Week 1

- Overview: what is diophantine geometry.
  (study of solutions in algebraically non-closed fields)

△ Module 01: Weil's height, Roth's theorem.

Height => what's height? why is it useful?
function: characterize the complexity of algebraic objects.
tells you size of solutions to diophantine equations
introduce heights for proj. varieties / then of polynomials.
then weil heights.
Siegel's lemma => Roth's theorem.

Diophantine Approximation => Roth's theorem:
Approximate algebraic numbers / $\mathbb{Q}$ by algebraic numbers in K.

△ Module 02, 03 Mordell - Weil.
Finite generation of group of rational points of an abelian variety defined over # field.

for this we need (Module 2)
① Abelian Variety.
② Néron - Tate Ht (for strong mordell - Weil).
③ Hilbert's irreducibility theorem

Then. Weak Mordell - Weil (& Chevalley Weil)
Strong Mordell - Weil

△ Module 04: Falting's theorem, abc - conjecture.
- Vojta's divisor
- Falting's theorem
- Belyi's Theorem
- abc - conjecture. (Introduce what this is).
No Inter-Universal Teichmüller theory etc.

△ Module 05: Nevalinna's theory, Vojta's conjecture
- Complex analysis based.
- Vojta's dictionary & conjectures.
- abc - theorem for function field.

△ (If interested) Zhang's theory on small pts.

Today's Goal!
- Introduce Height and Weil Height.

① Absolute Value: $|\cdot| : K \to \mathbb{R}$.
(a) $|x| \geq 0$ and $|x| = 0$ iff $x = 0$
(b) $|xy| = |x| \cdot |y|$
(c) $|x+y| \leq |x| + |y|$ ⇐ Archimedean
and (c') $|x+y| \leq \max\{|x|, |y|\}$ ⇐ Non-archimedean

Place: Equivalence class of $|\cdot|$
$|\cdot|_1 \sim |\cdot|_2 \iff |x|_1 = |x|_2^s$ for $\forall x \in K$, $s>$

$w|v$: w lies over v. If w place of L

v place of K, L/K, and $\forall$ representative of w is representative of v.

- Completion of K w.r.t. v : (K_v, w) ← field, ← place
  ① $w|v$
  ② Topology of K_v induced by w is complete.
  ③ K is dense in K_v

(1.3.2) **Cor** If $[L:K] < \infty$ finite-dimensional separable field Extension, then $\sum_{w|v} [L_w : K_v] = [L:K]$

where sum ranges over all places w of L with $w|v$.

$[L_w : K_v]$: local degree of L/K in w.

Now let K be field with non-trivial absolute value ($\neq 1$ everywhere).

Let $[L:K] < \infty$ separable extension with $w|v$.

Def $\|x\|_w := |N_{L_w/K_v}(x)|_v$

$|x|_w := |N_{L_w/K_v}(x)|_v^{1/[L:K]}$ for $x \in L$

(1.3.7) **Lem** Let $x \in K^\times$, $y \in L^\times$, then

$\sum_{w|v} \log |x|_w = \log |x|_v$ (1) $\sum_{w|v} \log \|y\|_w = \log |N_{L/K}(y)|_v$ (2)

Pf 1.3.2 ——— (1)

$N_{L/K}(y) = \prod_{w|v} N_{L_w/K_v}(y)$ ——— (2).

Height in Projective & Affine space

(Height): $\overline{\mathbb{Q}}$: an algebraic closure of $\mathbb{Q}$.
consider $\mathbb{P}^n_{\overline{\mathbb{Q}}}$: projective space.

Now we define a function called <u>height</u>

which measures "algebraic complication" needed to describe P.

Def Formally: $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$ pt. let $\vec{x} = (x_0 : x_1 : \cdots : x_n)$ be the homogeneous coordinate in K, where K number field. Then:

$$h(\vec{x}) = \sum_{v \in M_K} \max_j \log |x_j|_v .$$

side: $M_{\mathbb{Q}} := \{ |\cdot|_p \mid p \text{ prime or } p = \infty \}$

M_K: associated set of places and normalized absolute values, consisting of representatives $|\cdot|_w$ of $w|v$ for $v \in M_{\mathbb{Q}}$.

