A CHARACTERIZATION OF THE GOOD REDUCTION OF MUMFORD CURVES

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ABSTRACT. Mumford defines a Shimura curve of Hodge type, parameterizing complex abelian fourfolds. In this paper, we study the good reduction of such a curve in positive characteristic and gives a complete characterization.

1. INTRODUCTION

This paper aims at characterizing the good reduction of certain Shimura varieties of Hodge type. This description will serve as a main example in our later work in defining Shimura varieties in positive characteristic.

Recall the Hodge group of an abelian variety $A$ is the largest $\mathbb{Q}$–subgroup of $GL(H^1(A, \mathbb{Q}))$ which leaves all Hodge cycles invariant. And Mumford defines in [14] that a Shimura variety of Hodge type as a moduli scheme of abelian varieties (with a suitable level structure) whose Hodge group is contained in a prescribed Mumford-Tate group.

Further, Mumford shows in [13] an explicit example of Shimura curve of Hodge type and as far as the author knows, it is the simplest example. He constructs a simple algebraic group $G$ over $\mathbb{Q}$ which is the $\mathbb{Q}$–form of the real algebraic group $SU(2) \times SU(2) \times SL(2)$, a cocharacter $h$ of $G$ in $\mathbb{R}$

$$h : S_m(\mathbb{R}) \longrightarrow G(\mathbb{R})$$

$$e^{i\theta} \mapsto I_4 \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and an 8 dimensional absolute irreducible rational representation $V$ of $G$.

The Shimura datum $(G, h)$ defines Shimura curves of Hodge type. With proper level structures, they can be embedded into the moduli scheme of 4 dimensional polarized abelian varieties.

Generalizing the construction (see [18]), one is able to define Shimura curves of Hodge type, parameterizing $2^m$ dimensional polarized abelian varieties (see Section 2). We call such Shimura curves (with its universal family) the Mumford curves, denoting as $M$ and let $A \longrightarrow M$ be the universal family of abelian varieties over $M$.

Mumford curves play significant roles amid smooth Shimura curves of Hodge type. Theorem 0.8 in [12] shows the universal family over a Shimura curve has strictly maximal Higgs field. And Theorem 0.5 in [18] shows up to powers and isogenies, the only smooth families of abelian varieties over curves with maximal Higgs field are Mumford curves.

Our main theorem is as follows.

Definition 1.1. For any prime number $p$ and integer $m$, we say the datum $(X \longrightarrow C, k)$ satisfies the $(*,p,m)$ if they satisfy the following properties:
(1) \( k \) is an algebraically closed field of characteristic \( p \),
(2) \( C \) is a proper smooth curve over \( W(k) \) and \( X \to C \) is a family of polarized abelian varieties of dimension \( 2^m \) over \( C \),
(3) there exists a versally deformed height 2 BT group \( G \) and a height \( 2^m \) etale BT group \( H \) over \( C \) such that \( X[p^\infty] \cong G \otimes H : = \colim_n (G[n] \otimes H[n]) \).
(4) the reduction of \( X \to C \) at \( k \) is generically ordinary.

**Theorem 1.2.** Let \( A \to M \) be a Mumford curve, parameterizing polarized abelian varieties of dimension \( 2^m \). For infinitely many prime \( p > 2 \), there exists datum \( (X \to C, k) \) satisfying \((*,p,m)\) (see definition 1.1) and
\[
(X \to C) \otimes_{W(k)} C = (A \to M).
\]

**Remark 1.3.** In the proof, we can see that the Dieudonne crystal \( \mathbb{D}(X/C) \) associated to abelian scheme \( X \) satisfies that
\[
\mathbb{D}(X/C) \cong V \otimes T
\]
where \( V \) is a Dieudonne crystals of rank 2 with maximal Higgs field and \( T \) is a unit root crystal of rank \( 2^m \).

**Remark 1.4.** For the choice of \( p \), it suffices to require that
\begin{enumerate}
  \item \( p > 2 \), see remark 7.8
  \item \( M \) admits a good reduction at place \( p \), see 2.5
  \item the reflex field of \( M \) is splitting over \( p \), see remark 6.10
\end{enumerate}

The paper is to prove the theorem 1.2. Using [6], we know the condition (3) in definition 1.1 is equivalent to prove the crystalline Dieudonne module associated to \( X/C \) is a tensor product of a rank 2 Dieudonne crystal and a rank \( 2^m \) unit root crystal. By Lefschetz principle 2.5, the Mumford curve with the universal family can be descent to a Witt ring whose special fiber \( \bar{X}/\bar{C} \) is smooth. From the definition of Mumford curve over \( \mathbb{C} \), the Dieudonne crystal \( \mathbb{D}(\bar{X}/\bar{C}) \) of the abelian scheme \( \bar{X} \) is a tensor product of \( m \) rank 2 crystals:
\[
\mathbb{D}(\bar{X}) \cong V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1}.
\]

The last and most difficult part is to decompose the Frobenius of \( \mathbb{D}(\bar{X}) \) as a tensor product, which is proved using the generically ordinary property (8.2).

Roughly, in Section 2, we introduce the basic notions and using Lefschetz principle, we can descend a Mumford curve and show that the crystal \( \mathbb{D}(\bar{X}) \) admits the tensor decomposition. In Section 3, we recall some basic results in Tannakian categories. Section 4 lists the notations in the paper. In Section 5, we use Tannakian category to describe the structure of the rank 2 crystals. Via some analysis on the Tannakian groups corresponding to the rank 2 crystals in Section 6, we deduce that the Frobenius has a tensor decomposition. And then we use a result in Section 7 to construct the rank 2 and rank \( 2^m \) Dieudonne crystals. In Section 9, we study the BT groups corresponding to the two Dieudonne crystals.

In the paper, we use \( \bar{Y} \) to denote an object defined over \( k \). In particular, if \( X \) is already defined over \( W(k) \), then \( \bar{X} \) denote the special fiber of \( X \).
2. Mumford curve and its reduction

2.1. Mumford curve over $\mathbb{C}$. ([13])

Let $K$ be a totally real field of degree $m + 1$ and $D$ be a quaternion division algebra over $K$ which splits only at one place. One has the following result from (Lemma 5.7, [18]).

