Notes for Bump, Automorphic Forms

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1 Chapter 4: Representations of GL(2) over a p-adic Field

1.1 Smooth and Admissible Representations

1.1.1 Notations

For our discussion here, F will be a non-Archimedean local field. Let $\mathfrak{o}$ be its ring of integers and $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$. Fix a uniformizer ϖ of $\mathfrak{p}$. Let $q = |\mathfrak{o}/\mathfrak{p}|$ be the cardinality of the residue field. We denote by $\int_F dx$ the additive Haar measure on F normalized so that $\mathfrak{o}$ has volume 1. We also denote by $\int_{F^\times} d^\times x$ normalized so that $\mathfrak{o}^\times$ has volume 1. In fact,

$$d^\times x = (1 - q^{-1})^{-1} |x|^{-1} dx.$$

1.1.2 Definitions

We recall some definitions from topology.

Definition 1.1 (Neighborhood Base). If X is a topological space and $x \in X$, a neighborhood base at x is a collection $\mathcal{U}$ of neighborhoods of x such that every neighborhood of x contains an element of $\mathcal{U}$.

By a *totally disconnected locally compact topological space* we mean a Hausdorff topological space X such that every point of X has a neighborhood base consisting of sets that are both open and compact. Note that in the p-adic topology, F is totally disconnected and locally compact topological group. In fact, any closed subset of the affine space F^n is totally disconnected and locally compact. In particular, these include $GL(n, F)$ and its closed subgroups.
It is a general fact that if G is a totally disconnected and locally compact topological group, then compact and open subgroups of G form a neighborhood base at identity. If G is in fact compact, then compact and open normal subgroups form a neighborhood base. If $G = GL(n, F)$, a neighborhood base at identity can be $K(\varpi^n)$, $n \in \mathbb{Z}_{\geq 0}$, which consist of elements of $GL(n, \mathfrak{o})$ congruent to the identity modulo ϖ^n.

Definition 1.2 (Smooth Representations). Let G be a totally disconnected and locally compact group, and let (π, V) be a representation of G. Here V is a possibly infinite dimensional complex vector space and we do not impose any topology on V. We say that π is *smooth* if for any $v \in V$, the stabilizer $\{g \in G \mid \pi(g)v = v\}$ is open.

Definition 1.3 (Admissible Representations). If π is smooth and for any open subgroup $U \subset G$, the space V^U of vectors fixed by U is finite dimensional, then π is called *admissible*.

1.1.3 Decomposition of Smooth Representations

Smooth representations have a nice decomposition into an algebraic direct sum of isotypic components. This decomposition also gives an alternative characterization to admissible representations. We need to introduce some language before formulating this statement precisely.

Let Γ be a compact totally disconnected group (in particular, Γ can be finite). Let $\widehat{\Gamma}$ be the set of equivalence classes of finite-dimensional irreducible representations of Γ whose kernel is open and hence of finite index. For $\rho \in \widehat{\Gamma}$, we can by abuse of notation denote by (ρ, V_ρ) a representation in the equivalence class of ρ.

Suppose for now that Γ is finite and (π, V) is a representation of Γ on a possibly infinite-dimensional vector space. If $\rho \in \widehat{\Gamma}$, let $V(\rho)$ be the sum of all Γ-invariant subspaces of V that are isomorphic to V_ρ as Γ-modules. We call $V(\rho)$ the *ρ-isotypic subspace*. By the theory of representations of finite groups[1], we know that $\widehat{\Gamma}$ is finite for finite Γ. We have a decomposition

$$V = \bigoplus_{\rho \in \widehat{\Gamma}} V(\rho).$$

We can generalize this decomposition to smooth representations of a totally disconnected locally compact group G. By the general topological fact above, we know that G has an open compact subgroup K. We also know that the compact and open normal subgroups of K form a neighborhood base at the identity in K and hence in G.
Let $\rho \in \widehat{K}$, then the kernel K_ρ of ρ is a compact open normal subgroup of K. Since K_ρ is of finite index in K, we can treat ρ as a representation of the finite group K/K_ρ.
Let (π, V) be a smooth representation of G. Generalizing the previous notation for finite groups, we let $V(\rho)$ denote the sum of all K-invariant subspaces that are isomorphic to ρ as K-modules. We have the following decomposition:

Proposition 1.1. *Let (π, V) be a smooth representation of G. Then*

$$V = \bigoplus_{\rho \in \widehat{K}} V(\rho).$$

The representation π is admissible if and only if each of the $V(\rho)$ is finite-dimensional.