1.3.12 for more information

Lem $h(\vec{x})$ independent of choice of K.

Pf Let L be another number field. WLOG, assume $K \subset L$.
Then $\sum_{w \in M_L} \max_j \log |x_j|_w = \sum_{v \in M_K} \sum_{w|v} \max_j \log |x_j|_w$

Lem $h(\vec{x})$ independent of choice of coordinates.

Pf let $\vec{y}$ be another coordinate.
then we may assume $(x_0, \cdots, x_n, y_0, \cdots, y_n) \in K$
There is $\lambda \in K$ $\lambda \neq 0$ with $\vec{y} = \lambda \vec{x}$

Then $h(\vec{y}) = \sum_{v \in M_K} \max_j \log |y_j|_v = \sum_{v \in M_K} \log |\lambda|_v + \sum_{v \in M_K} \max_j \log |x_j|_v$

but for $\forall x \in K$ $\prod_{v \in M_K} |x|_v = 1$ by product formula.

Def $h(P)$ = absolute log height (or simply height).
and multiplicative Height $H(P) = e^{h(P)}$.

Rmk ① if $x_i \in \mathbb{Q}$, then $h(P) = \max_j \log |x_j|_\infty$
because non-archimedean have no contribution,
since we can take $\gcd(x_i) = 1$

② if $P \in \mathbb{A}^n_{\bar{\mathbb{Q}}}$, then embed
$P = (x_1, \cdots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$

$\log^+ t = \max(0, \log t)$. $h(x_1, x_2, \cdots, x_n) = \sum_{v \in M_K} \max_j \log^+ |x_j|_v$

$h(\alpha) = \sum_{v \in M_K} \log^+ |\alpha|_v$, $\alpha \in K$ algebraic #.
(K # field)

1.5.9

Theorem (Kronecker) Tells us when height is 0.
$\zeta \in \bar{\mathbb{Q}}^\times$ $h(\zeta) = 0 \Leftrightarrow \zeta$ is root of unity

$\Rightarrow$: If ζ is r.o.u. $\Rightarrow |\zeta|_v = 1 \Rightarrow h(\zeta) = 0$.
$\Leftarrow$: $h(\zeta) = 0 \Rightarrow |\zeta|_v \leq 1$ for $\forall v \in M_K$

$\Rightarrow \zeta$ is algebraic integer.
$\Rightarrow$ if $\deg(\zeta)=d$ denote $\zeta_1, \dots, \zeta_d$ be all conjugates.
consider $s_i(\zeta^m)$ $i=0, \dots, d$ of $\zeta_1^m, \dots, \zeta_d^m$
elementary symmetric poly.
$s_0 = 1$ $s_1 = \sum_{i=1}^{d} \zeta_i$ $s_2 = \sum_{1 \le i < j \le d} \zeta_i \zeta_j$ ...

$s_i(\zeta^m) \in \mathbb{Z}$. since ζ^m algebraic integer
(by Vieta's thm).
$$\Rightarrow \sum_{i=0}^{d} |s_i(\zeta^m)| \le \sum_{i=0}^{d} \binom{d}{i} = 2^d.$$
But $\zeta^1, \zeta^2, \dots, \zeta^k$ only finitely-many
so $\exists$ m, n such that $m > n$, same symmetric function
$\Rightarrow \zeta^m = \pi(\zeta^n)$ for π permutation.
$\zeta_i^{m^k} = \zeta_{\pi^k(i)}^{n^k}$ take k s.t. $\pi^k = 1$
then $\zeta^{m^k - n^k} = 1$ and $m^k - n^k > 0$ Done.

$S \subset M_K$ finite S_∞ all archimedean places.
Now: S-integers and S-units.
Def $\alpha \in K$ S-integer if $|\alpha|_v \le 1$ for $\forall v \notin S$.
S-inter of K forms $\mathcal{O}_{S,K}$ subring
units in $\mathcal{O}_{S,K}$ are S-units and form
a group $U_{S,K}$
$\alpha \in \mathcal{O}_{S,K}$ unit $\Leftrightarrow$ $|\alpha|_v = 1$ for $v \notin S$.