**Lemma 2.2.** Let $K$ and $D$ be as above. It holds that either

1. Cor$_{K|\mathbb{Q}}(D) \cong M_{2m+1}(\mathbb{Q})$ and $m + 1$ is odd, or;
2. Cor$_{K|\mathbb{Q}}(D) \not\cong M_{2m+1}(\mathbb{Q})$. Then

$$\text{Cor}_{K|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}) \cong M_{2m+1}(\mathbb{Q}(\sqrt{b})).$$

So we can fix an embedding of $\mathbb{Q}$–algebras: Cor$_{K|\mathbb{Q}}(D) \hookrightarrow M_{2m+1+\epsilon(D)}(\mathbb{Q})$ where $\epsilon(D) = 0, 1$ depending only on $D$. And then $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \times \cdots \times \mathbb{H} \times M_2(\mathbb{R})$.

Let $^\ast$ be the standard involution of $D$, and let

$$G = \{ x \in D^* | x\bar{x} = 1 \}.$$

Then $G$ is a simple algebraic group over $\mathbb{Q}$ which is the $\mathbb{Q}$–form of the real algebraic group

$$SU(2)^m \times SL(2, \mathbb{R}).$$

The group $G$ admits a natural 8 dimensional rational representation $V$ whose real form is

$$\rho : SU(2)^m \times SL(2) \longrightarrow SO(2^{m+1}) \times SL(2)$$

acting on $\mathbb{R}^{2^{m+1}+\epsilon(D)}$.

Note $G_\mathbb{C} = SL(2, \mathbb{C})^{m+1}$. And then $V_\mathbb{C}$ is the tensor of $m + 1 + \epsilon(D)$ copies of standard representation $\mathbb{C}^2$ of $SL(2, \mathbb{C})$.

Since the value of $\epsilon$ does not affect our proof, for simplicity, let us assume $\epsilon(D) = 0$.

Let

$$h : S_m(\mathbb{R}) \longrightarrow G(\mathbb{R})$$

$$e^{i\theta} \mapsto I_{2m} \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$ 

Then $(G, h)$ defines a Shimura datum. So is $(\rho(G), \rho \circ h)$. Generically $\rho(G)$ is the Hodge group of $V$.

Let $K \subset G_\mathbb{R}$ be the stabilizer of $h$. Then $K$ is a maximal compact subgroup of $G_\mathbb{R}$ and hence congruent to $SO(2) \times SU(2)^{m}$. So $G_\mathbb{R}/K \cong Sp(1, \mathbb{R})/SO(2, \mathbb{R}) \cong \mathfrak{h}$ the upper half plane. Since ker $\rho \subset \text{stab}(h)$, we have

$$\rho(G)/\text{stab}(\rho \circ h) = G_\mathbb{R}/K = \mathfrak{h}.$$ 

Choose $\Gamma \subset G_\mathbb{R}$ be the arithmetic subgroup such that $\Gamma$ acts freely and properly discontinuous on $\mathfrak{h}$. Note ker $\rho \subset Z(G)$ and then fixes $h$, $\Gamma \hookrightarrow \rho(G(\mathbb{R}))$.

The (1-dimensional) Shimura varieties defined by $(G, h)$ are called Mumford curves. With a suitable level structure $\Gamma$, a Mumford curve is proper and smooth. On one hand, it is a Shimura curve of Hodge type. Choose $\Gamma$ small enough, and then $M$ admits an embedding to $\mathcal{A}_{2m+1,n}$. Since $\mathfrak{h}$ is simply connected, $\pi_1(M) = \Gamma$. On the other hand, $V$ induces a monodromy $V_\mathbb{C}$. Further, the tensor components $\mathbb{C}^2$ of $V_\mathbb{C}$ also admit representations of $\pi_1(C) = \Gamma$ and hence also a monodromy.

From [11], we know the following relation between Hodge group and the algebraic monodromy.
Definition 2.3. For any monodromy in $GL(n)$, the algebraic monodromy is defined to be the Zariski closure of the monodromy in $GL(n)$. The connected algebraic monodromy is the connected component of the identity of the algebraic monodromy.

Proposition 2.4. The algebraic monodromy of $\mathbb{C}^2$ is $SL(2, \mathbb{C})$ and that of $V_{\mathbb{C}}$ is the image of $SL(2, \mathbb{C})^{x+m+1}$ in $Aut(V_{\mathbb{C}})$.

Proof. Note the monodromy on $V_{\mathbb{Q}}$, induced by the representation of $G$, is tensor of three copies of monodromy $\mathbb{C}^2$. Let $K$ be the algebraic monodromy group of $\mathbb{C}^2$, then $K \subset SL(2, \mathbb{C})$.

By [1], the connected algebraic monodromy on $V_{\mathbb{Q}}$ is a normal subgroup in the Hodge group $\rho(G)$. Since $G$ is simple, $\rho(G)$ is also simple over $\mathbb{Q}$. Thus the connected algebraic monodromy is $\rho(G)$. The connected complex algebraic monodromy of $V_{\mathbb{C}}$ is $\rho(G)$. Since $\rho(G)_{\mathbb{C}} = im (SL(2, \mathbb{C})^{x+m+1} \rightarrow Aut(V_{\mathbb{C}}))$ is connected, the connected complex algebraic monodromy of $V_{\mathbb{C}}$ is $\rho(G)$. Since $\rho(G)_{\mathbb{C}} = im (SL(2, \mathbb{C})^{x+m+1} \rightarrow Aut(V_{\mathbb{C}}))$.

Then necessarily, $K = SL(2, \mathbb{C})$.

2.5. Lefschetz principle. By Lefschetz principle, the morphism $A \rightarrow M$ can descend from $K$ to a ring $R$ finite type over $\mathbb{Z}$. Throwing away finite places, we can assume $R$ is smooth over $\mathbb{Z}$. Let $k$ be a residue field of $R$ with characteristic $> 2$ such that $M$ admits a good reduction over $k$. We have the a lifting from $\text{Spec } W_n(k)$ to $\text{Spec } R$:

$$
\begin{array}{c}
\text{Spec } k \\
\downarrow \\
\text{Spec } W_n(k) \\
\downarrow \\
\text{Spec } \mathbb{Z}
\end{array}
$$

Hence we find a morphism $\text{Spec } W(k) \rightarrow \text{Spec } R$.