Proof. One first show that V is the sum of the spaces $V(\rho)$. Since π is smooth, $v \in V$ is fixed by an open compact subgroup K_0 of K. In fact, K_0 can be assumed to be normal. Thus, with $\Gamma = K/K_0$,

$$v \in V^{K_0} = \bigoplus_{\rho \in \widehat{\Gamma}} V(\rho) \subset \sum_{\rho \in \widehat{K}} V(\rho).$$

The sum can be seen to be direct by reducing to the finite group case.
Note that $V(\rho) \subset V^{\ker \rho}$. Since $\ker \rho$ is open by assumption, if π is admissible, we would have $V^{\ker \rho}$ to be finite dimensional. Conversely, if π is not admissible, then there exists some open normal subgroup K_0 of K such that V^{K_0} is infinite-dimensional. This leads to a contradiction because V^{K_0} decomposes into a finite direct sum of $V(\rho)$ over $\rho \in \widehat{K/K_0}$. □

[1]The sum of the square of the dimension of spaces in $\widehat{\Gamma}$ equals to the size of Γ

We also want to define the contragradient representation of a smooth representation (π, V). If $\hat{v} : V \to \mathbb{C}$ is a linear functional, we write $\langle v, \hat{v} \rangle$ for $\hat{v}(v)$ when $v \in V$. We say that the linear functional $\hat{v}$ is *smooth* if there exists an open neighborhood U of the identity in G such that

$$\langle \pi(g)v, \hat{v} \rangle = \langle v, \hat{v} \rangle$$

whenever $g \in U$ and for all $v \in V$. Let $\widehat{V}$ be the space of smooth linear functionals on V. We define the *contragradient representation* $(\hat{\pi}, \widehat{V})$, where the action of g on $\widehat{V}$ is defined by

$$\langle v, \hat{\pi}(g)\hat{v} \rangle := \langle \pi(g^{-1})v, \hat{v} \rangle.$$

To check that $\hat{\pi}(g)\hat{v} \in \widehat{V}$, we need to find an open neighborhood U' of G such that

$$\langle \pi(g')v, \hat{\pi}(g)\hat{v} \rangle = \langle v, \hat{\pi}(g)\hat{v} \rangle$$

whenever $g' \in U'$ and for all $v \in V$. Since $\hat{v} \in \widehat{V}$, we can just take $U' = gUg^{-1}$, where U is the neighborhood for $\hat{v}$, as the criterion by the definition of $\hat{\pi}$ translates to

$$\langle \pi(g^{-1}g')v, \hat{v} \rangle = \langle \pi(g^{-1})v, \hat{v} \rangle.$$

Moreover, to show that $(\hat{\pi}, \widehat{V})$ is a smooth representation, we need to show that for all $\hat{v}$, the stabilizer $\{g \in G \mid \hat{\pi}(g)\hat{v} = \hat{v}\}$ is open. Note that the stabilizer is a subgroup of G, so it suffices to show that there exists an open subgroup U of G that stabilizes $\hat{v}$. That follows exactly from the definition of smooth vectors. By the summand in Proposition 1.1, the dual space V^* of V is the direct product $V^* = \prod_{\rho \in \widehat{K}} V(\rho)^*$. We know that $\widehat{V} \subset V^*$. In fact, $\hat{v} \in \widehat{V}$ iff. it is zero on all but finitely many[2] of the summands in 1.1. Thus, $\widehat{V}$ can be identified with

$$\widehat{V} = \bigoplus_{\rho} V(\rho)^*$$

From this identification, we can see that if (π, V) is admissible, then so is $\hat{\pi}$ and $\hat{\hat{\pi}} \cong \pi$.

1.1.4 Hecke Algebra and Applications

We will now introducing the Hecke Algebra, which allows us to view a smooth representation of a totally disconnected and locally compact topology group G from a different perspective.