Def: $\phi: U_{S,K} \longrightarrow \mathbb{R}^{|S|}$ $\alpha \mapsto (\log |\alpha|_v)_{v \in S}$.

Thm (Dirichlet). S as above. ϕ is a lattice of maximal rank $|S|-1$ in $\sum_{v \in S} y_v = 0$

Hence $U_{S,K} \cong \mu_K \times \mathbb{Z}^{|S|-1}$

Pf By product formula $\sum_{v \in S} y_v = 0, \vec{y} \in \mathbb{R}^{|S|}$

By Kronecker $\ker(\phi) = \mu_K$

Prop If $P_1, \ldots, P_r$ are pts $\mathbb{A}^n_{\overline{\mathbb{Q}}}$ then.

$$h(P_1 + P_2 + \cdots + P_r) \leq h(P_1) + \cdots + h(P_r) + \log r.$$

Here addition is coordinate-wise addition.

Pf $h(P_1 + \cdots + P_r) = \sum_{v \in M_K} \max_j \log^+ |x_j^{(1)} + \cdots + x_j^{(r)}|_v$.

Non-archimedean $|x_j^{(1)} + \cdots + x_j^{(r)}|_v \leq \max_k |x_j^{(k)}|_v$.

Archimedean.

$|x_j^{(1)} + \cdots + x_j^{(r)}|_v \leq |r|_v \max_k |x_j^{(k)}|_v$.

$\sum_{v|\infty} \log |r|_v = \log r$.

$\Rightarrow h(P_1 + \cdots + P_r) \leq \log r + \sum_{v \in M_K} \max_{j,k} \log^+ |x_j^{(k)}|_v$

$\leq \log r + h(P_1) + \cdots + h(P_r)$.

Prop $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}(\mathbb{A}^n_{\overline{\mathbb{Q}}})$. if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

and P has (x_j) in $\overline{\mathbb{Q}}$ $\sigma(P)$ has $(\sigma(x_j))$, then

$h(P) = h(G(P))$

<u>Lem</u> $\alpha \in k\setminus\{0\}$, $\lambda \in \mathbb{Q}$ $h(\alpha^\lambda) = |\lambda| \cdot h(\alpha)$.

and $h(1/\alpha) = h(\alpha)$

<u>Pf</u> $\log|\alpha|_v = \log^+|\alpha|_v - \log^+|1/\alpha|_v$.

$$\sum_v \Rightarrow 0 = \sum(\ldots).$$

Thus by <u>Lem</u>, fundamental Inequallity

$$-h(\alpha) \leq \sum_{v \in S} \log|\alpha|_v \leq h(\alpha).$$

($S \subset M_k$ finite set of places, $\alpha \in k\setminus\{0\}$).

Now let $[L:k] < \infty$. Let $S \subset M_L$ finite set of places.

Diophantine approximation: approximate $\alpha \in L$ by $\beta \in k$ at all $w \in S$.

Denote $\|\cdot\|_{w,k} = |N_{L_w/k_v}(x)|_v$

for $x \in L$.

so <u>Thm</u> (Liouville's inequallity).

$$(2H(\alpha)H(\beta))^{-[L:k]} \leq \prod_{w \in S} \|\alpha - \beta\|_{w,k} \leq (2H(\alpha)H(\beta))^{[L:k]}$$

(Aside: function field Heights)

k field: $F = k(X)$ function field of irreduc. proj variety X/k, regular in codimension 1. $P \in \mathbb{P}^n(F)$. $P = (f_0 : \dots : f_n)$

$$h(P) = -\sum_{Z} \deg(Z) \min_{j} \operatorname{ord}_Z(f_j).$$

Z ranges over all prime divisors and degree w.r.t. fixed ample class.

f : rational function $f \in k(X)^\times$.

$$h(f) = h((1:f)) = -\sum_{Z} \deg(Z) \min(0, \operatorname{ord}_Z(f)).$$

$h(f) = 0$ iff f has no poles. (In $\mathbb{C}$, it's holomorphic)

$h(f) = h(f^{-1}) \Rightarrow \operatorname{div}(f) = 0$.