Let $X \rightarrowbar C$ be the base change $A \rightarrow M$ from $\text{Spec } R$ to $\text{Spec } W(k)$. Let $\tilde{X}/\tilde{C}$ be the special fiber of $X/C$. Let $\mathcal{E}$ be the Hodge bundle $\mathcal{E} = R^1\pi^*dR(\Omega_{X/C})$ and $\mathcal{E}$ admits the Gauss-Manin connection. Using ( [3], theorem 6.6), the category of crystals on $\tilde{C}$ is equivalent to the category of modules with integrable connection. And the Hodge bundle $\mathcal{E}$ corresponds to the Dieudonne crystal $R^1\tilde{\pi}_{\text{cris}}(\mathcal{O}_{\tilde{X}})$. Let denote the crystal still as $\mathcal{E}$. Denote the descent of $\mathbb{C}^2$ to $C$ as $\mathcal{V}_i$. By [2.4], $\mathcal{V}_i$ also correspond to crystals and denote them as $\mathcal{V}_i$ as well. Then as crystals

$$
\mathcal{E} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.
$$

3. TANNAKIAN CATEGORY

Definition 3.1. Let $k$ be a field. A Tannakian category $T$ is a $k$–linear neutral rigid tensor abelian category with an exact fiber functor $\omega : T \rightarrow Vect_k$.

Theorem 3.2. ([7], theorem 2.11) For any Tannakian category $T$, there exists an $k$–algebraic group $G$ such that $T$ is equivalent to $\text{Rep}_k(G)$ as tensor categories.
Example 3.3. Choose a point \(c \in C\), the category of all MIC on \(C\) with fiber functor \(\mathcal{F} \rightarrow \mathcal{F}_c\) form a Tannakian category. By theorem 3.2, it corresponds to \(\text{Rep}_C(G_{\text{univ}})\).

By Riemann-Hilbert correspondence, the category of MIC is equivalent to \(\text{Rep}(\pi_1(C))\). The algebraic group \(G_{\text{univ}}\) can be constructed from \(\pi_1(C)\) by the following:

\[
G_{\text{univ}} = \lim H
\]

where \(H\) lists the Zariski closure of image of \(\pi_1(C)\) in \(GL(W)\) for all complex representations \(W\). Note the system of \(H\) is projective. So the image of \(G_{\text{univ}} \rightarrow \text{Aut}(W)\) is exactly the Zariski closure of the image of \(\pi_1(C)\) in \(GL(W)\).

Example 3.4. ([17] VI 3.1.1, 3.2.1) Inverting \(p\) in the category \(\text{Cris}(C)\), we obtain \(\text{Isocris}(C)\). Similar to example 3.3, the category \(\text{Isocris}(C)\) forms a Tannakian category, with the obvious fiber functor. Hence there exists a \(B(k)\)-affine group scheme \(H_{\text{univ}}\) such that the following two categories are equivalent.

\[
\{\text{finite locally free isocrystals on } C/W(k)\} \longleftrightarrow \text{Rep}_{B(k)}(H_{\text{univ}}).
\]

An object \(\mathcal{F}'\) in \(\text{Isocris}(C)\) is called effective if it is from an object \(\mathcal{F}\) in \(\text{Cris}(C)\), i.e. \(\mathcal{F}' = \mathcal{F} \otimes B(k)\). For any morphism \(f : \mathcal{F} \otimes B(k) \rightarrow \mathcal{G} \otimes B(k)\) between effective objects in \(\text{Isocris}(C)\), there exists \(m \in \mathbb{Z}\) such that \(p^m f : \mathcal{F} \rightarrow \mathcal{G}\) is a morphism in \(\text{Cris}(C)\).

Note different from ([17], VI 3.1.1, 3.2.1), \(\text{Isocris}(C)\) denotes just the isocrystals, not the \(F\)-isocrystals. So \(H_{\text{univ}}\) is a affine group scheme over \(B(k)\).

We will need another result later.

Proposition 3.5. For any Tannakian category \(T\) and \(W,V \in T\), let \(<W>\) denote the Tannakian subcategory generated by \(W\), with Tannakian group \(G_W\). Similarly, \(<V>\) = \(\text{Rep}_k(G_V), <W,V> = \text{Rep}_k(K)\). Then there exists a natural injection \(K \hookrightarrow G_W \times G_V\).

Proof. Since \(W,V \in \text{Rep}(K)\), by ([7], 2.21), \(K\) admits surjections onto \(G_W\) and \(G_V\). Then \(K\) admits a map \(K \rightarrow G_W \times G_V\). The induced morphism \(\text{Rep}(G_W \times G_V) \rightarrow \text{Rep}(K)\) satisfies ([7], 2.21(2)). So the map is injective.

\[\square\]

In section 5, we concentrate on the Tannakian duality of \(\mathcal{V}_i\) for each \(i\).

4. Notations

We summarize the notations and fix them till the end.
$X \rightarrow C$ the descent of the Mumford curve with the family of abelian fourfolds $A/M$ to Spec $W(k)$.

$E, V_i, \mathcal{V}_i$ $\mathcal{E} = R^1\pi_{\text{cris}}^*(\mathcal{O}_X)$ and $\mathcal{E} \cong V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1}$, $\mathcal{E}^\sigma \cong V_1^\sigma \otimes V_2^\sigma \otimes V_3^\sigma \cdots \otimes V_{m+1}^\sigma$.

$E, W_i, V_i$ $(E, W_i, V_i)$ are the Tannakian dualities of $(\mathcal{E}, V_i^\sigma, V_i)$, respectively.

$G$ the reductive group defining the Mumford curves

$H_{\text{univ}}$ the Tannakian group of the category of finitely locally free isocrystals on $C/W(k)$.

$H_i$ the Tannakian group of the subcategory generated by $\mathcal{E}$, i.e. $\text{im } (H_{\text{univ}} \rightarrow \text{Aut}(E))$.

$H_i$ the Tannakian group of the subcategory generated by $V_i$, i.e. $\text{im } (H_{\text{univ}} \rightarrow \text{Aut}(V_i))$.

$G_i$ the Tannakian group of the subcategory generated by $V_i^\sigma$, i.e. $\text{im } (H_{\text{univ}} \rightarrow \text{Aut}(W_i))$.

$G_i^0$ the connected component of the identity in $G_i$.

$H'$ the Tannakian group of the subcategory generated by $\{\mathcal{E}^\sigma\}$.

$G'$ $G' = \text{im } (H_{\text{univ}} \rightarrow \prod G_i)$.