If X is a totally disconnected topological space, we call a function f on X smooth if it is locally constant. Let $\mathcal{H}$ be the space of smooth compactly supported complex-valued functions on G. Assume G to be unimodular, which means the left and right measure coincide. Make $\mathcal{H}$ an algebra under convolution:

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh^{-1})\phi_2(h)dh.$$

We call this the *Hecke Algebra* and it does not have a unit. If K_0 is a compact open subgroup of G, let $\mathcal{H}_{K_0}$ denote the subspace of K_0-biinvariant functions in $\mathcal{H}$. It is easily seen that $\mathcal{H}_{K_0}$ is closed under convolution. Moreover, it has the following identity element:

$$\epsilon_{K_0} = \begin{cases} \mathrm{vol}(K_0)^{-1} & \text{if } g \in K_0 \\ 0 & \text{otherwise.} \end{cases}$$

[2] If it is zero on all but finitely many, take the intersection of the kernels of ρ. For the reverse direction, show that it is zero outside of V^U, with U being the neighborhood in the definition of smooth linear functional.

Given a representation (π, V) of G, we can construct a representation of $\mathcal{H}$ as follows.
If $\phi \in \mathcal{H}$, we define

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v dg.$$

It can be checked through relatively tedious calculation that it satisfies

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2).$$

The following proposition gives us some insights into the relation between representations of G and representations of $\mathcal{H}$.

Proposition 1.2. *Let (π, V) be a smooth representation of G. Assume that V is nonzero. The following are equivalent:*

(i) The representation π is irreducible.

(ii) V is simple as a $\mathcal{H}$-module.

(iii) V^{K_0} is either zero or simple as a $\mathcal{H}_{K_0}$-module for all open subgroups K_0 of G.

Proof. We first show that (i) and (ii) are equivalent. Since a G-invariant subspace is also $\mathcal{H}$ invariant, (ii) implies (i) is clear. For the reverse direction, we want to start with a $\mathcal{H}$-invariant subspace and show that it is G-invariant. This relies on the fact that we can describe the action of G using characteristic functions in $\mathcal{H}$.
The implication (iii) implies (ii) relies on the fact that $V = \bigcup V^{K_0}$, which follows from the fact that π is smooth.
It remains to show (ii) implies (iii). Let $W_0 \subset V^{K_0}$ be a proper nonzero $\mathcal{H}_{K_0}$ module. It suffices to show that $\pi(\mathcal{H})W_0 \cap V^{K_0} = W_0$. In that case, by the observation above we know that $\pi(\mathcal{H})W_0$ is a proper nonzero subspace of V, which is $\mathcal{H}$ invariant by construction. For the non obvious inclusion, one makes recursive use of the fact that for a K_0 fixed vector v, $\pi(\epsilon_{K_0})v = v$. Moreover, for $\phi \in \mathcal{H}$, $\epsilon_{K_0} * \phi * \epsilon_{K_0} \in \mathcal{H}_{K_0}$. □

Now we work towards some theorems that describe the relationships between certain irreducible admissible representations using distributions on G.
Recall the definition of an intertwining operator between two representations. We have Schur's lemma for irreducible admissible representations, which states the following.

Proposition 1.3 (Schur's Lemma)**.** *Let (π, V) be an irreducible admissible representation of the totally disconnected locally compact group G. Let $T : V \to V$ be an intertwining operator for π. Then there exists a complex number c such that $T(v) = cv$ for all $v \in V$.*

The reason why we can remove the finite-dimensionality condition on V is because that admissibility of π ensures the existence of a finite-dimensioanl subspace of V that T acts on. An immediate consequence is that the center of G acts by scalars on the irreducible admissible representation (π, V). Thus, if $G = GL(n, F)$, there exists a quasicharacter ω of $F^\times$, called the *central quasicharacter* such that

$$\pi(zI_n)v = \omega(z)v.$$

Note that a quasicharacter is a character when assumed to be unitary.
We also have a result about the canonical pairing between V and $\widehat{V}$ between an admissible representation and its contragradient.

Proposition 1.4. *Let (π, V) be an admissible representation of the totally disconnected locally compact group G, and let $(\hat{\pi}, \widehat{V})$ be the contragradient. Let K_0 be an open and compact subgroup of G. Then the canonical pairing between V and $\widehat{V}$ induces a nondegenerate pairing between V^{K_0} and $\widehat{V}^{K_0}$.*

We now define what we call the *character* of an admissible representation of a totally disconnected locally compact group G, which also plays an important role in the study of representations, just like the case of finite groups. We will define the character as a distribution on G.

