{D}} \circ \varphi(P') = \lambda_{\varphi^*(\hat{D})}(P'), \quad P' \in X' \setminus \varphi^{-1}\mathrm{Supp}(D)$$

(c) If $f$ is a rational function on $X$, not identically zero on any irreducible component, and $\hat{D}(f)$ denotes the Néron divisor with trivial metric on $\mathcal{O}(D(f)) = \mathcal{O}(X)$, then

$$\lambda_{\hat{D}(f)}(P) = -\log|f(P)|, \quad P \notin \mathrm{Supp}(D).$$

(d) If $\|\cdot\|'$ is another locally-bounded metric on $\mathcal{O}(D)$, then

$$\lambda_{(D,\|\cdot\|')}(P) - \lambda_{(D,\|\cdot\|)}(P) = \log \rho$$

where $\rho$ is the norm of $1 \in \Gamma(X, \mathcal{O}_X)$ with respect to locally bounded metric $\|\cdot\|/\|\cdot\|'$