$K$ the Tannakian group of the subcategory generated by $\{V_1^\sigma, V_2\}$.

5. Tensor Decomposition of $\mathcal{E}$ and $\mathcal{E}^\sigma$ as Isocrystals

The Dieudonne crystal $\mathcal{E}$ over the good reduction of the Mumford curve $\bar{X}/\bar{C}$ admits the Frobenius map:

$$\mathcal{E}^\sigma \xrightarrow{F} \mathcal{E}.$$ 

Then we have

(1) $F : V_1^\sigma \otimes V_2^\sigma \otimes V_3^\sigma \cdots \otimes V_{m+1}^\sigma \otimes B(k) \xrightarrow{\cong} V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1} \otimes B(k)$.

where $B(k)$ be the fractional field of $W(k)$. By example 3.4, the category of isocrystals over $\bar{C}$ is Tannakian.

5.1. For $V_i \otimes B(k)$.

**Proposition 5.2.** For each $i$, $H_i \cong SL(2, B(k))$ and $H_{\text{univ}} \rightarrow \prod H_i$ is surjective.

**Proof.** Since by (3), theorem 6.6, the crystals on $C$ are exactly vector bundles with connections, $\text{Rep}_C(H_{\text{univ}} \otimes \mathbb{C})$ is a full subcategory of $\text{Rep}_C(G_{\text{univ}})$. By functorality, $G_{\text{univ}} \rightarrow \text{Aut}(E \otimes \mathbb{C})$ factors through $H_{\text{univ}} \otimes \mathbb{C}$. By 2.4, $H \otimes \mathbb{C} = \text{im } (G_{\text{univ}} \rightarrow \text{Aut}(V)) = \text{im } (SL(2, \mathbb{C})^{\times m+1} \rightarrow \text{Aut}(\mathbb{C}^{2^{\times m+1}})$

$H_i \otimes \mathbb{C} = \text{im } (G_{\text{univ}} \rightarrow \text{Aut}(\mathbb{C}^2)) = SL(2, \mathbb{C})$.

The group $H_i$ is a $B(k)$–form of $SL(2)$ and admits a faithful two dimensional representation. Therefore $H_i \cong SL(2, B(k))$. 

6
Hence $H = \text{im} \left( H_{\text{univ}} \rightarrow \prod_i H_i \rightarrow \text{Aut}(E) \right)$ is the same as $\prod_i H_i \rightarrow \text{Aut}(E)$, after tensoring with $\mathbb{C}$. Since it is faithfully flat, it is also true over $B(k)$ and $H = \text{im} \left( \prod_i H_i \otimes B(k) \rightarrow \text{Aut}(E) \right)$. Further, since the kernel of $(\prod_i H_i \rightarrow \text{Aut}(E))$ is finite, $\text{im} \left( H_{\text{univ}} \rightarrow \prod_i H_i \right)$ is a subgroup of $\prod_i H_i$, with the same dimension. Since $\prod_i H_i = SL(2, B(k)) \times m+1$ are connected, $\text{im} \left( H_{\text{univ}} \rightarrow \prod_i H_i \right) = \prod_i H_i$, i.e. $H_{\text{univ}} \rightarrow \prod_i H_i$ is surjective.

5.3. For $V^* \otimes B(k)$.

**Lemma 5.4.** For any $g \in GL(2)$, as a variety, the centralizer $Z(g)$ of $g$ has dimension $\geq 2$.

**Proof.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The centralizer of $g$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implies

$$bz = cy, (a - d)y = b(x - w).$$

Note $\dim GL(2) = 4$. As a subvariety of $GL(2)$, $Z(g)$ has dimension at least 2. □

Note $G'$ is the image of $H_{\text{univ}} \rightarrow \prod G_i$, the projections $G' \rightarrow G_i$ are surjective for each $i$.

**Proposition 5.5.** For each $i$, $G_i = GL(2, B(k))$ or $G_i^0 \cong SL(2, B(k))$.

**Proof.** Let $H'$ be $\text{im} \left( G' \rightarrow \text{Aut}(E) \cong GL(8) \right)$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
G_1 \times G_2 \times G_3 \cdots \times G_{m+1} & \xleftarrow{G'} & SL(2)^{\times m+1} \\
\downarrow & & \downarrow \\
GL(2)^{\times m+1} & \xrightarrow{\text{Twisted by } F} & GL(2^{m+1})
\end{array}
$$

Note $SL(2)^{\times m+1} \rightarrow GL(2^{m+1})$ may not be the usual inclusion. The right triangle can be specified as

$$
\begin{array}{ccc}
G' & \xrightarrow{\text{Twisted by } F} & SL(2)^{\times m+1} \\
\downarrow & & \downarrow \\
H' & \xrightarrow{\text{Twisted by } F} & GL(2^{m+1})
\end{array}
$$

Since $SL(2)$ is semisimple, so is $H'/Z(H')$. Since $\ker(G' \rightarrow GL(2^{m+1})) \subset \ker(GL(2)^{\times m+1} \rightarrow GL(2^{m+1}))$, the kernel of $G' \rightarrow H'$ consists of just central elements. The group $G''$ is an extension of central elements and a semisimple group. Therefore $G''$ is reductive and $H'$ is the adjoint group of $G'$. Further, $G' \rightarrow H'$ induces a morphism from the derived
Conjugated by some element in $H'$ which further induces a surjection to $H'/Z(H')$. Note $[G', G']$ is also semisimple.

If the projection of $[G', G']$ to some factor has dimension less than 3, then one of the projection must have dimension 4 because $\dim H' = 2^m$. So one of the projection is $GL(2)$. Since the kernel of $G' \to H'$ is finite, $H'$ would have infinite centers, contradiction.

We can now assume the projection of $[G', G']$ to each factor has precisely dimension 3. And hence each factor has the form $SL(2) \times \mu_2$. Then we have a lifting

$$SL(2)^{\times m+1} \to [G', G'] \subset G'$$

such that the right triangle is commutative

$$\begin{array}{ccc}
G_1 \times G_2 \times G_3 \cdots \times G_{m+1} & \xleftarrow{\psi} & G' \\
GL(2)^{\times m+1} & \xrightarrow{\text{twisted by } F} & GL(2^{m+1})
\end{array}$$

Now we classify the elements with finite kernel in $\text{Hom}(SL(2)^{\times m+1}, GL(2)^{\times m+1})$.