If X normal then X no poles $\Rightarrow$ X regular. hence $X = c$ on irreduc components $X_{\bar{k}}$.

$h(f) = 0 \Leftrightarrow f$ locally constant on X.

Heights of Polynomials

Def (height of a polynomial.

$$f(t_1, \dots, t_n) = \sum_{j_1, \dots, j_n} a_{j_1 \dots j_n} t_1^{j_1} \cdots t_n^{j_n} = \sum_{\vec{j}} a_{\vec{j}} t^{\vec{j}}$$

with $a_{\vec{j}} \in K$ is $h(f) = \sum_{v \in M_K} \log |f|_v$

where $|f|_v := \max_j |a_{\vec{j}}|_v$ is the Gauss norm for any places v.

Prop $f(t_1, \dots, t_n)$, $g(s_1, \dots, s_m)$

$h(fg) = h(f) + h(g)$.

Lem (Gauss). If v is non-archimedean

$|fg|_v = |f|_v |g|_v$.

Mahler measure.

$$M(f) := \exp\left(\int_{\mathbb{T}^n} \log |f(e^{i\theta_1}, \dots, e^{i\theta_n})| \, d\mu_1 \cdots d\mu_n \right)$$

$\mathbb{T} := \{ e^{i\theta} \mid 0 \le \theta < 2\pi \}$ $\quad d\mu = \frac{1}{2\pi} d\theta$.

$M(fg) = M(f) M(g)$.

$f(t) = a_d t^d + \dots + a_0$

$f(t) = a_d (t - \alpha_1) \cdots (t - \alpha_d)$.

$\log |t - \alpha|$ mean value: $\log^+ |\alpha|$.

so $M(t - \alpha) = \log^+ |\alpha|$.

Prop (Jensen's formula).

$$\log M(f) = \log |a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j|.$$

Connection between Mahler measure and the Height and gives a bound for the absolute norm of an algebraic #.

Prop $\alpha \in \overline{\mathbb{Q}}$, f its minimal polynomial over $\mathbb{Q}$.
Then $\log M(f) = \deg(\alpha)\, h(\alpha)$.
In particular $\log |N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)| \leq \deg(\alpha)\, h(\alpha)$.

Pf $d = \deg(\alpha)$, write $f(t) = a_d t^d + \dots + a_0$
choose $\alpha \in K$ and $K/\mathbb{Q}$ is galois. then
$(\sigma\alpha)_{\sigma \in G}$ contains every conjugate of α exactly
$[K:\mathbb{Q}]/d$ times. Gauss's lemma gives

$$|a_d|_v \prod_{\sigma \in G} \max(1, |\sigma\alpha|_v)^{d/[K:\mathbb{Q}]} = 1$$

for any non-archimedean $v \in M_K$

$$[K:\mathbb{Q}]\, h(\alpha) = \sum_{v \in M_K} \sum_{\sigma \in G} \log^+ |\sigma\alpha|_v$$

$$= \sum_{v | \infty} \sum_{\sigma \in G} \log^+ |\sigma\alpha|_v - \frac{[K:\mathbb{Q}]}{d} \sum_{v \nmid \infty} \log |a_d|_v$$

$$= \frac{[K:\mathbb{Q}]}{d} \sum_{v|\infty} \left(\log |a_d|_v + \sum_{j \geq 1}^{d} \log^+ |\alpha_j|_v \right).$$

by product formula & collected elements $\sigma\alpha$ into the conjugates $j = 1, \dots, d$ of α.
So first claim follows by Jensen's Formula.

Now by $\boxed{1.3.7}$, $\log |N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)|$ $\sum_{v|\infty} \sum_{j=1}^{d} \log^+ |\alpha_j|_v$.

Def for $p \geq 1$ finite denote by $l_p(f)$ the norm

$$l_p(f) := \left(\sum_{j=0}^{d} |a_j|^p \right)^{1/p}.$$

$p = \infty$. $\quad l_\infty(f) = \max |a_j| = |f|_\infty$.