For a totally disconnected locally compact topological space, a distribution on X is a linear functional on $C_c^\infty(X)$, the ring of smooth compactly supported functions $X \to \mathbb{C}$. We denote by $\mathfrak{D}(X)$ the space of distributions on X. In the case of G, we know $C_c^\infty(G) = \mathcal{H}$, and a distribution on G is just a linear functional on the Hecke algebra.

We first need to define trace of some linear operators over possibly infinite dimensional vector space. If $f : V \to V$ is an endomorphism of a possibly infinite-dimensional vector space, we say f has finite rank if its image is finite dimensional. Let U be a finite dimensional subspace of V that contains the image of f, then $f|_U$ is an endomorphism of U and we define $\mathrm{Tr}(f)$ to be the trace of $f|_U$.

Now let (π, V) be an admissible representation of G. If $\phi \in \mathcal{H}$, it is locally constant and compactly supported, and therefore, $\phi \in \mathcal{H}_{K_0}$ for some compact open subgroup K_0. The operator $\pi(\phi)$ has its image in the space of K_0-fixed vectors, which is finite dimensional because of admissibility. Thus, $\pi(\phi)$ has finite rank and we can define the character of the representation $\chi : \mathcal{H} \to \mathbb{C}$ by $\chi(\phi) = \mathrm{Tr}(\pi(\phi))$.

We give a sufficient condition for two irreducible admissible representations to be isomorphic to each other.

Theorem 1.1. *Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of the totally disconnected compact group G. If the characters of π_1 and π_2 agree, then the two representations are isomorphic.*

The proof requires the following two facts.

Proposition 1.5. *Let R be an algebra over a field k. Let E_1, E_2 be simple R-modules that are finite dimensional over k. For each $r \in R$, multiplication by r induces endomorphisms $m_i(r)$ of E_i. If $\mathrm{Tr}(m_1(r)) = \mathrm{Tr}(m_2(r))$ for all $r \in R$, then E_1 and E_2 are isomorphic as R-modules.*

Proposition 1.6. *Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of the totally disconnected compact group G. If $V_1^{K_1} \cong V_2^{K_1}$ as $\mathcal{H}_{K_1}$-modules for every open and compact subgroup K_1 of G, then $\pi_1 \cong \pi_2$.*

Note that this is in fact true if there exists one such K such that the condition of the proposition is satisfied.

Proof of Theorem 1.1. Applying the first fact to $k = \mathbb{C}$, $R = \mathcal{H}_{K_1}$, $E_i = V_i^{K_1}$, we obtain that the E_i's are isomorphic as R-modules, which is exactly what requires for the two representations to be isomorphic by the second fact. □

Remark 1.1.1. This theorem illustrates the applications enabled by the Hecke algebra modules structure. In the proof of the second fact, one starts with an $\mathcal{H}_{K_1}$ module isomorphism and extend it to its open subgroups in a compatible way. One then defines the $\mathcal{H}$-module homomorphism using the $\mathcal{H}_{K_1}$ module isomorphisms developed. Finally the $\mathcal{H}$-module isomorphism obtained is translated into an intertwining operator by representing the action of G with the action of $\mathcal{H}$.

With this sufficient condition, we can explicitly describe the contragradient representation we defined earlier.

Theorem 1.2. *Let $G = GL(n, F)$ and let (π, V) be an irreducible admissible representation of G.*

(i) Define a representation (π_1, V) on the same space by $\pi_1(g) = \pi({}^{\top}g^{-1})$. Then $\hat{\pi} \cong \pi_1$.

(ii) Suppose that $n = 2$. Let ω be the central quasicharacter of π. Define a representation (π_2, V) on the same space by $\pi_2(g) = \omega(\det(g))^{-1}\pi(g)$. Then $\hat{\pi} \cong \pi_2$.

Remark 1.2.1. The proof of this theorem uses the result below on the distributions of G. The proof of that result requires more knowledge on distributions than we have developed so far.

Proof. Following Theorem 1.1, one wants to compare the characters of two representations. They can be shown to be the same using the fact that the character as a distribution is invariant under conjugation and transpose. □

An easy consequence of the theorem is that π is irreducible iff. $\hat{\pi}$ is irreducible.