First, recall that $\text{Hom}(SL(2), GL(2))$ consists of trivial morphism and the usual inclusion conjugated by some element in $GL(2)$. For any morphism $f \in \text{Hom}(SL(2)^{\times m+1}, GL(2)^{\times m+1})$, restricting to each factor of $SL(2)$ gives three inclusions $SL(2) \to GL(2)$. Explicitly,

$$(g_1, 1, 1, \cdots) \mapsto (\psi_{11}(g_1), \psi_{12}(g_1), \psi_{13}(g_1), \cdots)$$

$$(1, g_2, 1, \cdots) \mapsto (\psi_{21}(g_2), \psi_{22}(g_2), \psi_{23}(g_2), \cdots)$$

$$(1, 1, g_3, \cdots) \mapsto (\psi_{31}(g_3), \psi_{32}(g_3), \psi_{33}(g_3), \cdots).$$

Then $\psi_{11}(g_1)$ and $\psi_{21}(g_2)$ commute for any $g_i \in SL(2)$. Note all the automorphisms of $SL(2)$ are inner. So if neither is identity, then there exists $h, k \in GL(2)$ such that $\psi_{11}, \psi_{21}$ are conjugation by $h$ and $k$, respectively. Then for any $g_i \in SL(2)$,

\begin{align}
(2) \quad & h g_1 h^{-1} k g_2 k^{-1} = k g_2 k^{-1} h g_1 h^{-1} \\
(3) \quad & k^{-1} h g_1 h^{-1} k g_2 k^{-1} h = g_2 k^{-1} h g_1,
\end{align}

Since by lemma 5.4, $Z(k^{-1}g)$ in $GL(2)$ has dimension at least 2, we can choose $g_2 \in SL(2)$ such that $g_2 \neq \pm I$ and $g_2 \in Z(k^{-1}g)$. But then from (2), $g_2$ has to commute with $g_1$, i.e. $g_2 \in Z(SL(2)) = \pm 1$, contradiction. Hence at least one of $\psi_{11}$ and $\psi_{21}$ is identity. Further, each column $\psi_{ki}$ has at least two identities. So each factor $SL(2)$ is embedded into exactly one of the three copies of $GL(2)$ trivially to the other two.

Then $\dim G_i = 3$ or 4. If $\dim G_i = 4$, then $G_i = GL(2)$. If $\dim G_i = 3$, then $G_i \cong SL(2)$ and the isomorphism is a conjugation by some element in $GL(2)$. In the latter case, $\det(G_i) < \mathbb{G}_m$ and thus $\det G_i = \mu_2^n$ for some integer $n$. So precisely, $G_i = \det^{-1}(\mu_2^n)$ and $Z(G_i) = \mu_n$.

\[ \square \]

6. Tensor Decomposition of $F$ . . .

6.1. between isocrystals. We still use Tannakian duality to decompose $F \otimes B(k)$ into tensor products.
Proposition 6.2. There exists a permutation \( s \in S_{m+1} \), rank 1 crystals \( \mathcal{L}_i \) with \( \otimes_i \mathcal{L}_i \cong \mathcal{O}_C \) and isomorphisms

\[
\phi_i : \mathcal{V}_i^s \otimes B(k) \longrightarrow \mathcal{V}_s(i) \otimes \mathcal{L}_i \otimes B(k)
\]
such that

\[
F = \otimes_i \phi_i.
\]

6.2.1. Proof of proposition [6.2] From the proof of proposition [5.5] for each \( i \), there exists a unique \( H_j = SL(2) \) such that \( H_j \hookrightarrow G_i \). This inclusion is an isomorphism between \( H_j \) and \( G_0 \), which is just a conjugation by some \( l \in GL(2) \). Without loss of generality, assume \( i = 1 \) and \( j = 2 \).

Note we have the following diagram:

\[
\begin{array}{ccc}
H_{\text{univ}} & \xleftarrow{f_1} & H_1 \\
& \searrow f_2 & \\
& H_2 & \\
& \nearrow f_1 & \\
& PGL(2) & \\
\end{array}
\]

The morphism \( f_1 \) is the usual quotient by the center \( SL(2) \longrightarrow PGL(2) \). Yet the morphism \( f_2 \) is twisted by the conjugation by \( l : f_2 = f_1 \circ C_l \).

Claim: this diagram is commutative.

Proof. We have

\[
\begin{array}{ccc}
G_1 \times G_2 \times G_3 \cdots \times G_{m+1} & \xleftarrow{\text{twisted by } F} & \prod H_i \\
& \xrightarrow{\text{twisted by } F} & \prod H_i \\
GL(2)^{\times m+1} & \longrightarrow & GL(2^{m+1})
\end{array}
\]

For any \( h \in H_{\text{univ}} \), let \( (h_1, h_2, h_3, \cdots, h_{m+1}) \) be the image of \( h \) in \( \prod H_i \) and \((g_1, g_2, g_3, \cdots, g_{m+1})\) image in \( G' \). Then \( \prod H_i \longrightarrow G' \) permutes the factors and sends \((h_1, h_2, h_3, \cdots)\) to \((h_2 l^{-1}, \cdots)\). Then \((h_2 l^{-1}, \cdots)\) and \((g_1, g_2, g_3, \cdots)\) have the same image under \( GL(2)^{\times m+1} \longrightarrow GL(2^{m+1}) \). Hence \( C_l(h_2) = t g_1 \) for some scalar \( t \in B(k) \). In particular, \( f_2(h_2) = f_1(g_1) \). The claim is true.

Then \( H_{\text{univ}} \longrightarrow G_1 \times H_2 \) factors through the limit of

\[
\begin{array}{ccc}
G_1 & \xrightarrow{f_1} & H_1 \\
& \nearrow f_2 & \\
& H_2 & \\
& \searrow f_1 & \\
& PGL(2) & \\
\end{array}
\]
Lemma 6.3. the limit of the diagram is $H_2 \times Z(G_1) = SL(2) \times \mu_n$ or $SL(2) \times \mathbb{G}_m$ with

$$H_2 \times Z(G_1) \to H_2 \quad \text{and} \quad H_2 \times Z(G_1) \to G_1$$

$$(h, k) \mapsto h \quad \text{and} \quad (h, k) \mapsto (kh^{-1})_l.$$  

Proof. we can prove it directly: for any $K'$ fitting in the diagram

we construct the map

$$K' \to Z(G_1) \times SL(2)$$

$$k \mapsto (s_1(k)C_1(s_2(k))^{-1}, s_2(k)).$$

Since the lower triangle is commutative, the map is well defined and obviously it is unique.  