Lem If $f(t)$ is above, then $M(f) \le \ell_1(f)$.
Moreover

$$\binom{d}{\lfloor d/2 \rfloor}^{-1} \ell_\infty(f) \le M(f) \le \ell_2(f) \le (d+1)^{1/2} \ell_\infty(f)$$

Pf See [Bom] 1.6.7.

The consequence is Northcott's theorem.
(We will use it in proof of Roth's theorem).

Thm There are only finitely-many algebraic numbers of bounded degree and bounded height

Proof α algebraic degree d. $h(\alpha) \le \log H$
Let $f(t) = a_d t^d + \cdots + a_0$ be minimal polynomial of α over $\mathbb{Z}$. Now by Lem
$\max |a_i| \le 2^d M(f)$. So coefficients of f are bounded by $(2H)^d$. But there are $d+1$ integer coefficients $\Rightarrow$ $(2\lfloor (2H)^d \rfloor + 1)^{d+1}$ distinct polynomials f. But f has d roots $\Rightarrow$ # alg integers of degree d height at most H is at most

$$d\,(2\lfloor (2H)^d \rfloor + 1)^{d+1} \le (5H)^{d^2+d}.$$

Next we also note Gelfond's lemma

Lem (Gelfond). If $f_1, \ldots, f_m$ are complex polynomials in n variables and set $f = f_1 \cdots f_m$, then

$$2^{-d}\prod_{j=1}^{m} l_\infty(f_j) \le l_\infty(f) \le 2^{d}\prod_{j=1}^{m} l_\infty(f_j).$$

where d is the sum of partial degrees of f

Lem 16.10 Let $f(x_1,\dots,x_n)$ be poly of complex coefficients and partial degrees $d_1,\dots,d_n$.

Then $\prod_{j=1}^{n}(d_j+1)^{-1/2} M(f) \le l_\infty(f) \le \prod_{j=1}^{n}\binom{d_j}{\lfloor d_j/2\rfloor} M(f)$

Lem Let $a\le A$, $b\le B$, d be natural numbers.

$\binom{A}{a}\binom{B}{b} \le \binom{A+B}{a+b}$, $\binom{d}{\lfloor d/2\rfloor}(d+1)^{1/2} \le 2^{d}$.

Thm Let $f_1,\dots,f_m$ be polynomials in n variables with coefficient in $\overline{\mathbb{Q}}$, and let d be the sum of partial degrees of $f = f_1 f_2 \cdots f_m$

then $-d\log 2 + \sum_{j=1}^{m} h(f_j) \le h(f) \le d\log 2 + \sum_{j=1}^{m} h(f_j)$

For more on lower/upper bound, read chap 1.

Weil Heights

Def (Weil Divisor) Formal linear combination of codim 1 irreducible subvarieties

(Cartier Divisor) collection of rational functions f_i on U_i where $X = \bigcup U_i$ open cover.

such that f_i/f_j has no poles, no zeros on $U_i \cap U_j$, up to "$\sim$"

where $(U_i, f_i) \sim (W_j, g_j)$.

if $\exists \{Z_k\}$ as a common refinement of $\{U_i\}$ and $\{W_j\}$, and if $Z_k \subset U_i$

$Z_k \subset W_j$, then $(f_i|_{Z_k})/(g_j|_{Z_k})$ has no zeros and no poles.

Now: K is a field, $|\cdot|$ fixed on $\overline{K}$

Let X be irreducible projective variety.

Now let D cartier divisor, $\mathcal{O}(D)$, meromorphic section s_D. For construction of $\mathcal{O}(D)$, s_D.

ample

check Appendix of [Bom]. Note cartier divisor $D(s_D)$ of s_D is equal to D.

Took L, M: base-point free line bundles on X such that $\mathcal{O}(D) \cong L \otimes M^{-1}$

Base-point free $\Rightarrow$ generating global section $s_0, \dots, s_n$ of L and $t_0, \dots, t_m$ of M

call $\mathcal{D} = (s_D; L; \vec{s}; M; \vec{t})$

presentation of cartier divisor D.

Def If $P \notin \mathrm{Supp}(D)$, define

$$\lambda_D(P) := \max_k \min_l \log \left| \frac{s_k}{t_l s_D}(P) \right|$$

here $t_l s_D$ means $t_l \otimes s_D$, and s_k / s' means $s_k \otimes (s')^{-1}$. Hence $s_k / (t_l s_D)$ is a rational function on X.

We call $\lambda_D(P)$ local heights of $P \in X$ relative to D.

Ex f non-zero rational function on X with Cartier divisor $D := D(f)$. $\mathcal{O}(D) = \mathcal{O}_X$ and f is meromorphic section of $\mathcal{O}(D)$

Thus there's λ_f relative to D, given by presentation $(f; \mathcal{O}_X, 1; \mathcal{O}_X, 1)$.

for $P \notin \mathrm{Supp}(D)$ $\lambda_f(P) = -\log|f(P)|$

and if g is rational function $g \neq 0$ on X

$\lambda_{fg} = \lambda_f + \lambda_g$ $\lambda_{f^{-1}} = -\lambda_f$

Def If D_1, D_2 divisors $\to$

$D_i = (s_{D_i}; L_i, \vec{s}_i; M_i; t_i)$.

$\vec{s}_1 \vec{s}_2 = (s_{1k} s_{2k'}) \to L_1 \otimes L_2$

$\vec{t}_1 \vec{t}_2 = (t_{1l} t_{2l'}) \to M_1 \otimes M_2$

$\lambda_{D_1 + D_2}$ is define as local height to

~~global~~ relative to presentation

$$D_1 + D_2 = (S_{D_1}, S_{D_2}; L_1 \otimes L_2, \vec{s_1}\vec{s_2}; M_1 \otimes M_2; \vec{t_1}\vec{t_2})$$

And with this presentation

$$\lambda_{D_1+D_2}(P) = \lambda_{D_1}(P) + \lambda_{D_2}(P).$$

for $P \in X$, $P \notin \mathrm{Supp}(D_1) \cup \mathrm{Supp}(D_2)$

& $\lambda_{-D}(P) = -\lambda_D(P)$. for $P \in X \setminus \mathrm{Supp}(D)$.

Def (Pull-back). $D = (S_D; L, \vec{s}; M, \vec{t})$.

Let $\pi : Y \to X$ dominant morphism.

Then $\pi^* D = (\pi^* S_D; \pi^* L, \pi^* s; \pi^* M, \pi^* \vec{t})$

is a presentation of $\pi^* D$

$\lambda_{\pi^* D}(P) = \lambda_D(\pi(P))$ for $\forall P \in Y$ with $\pi(P) \notin \mathrm{Supp}(D)$.

(for all $\pi : Y \to X$ of irreduc. proj. variety such $\pi(Y) \not\subset \mathrm{Supp}(D)$, above works).

Def The set $E \subset U(\bar{K})$ is bounded in U if for $\forall f \in k[U]$ the function $|f|$ is bounded on E.

where U is affine variety over k.

Lem Let $\{f_1, \ldots, f_N\}$ be the generators of $k[U]$

as a k-algebra. If $\sup_{P \in E} \max_{j=1,\dots,N} |f_j(P)| < \infty$

then E is bounded.

Next we show two presentation cannot be "too far away"

Thm Let X be a projective variety $/k$ and let $\mathcal{D}, \mathcal{D}'$ be two presentations of D.

Then $|\lambda_{\mathcal{D}} - \lambda_{\mathcal{D}'}| \leq \gamma$ for $\gamma < \infty$ constant.

To show this we'll need the lemma

Lem If $\{U_i\}$ is a finite affine open covering of the affine k-variety U and if E is bounded in U, then there are bounded subsets E_ℓ of U_ℓ such that $E = \bigcup_\ell E_\ell$.

Rmk By Hilbert's Nullstellensatz, γ in theorem above is effectively computable in terms of $\mathcal{D}$ and $\mathcal{D}'$.

Global Heights

Now let $K \subset F \subset \bar{K}$, where F is a number field. Let $P \in X(F) \setminus \operatorname{supp}(D)$.