Consider the Tannakian category generated by $\{V_1^\sigma \otimes B(k), V_2 \otimes B(k)\}$. Then it is isomorphic to $\text{Rep}(K')$ for some algebraic group $K'$. Recall $W_1$ and $V_2$ are the corresponding objects of $V_1^\sigma \otimes B(k)$ and $V_2 \otimes B(k)$ in $\text{Rep}(H_{\text{uni}})$, respectively. By proposition 3.5

$$K = \text{im} (H_{\text{uni}} \to \text{Aut}(W_1) \times \text{Aut}(V_2)) \subset G_1 \times H_2.$$  

Hence by lemma 6.3 $K \subset H_2 \times Z(G_1) = SL(2) \times Z(G_1)$. 

If $G_1 = GL(2)$, then $\dim K = 4$ and by $GL(2)$ connected, $K = SL(2) \times \mathbb{G}_m$. 

If $G_1^0 = SL(2)$ and $Z(G_1) = \mu_n$, then $\dim K^0 = 3$ and hence $K^0 = SL(2)$. It suffices to determine the number of the connected components of $K$. Let $\zeta$ be a generator of $\mu_n$. Then $\zeta$ and $-\zeta$ are in the same component of $G_1$.

(1) If $n \equiv 0 \pmod{4}$, then $-\zeta$ is also a generator of $\mu_n$. Hence $K$ has to be $SL(2) \times \mu_n$ to cover the whole $G_1$. 

(2) If $n \equiv 2 \pmod{4}$, then $\mu_n = \pm I \times \mu_2$ and hence $G_1 \cong SL(2) \times \mu_2$. So besides $SL(2) \times \mu_n$, $K$ also can be $G_1$.

In summary, $K = SL(2) \times \mathbb{G}_m$ or $SL(2) \times \mu_k$ for some $k$.

Hence as irreducible $K$ representation, $W_i$ is tensor of a $SL(2)$ representation and an irreducible $\mu_k$ or $\mathbb{G}_m$ representation, i.e. $V_i^\sigma \cong V_j \otimes L_i \otimes B(k)$.

That is the end of the proof of proposition 6.2.

6.3.1. Two corollaries. The following corollaries further determine the structure of the group $G_i$. However, we do not use them in the later arguments.

Corollary 6.4. For any $i$, $L_i^\otimes 2$ is trivial as isocrystal.

Proof. Since the monodromy on $\mathbb{C}^2$ has trivial determinant, the isocrystals $V_i$ all have trivial determinant, i.e. $\det V_i = \mathcal{O}_C$ as isocrystals. Then

$$\wedge^2 V_i^\sigma = (\wedge^2 V_i)^\sigma = \mathcal{O}_C.$$
Hence taking the determinant of $\phi_i$

$$\det \phi_i : \det V_i^\sigma \longrightarrow \det V_i \otimes L_i \otimes \mathcal{O}_C,$$

gives $L_i \otimes \mathcal{O}_C$ as isocrystal.

**Corollary 6.5.** Assumption as proposition 5.5: $G_i = SL(2)$ for each $i$.

**Proof.** Let $L_i$ be the one dimensional $H_{univ}$-representation corresponding to $L_i$. By corollary 6.4, $im(H_{univ} \longrightarrow GL(L_i)) = \mu_2$. Note the Tannakian category generated by $\{L_i, V_i\}$ is the same as $Rep(K)$:

$$< L_i, V_i >= < W_i, V_i >= Rep(K).$$

Then by proposition 3.5, $K \subset SL(2) \times \mu_2$. In the proof of proposition 6.2, we know $K$ always has the form of $SL(2) \times G_m$ or $SL(2) \times \mu_k$. Hence $K = SL(2)$ or $SL(2) \times \mu_2$. Either case gives $G_i = SL(2)$. □

6.6. **between crystals.** Since $E$ is an $F$-crystal, we still have $F : \otimes V_{i}^\sigma \longrightarrow \otimes V_{i}$. In the following, we base change from $W(k)$ to $k$ and use the properties of ordinary abelian varieties to further determine the permutation $s$.

**Lemma 6.7.** There exist $k_1, k_2 \in \mathbb{Z}$, such that $F$ is the tensor product of morphisms between crystals:

$$F = p_{i}^{k_1} \phi_1 \otimes p_{i}^{k_2} \phi_2 \otimes p_{i}^{k_3} \phi_3 \cdots \otimes p_{i}^{-k_1-k_2-\cdots-k_m} \phi_{m+1}.$$

**Proof.** Since $\phi_i (i = 1, 2)$ is a morphism between effective isocrystals, from example 3.4, there exists $k_i$ such that $p^{k_i} \phi_i$ is a morphism in $\text{Cris}(C)$. We can assume $p^{k_i} \phi_i \neq 0 \pmod{p}$ at each stalk. Then $p^{-k_1-k_2-\cdots-k_m} \phi_{m+1}$ is also a morphism in $\text{Cris}(C)$. In fact, for any $x \in C$ and $a_{m+1} \in V_{m+1}^\sigma$, we can find $a_1 \in V_1^\sigma_{|x}, a_2 \in V_2^\sigma_{|x}, \cdots$ such that

$$p^{k_i} \phi_i (a_i) \neq 0 \pmod{p}$$

for $1 \leq i \leq n$. Then $p^{-k_1-k_2-\cdots-k_m} \phi_{m+1}(a_{m+1}) \in V_{m+1}^\sigma$. Otherwise,

$$F(a_1 \otimes a_2 \cdots \otimes a_{n+1}) = p^{k_1} \phi_1 (a_1) \otimes_B (k) p^{k_2} \phi_2 (a_2) \cdots \otimes_B (k) p^{-k_1-k_2-\cdots-k_m} \phi_{m+1}(a_{m+1})$$

is not in $V_{1}^\sigma \otimes V_{2}^\sigma \cdots \otimes V_{m+1}^\sigma$. □

A straightforward corollary of lemma 6.7 is that

**Corollary 6.8.** Viewed as morphism between crystals, $F$ still preserves pure tensors.

Let

$$0 \longrightarrow \omega \longrightarrow E \longrightarrow \alpha \longrightarrow 0$$

be the weight 1 Hodge filtration associated to $X/C$. Then from the definition of Mumford curve, especially the action of Hodge group $G$ on $V$, we know $\omega$ is induced from a line bundle $\mathcal{L}$ in $V_1$, i.e.

$$\omega \cong \mathcal{L} \otimes V_2 \otimes V_3 \cdots \otimes V_{n+1}.$$

And $\alpha \cong V_1/\mathcal{L} \otimes V_2 \otimes V_3 \cdots \otimes V_{n+1}$.