Let $v \in M_F$ define **Local Height**

$$\lambda_D(P,v) := \max_k \min_l \log \left| \frac{s_k}{t_l s_D}(P) \right|_v .$$

Now let $p \in M_{\mathbb{Q}}$ s.t. v restrict to $\mathbb{Q}$ gives p. Let $|\cdot|_u$ for $\bar{K}$ with $u|v$ and $u|p$.

Then $\lambda_D(P,v) = \dfrac{[F_v : \mathbb{Q}_p]}{[F:\mathbb{Q}]} \lambda_D(P,u)$.

Ex The hyperplane $\{x_0 = 0\}$ in $\mathbb{P}^n_K$

have $D = (x_0; \mathcal{O}_{\mathbb{P}^n}(1), x_0, \dots, x_n; \mathcal{O}_{\mathbb{P}^n}, 1)$.

For $P \in \mathbb{P}^n(F)$ $x_0(P) \neq 0$, $v \in M_F$.

$$\lambda_D(P,v) = \max_k \log \left| \frac{x_k}{x_0}(P) \right|_v ,$$

and $h(P) = \sum_{v \in M_F} \lambda_D(P,v)$.

so this explains the "(local)" height.

Now λ_D local height of D on X
for $P \in X$ $\exists$ s_j and t_l s.t. $s_j(P) \neq 0$, $t_l(P) \neq 0$. Hence $\exists \neq 0$ sections s of $\mathcal{O}(D)$ such that P is not contained in the support of Cartier divisor $D(s)$.

Then $\mathcal{D}(s) = (s; L, \vec{s}; M, \vec{t})$ is a presentation of $D(s)$ and $\lambda_{D(s)} = \lambda_D + \lambda_f$

where f rational function s/s_D.

Now if $P \in X(F)$, $\lambda_{D(s)}(P,v)$ is finite for $\forall v \in M_L$. Because P is not in the support $D(s)$. Then we define global height of P rel $\lambda := \lambda_D$ by

$$h_\lambda(P) := \sum_{v \in M_F} \lambda_{D(s)}(P,v).$$

Prop The global height indep of choice of F and of the meromorphic section s.

Pf Lem 1.3.7 tells you indep of F.
Now for indep of s, it can be verified as follows. Let t be another $\neq 0$ meromorphic section of $\mathcal{O}(D)$ with $P \notin \mathrm{Supp}(D(t))$ then:

$$\lambda_{D(s)}(P,v) - \lambda_{D(t)}(P,v) = \lambda_{s/t}(P,v).$$

for $\forall v \in M_F$. On the other hand, the product formula shows global height relative to $\lambda_{s/t}$ is 0, so we are done.

Rmk Consequence: the global height relative to the natural local height of a non-zero function is identically 0. And also clear that $\lambda \mapsto h_\lambda$ is a group homomorphism.

Thm λ, λ' be local hts re Cartier divisor D, D' with $D - D'$ a principal divisor. Then $h_\lambda - h_{\lambda'}$ is a bounded function.

Now recall $\mathrm{Pic}(X) \cong \dfrac{\mathrm{CDiv}(X)}{\mathrm{Prin}(X)}$.

$\circledast$ $\quad cl(D) \mapsto cl(\mathcal{O}(D))$.

Now denote the real functions on X by $\mathbb{R}^X$ and space of bounded functions by $\mathcal{O}(1)$.

Def Let $\vec{c} \in \mathrm{Pic}(X)$ and chose a cartier divisor D with $\vec{c} = cl(\mathcal{O}(D))$ and local height λ relative to D. so image $h_{\vec{c}}$ of h_λ under projection

$$\mathbb{R}^X \longrightarrow \mathbb{R}^X / \mathcal{O}(1)$$

is independent of choice of D and λ. A representative of $h_{\vec{c}}$ is called a ht function associated to $\vec{c}$.

Advantage: Attractive functorial properties.
Disadvantage: Throw away finer properties of heights.

$\hookrightarrow$ "up to a constant problem" solved by Faltings's Height.

Thm The map: $\vec{h}: \mathrm{Pic}(X) \to \mathbb{R}^X / O(1)$ given by $\vec{c} \mapsto h_{\vec{c}}$ is a homomorphism. If $\varphi: Y \to X$ is a morphism of irreducible projective varieties over k, then

$$\vec{h}_{\varphi^* \vec{c}} = \vec{h}_{\vec{c}} \circ \varphi \quad \text{for } \forall \vec{c} \in \mathrm{Pic}(X)$$

Rmk Base-pt free line bundle has always non-negative ht function.

Prop D be an effective cartier divisor. Then $\exists$ local height λ relative to D such that for $\forall P \notin \mathrm{supp}(D)$ and for any w of $\bar{K}$, it holds $\lambda(P, w) \geq 0$.

Pf $\exists$ L, M of X bpf $O(D) \simeq L \otimes M^{-1}$.
choose global sections $t_0, \ldots, t_l$ of M.

Complete $s_D t_0, \ldots, s_D t_l$ to a family $s_0, \ldots, s_k$ of generating global section of L.

local height given by the presentation

$D = (s_D; L; s; M, t)$. is non-negative outside of support of D.

Weil Heights

We will see that any global height is the difference between two Weil Heights.

Let X be a projective Variety over $\overline{\mathbb{Q}}$

Def Let $\varphi: X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ be a morphism over $\overline{\mathbb{Q}}$. The Weil Height of $P \in X(\overline{\mathbb{Q}})$ relative to φ is defined by $h_\varphi(P) := h \circ \varphi(P)$, where h is the usual height on $\mathbb{P}^n_{\overline{\mathbb{Q}}}$.

Def If $\psi: X \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$ is another morphism over $\overline{\mathbb{Q}}$, the join $\varphi \# \psi$ is the morphism

$$X \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}_{\overline{\mathbb{Q}}}, \quad x \longmapsto (\varphi_j(x)\psi_k(x))$$

with lexicographic ordering on pairs (j,k)

Rmk If φ is a closed embedding, then $\varphi \# \psi$ is a closed embedding.

Prop If $\varphi: X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ and $\psi: X \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$ are morphisms over $\overline{\mathbb{Q}}$, then

$$h_{\varphi \# \psi} = h_\varphi + h_\psi.$$

Rmk Every Weil height may be viewed as a global height in the sense of previous section. $\exists\, \ell = l_0 x_0 + \dots + l_n x_n$ which does not vanish identically on any irreducible components of X. Then from previous section,

h_φ is the global height relative to the presentation $\varphi^*(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{Q}}}}(1), x_0, \dots, x_n, \mathcal{O}_{\mathbb{P}^n_{\overline{\mathbb{Q}}}}, 1)$

$\Leftarrow$ Conversely, every global ht is the difference of two Weil hts.

Let h_λ be the global height relative to the presentation-

$$\mathcal{D} = (s; L, s_0, \dots, s_n; M, t_0, \dots, t_m)$$

Consider morphisms

$$\varphi : X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}, \quad x \mapsto (s_0(x) : \dots : s_n(x))$$

and

$$\psi : X \to \mathbb{P}^m_{\overline{\mathbb{Q}}}, \quad x \mapsto (t_0(x) : \dots : t_m(x))$$

Then from independence of h_λ from s that

$$h_\lambda = h_\varphi - h_\psi.$$

Thm If $\varphi : X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ and $\psi : X \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$ are morphisms over $\overline{\mathbb{Q}}$ with $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \simeq \psi^*(\mathcal{O}_{\mathbb{P}^m}(1))$, then $h_\varphi - h_\psi$ is a bounded function.

Finally: (Northcott's theorem) Let X be proj. variety defined over $K/\mathbb{Q}$. Let $h_{\vec{c}}$ be the height function associated to an ample class $\vec{c} \in \operatorname{Pic}(X)$. Then the set

$$\{P \in X(\overline{K}) \mid h_{\vec{c}}(P) \leq C, [K(P) : K] \leq d\}$$

is finite for any constants $C, d \in \mathbb{R}$.

Rmk ① May introduce Weil height for any field with product formula, everything will be true except for Northcott's theorem

② Weil heights in the geometric case may be interpreted in terms of intersection theory, as a degree function.