Base change from $W(k)$ to $k$. Use $\bar{C}$ to denote the reduction of the Mumford curve over $k$ and $\bar{E}$ to denote the reduction of $E$. (For all, we add bar to denote the one reduced to $k$.)
Then the Frobenius $\mathcal{E}^{(p)} \xrightarrow{\bar{F}} \mathcal{E}$ factors through $\bar{\alpha}^{(p)}$ and then we have the conjugate spectral sequence:

$$0 \longrightarrow \bar{\alpha}^{(p)} \longrightarrow \bar{\mathcal{E}} \longrightarrow \bar{\omega}^{(p)} \longrightarrow 0.$$ 

Now we prove $s(1) = 1$ under some mild condition.

**Proposition 6.9.** If $\bar{C}$ intersects ordinary locus, then $s(1) = 1$.

**Proof.** Let $c$ be in the locus of the intersection of ordinary locus and $\bar{C}$. Then restricted to $c$, consider the composition $F' : \mathcal{E}^{(p)} \longrightarrow \bar{\alpha}$ in the following diagram

$$
\begin{array}{ccc}
\mathcal{E}^{(p)} & \xrightarrow{F} & \bar{\mathcal{E}} \\
\downarrow \phi & & \downarrow \pi \\
\bar{\alpha} & & \\
\end{array}
$$

Since $X_c$ is ordinary, $F'_c$ is surjective.

If $s(1) \neq 1$, Without loss of generality, suppose $s(1) = 2$. Note by lemma 6.7

$$F'(\mathcal{E}^{(p)}) = \pi \circ \bar{\mathcal{E}}(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})$$

$$= \pi \circ \otimes \bar{\phi}_1(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})$$

From the conjugate spectral sequence, $\bar{F}$ factors through $\bar{\alpha}^{(p)}$ and $\alpha = (\mathcal{V}_1/\mathcal{L}') \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{n+1}$, thus $\bar{\phi}_1 : \bar{\mathcal{V}}_1^{(p)} \longrightarrow \bar{\mathcal{V}}_2 \otimes \mathcal{L}_1$ factors through $\bar{\mathcal{V}}_1^{(p)}/\mathcal{L}'^{(p)}$, and the image of $\bar{\phi}_1$ has rank 1. But $\dim_{k} \mathcal{V}_2|_{c} = 2$. So $F'$ can not be surjective. \hfill $\square$

Hence $\phi_1 : \mathcal{V}_1^{(p)} \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}_1$.

**Remark 6.10.** According to [10], $\bar{C}$ intersects with the ordinary locus if $K$ splits over $\mathbb{Q}_p$. The Mumford curve $M$ is defined over the reflex field $K$, and let $p$ be the prime of $K$ over $p$.

Let $r = [K_p : \mathbb{Q}_p]$. Then by theorem 1.2 in [10], there are two Newton polynomials in $\bar{C}/k$, it is either $\{2m+1+(D) \times \frac{1}{2}\}$ or $\{2m+1-r+(D) \times 0, 2m+1-r+(D) \times \frac{1}{2}, \ldots, 2m+1-r+(D) \times 1\}$. So $\bar{C}$ intersects with ordinary locus if and only if $r = 1$, i.e. $K$ splits over $\mathbb{Q}_p$.

So there are infinitely prime $p$ over which the reduction of Mumford curve at $p$ is generically ordinary.

7. THE SURJECTIVITY OF $\sigma^* - \text{Id}$ ON THE PICARD GROUP

If we can find $\mathcal{L}'$ such that $\mathcal{L}_1^{-1} = (\sigma^* - \text{Id})(\mathcal{L}')$, then $\phi_1$ induces a morphism

$$\mathcal{V}_1^\sigma \otimes \mathcal{L}' \longrightarrow \mathcal{V}_1 \otimes \mathcal{L}'.$$ 

Replace $\mathcal{V}_1$ by $\mathcal{V}_1 \otimes \mathcal{L}'$. And then we can prove $\mathcal{V}_1$ is a Dieudonne crystal in next section.

Let $\sigma$ be the absolute Frobenius of $\bar{C}/k$ and Pic $(\bar{C}/W(k)_{\text{cris}})$ denote the groups of the rank 1 crystals on $\bar{C}$. The following general principle guarantees $\mathcal{L}_1 = (\sigma^* - \text{Id})(\mathcal{L}')$.

**Proposition 7.1.** The group endomorphism $\sigma^* - \text{Id}$ of Pic $(\bar{C}/W(k)_{\text{cris}})$ is surjective.
Lemma 7.3. \( \sigma^* - \text{Id} \) acts on \( W(k) \) surjectively.

Proof. Since \( k \) is algebraically closed, \( (\sigma^* - \text{Id}) \) acts on \( k \) surjectively. Then for any \( b \in W(k) \), we can find
\[
(\sigma^* - \text{Id})(a_0) = b + pb_1.
\]
And there exists \( a_1, a_2, a_3, \ldots \) such that
\[
(\sigma^* - \text{Id})(a_1) = b_1 + pb_2,
(\sigma^* - \text{Id})(a_2) = b_2 + pb_3,
(\sigma^* - \text{Id})(a_3) = b_3 + pb_4 \ldots
\]
Then \( (\sigma^* - \text{Id})(\sum_i p^i a_i) = b. \) In fact, since \( W(k) \) is \( p \)-adically complete, \( \sum p^i a_i \in W(k). \)
And \( (\sigma^* - \text{Id})(\sum p^i a_i) - b \) is contained in \( p^n W(k) \) for any \( n. \) Hence \( (\sigma^* - \text{Id})(a + \sum p^i a_i) - b = 0. \)

Now we recall the definition of Atiyah class. For the further explanation, please refer ([8], 10.1).
Let \( \mathcal{I} \) be the ideal sheaf of the diagonal set of \( X \times X \) and \( \mathcal{O}_{2\Delta} = \mathcal{O}_{X \times X}/\mathcal{I}^2. \)

Definition 7.4. For any smooth proper variety \( X \) and vector bundle \( V \) over \( X, \) the Atiyah class is the extension class of
\[
0 \rightarrow V \otimes \Omega^1_X \rightarrow p_{1*}(p_2^*V \otimes \mathcal{O}_{2\Delta}) \rightarrow V \rightarrow 0.
\]

Atiyah class is the unique obstruction to the existence of a connection on \( V. \)

From ([16] Remark 3.7), the Atiyah class of any line bundle coincides with its first Chern class. So line bundles with connections over a curve are exactly those of degree 0.

Lemma 7.5. The restriction of \( \sigma^* - \text{Id} \) to \( \text{Pic}^0(\bar{C}/k_{\text{cris}}) \) is surjective.

Proof. Note the rank 1 crystal on the site \( \bar{C}/k_{\text{cris}} \) is equivalent to a line bundle on \( \bar{C}/k \) with connection. And for any \( \mathcal{L} \in \text{Pic}(\bar{C}/k_{\text{cris}}), \sigma^*(\mathcal{L}) = \mathcal{L}^p. \) So it suffices to show that for any degree 0 line bundle with connection \( (\mathcal{L}, \nabla), \) there exists a line bundle with connection \( (L, \nabla_L) \) such that
\[
(L, \nabla_L)^{p-1} \cong (\mathcal{L}, \nabla).
\]

Since \( k \) is algebraically closed, the Jacobian \( \text{Jac}(\bar{C}/k) \) is a divisible group. Hence we always can find a line bundle \( L \in \text{Jac}(\bar{C}/k) \) such that \( L^{p-1} \cong \mathcal{L}. \)

Note the set of connections of \( \mathcal{L} \) is a torsor under
\[
\text{Hom}(\mathcal{L}, \mathcal{L} \otimes \omega_{\bar{C}}) = \Gamma(\omega_{\bar{C}}) = k^g.
\]
The same for \( L. \) For any two connections \( \nabla_L, \nabla'_L \) on \( L, \) let \( \nabla_L - \nabla'_L = h \in \Gamma(\omega_{\bar{C}}). \)
Thus to find the connection \( \nabla_L, \) it suffices to show the \( (p - 1)-\text{th} \) power is an injection from the connections on \( L \) to the connections on \( \mathcal{L}. \) Then for any local section \( \otimes_i s_i, \)
\[
((\nabla_L + h)^{p-1} - \nabla_L^{p-1})(\otimes_{i=1}^{p-1}s_i) = \sum_{i=1}^{p-1} \cdots \otimes h.s_i \otimes \cdots
= (p - 1)(\prod_i s_i)h(1 \otimes 1 \otimes \cdots \otimes 1).
\]

13
Hence for any connection $\nabla_{\mathcal{L}}$, if $g = \nabla_{\mathcal{L}} - \nabla_{\mathcal{L}}^{p-1}$, then $(\nabla_{\mathcal{L}} + \frac{g}{p-1})^{p-1} = \nabla_{\mathcal{L}}$. So $(\sigma^* - \text{Id})$ acts on $\text{Pic}^0(\bar{C}/k_{\text{cris}})$ surjectively.

**Lemma 7.6.** $\sigma^* - \text{Id}$ maps $H^1(\bar{C}/W_{\text{cris}}, \mathcal{O}_C)$ to itself surjectively.

First we quote a result from ([15], page 143):

**Proposition 7.7.** Let $k$ be an algebraically closed field with characteristic $p$ and $V$ be any vector space with a $p$–linear map $x \mapsto x^{(p)}$, there is a unique decomposition, invariant under $x \mapsto x^{(p)}$:

$$V = V_s \oplus V_n,$$

such that $V_s$ has a basis $x_1, \ldots, x_k$ for which $x_i^{(p)} = x_i$, and such that $x \mapsto x^{(p)}$ is a nilpotent map on $V_n$.

Then we can prove our lemma 7.6:

**Proof.** Let $C$ be any lifting of $\bar{C}$ to $W(k)$. Then by comparison theorem,

$$H^1(\bar{C}/W(k)_{\text{cris}}, \mathcal{O}_C) \cong H^1(C, \Omega^1_C) \cong W(k)^2\mathfrak{g}.$$ 

Let $N$ denote the free $W(k)$–module with $\sigma^*$ action. Then $V := N/pN$ is a $k$–vector space with $p$–linear action. By proposition 7.7,

$$V = V_s \oplus V_n.$$

On $V_n$, since $\sigma^*$ acts nilpotently, $(\sigma^* - \text{Id})$ is invertible and hence surjective. On $V_s$, by 7.3 we can find $\lambda$ such that $(\sigma^* - \text{Id})(\lambda) = 1$. Then for each $k$, $(\sigma^* - \text{Id})(\lambda x_k) = x_k$. Hence $(\sigma^* - \text{Id})$ acts on $V$ surjectively.

Back to $N$, for any $b \in N$, we can choose $a_0$ such that

$$(\sigma^* - \text{Id})(a_0) = b + pb_1.$$ 

Then choose $a_1$ such that

$$(\sigma^* - \text{Id})(a_1) = b_1 + pb_2.$$ 

Following this way, we can find $a_2, a_3, \ldots$.

Similar to the proof of lemma 7.3 we have

$$(\sigma^* - \text{Id})(a_0 + pa_1 + p^2a_2 + \cdots + p^na_n + \cdots) = b.$$ 

□

Now we can prove the theorem:

**Proof.** Note Pic $(\bar{C}/W(k)_{\text{cris}}) \cong H^1(\bar{C}/W(k)_{\text{cris}}, \mathcal{O}_C^*)$. And we have the sequence

$$0 \rightarrow (1 + p\mathcal{O}_{\bar{C}})^* \rightarrow \mathcal{O}_{\bar{C}}^* \rightarrow (\mathcal{O}_{\bar{C}}/p)^* \rightarrow 0.$$ 

And $(\sigma^* - \text{Id})$ acts on the long exact sequence. Since char $k > 2$, the exponential and logarithm maps converge and thus give an isomorphism between abelian groups

$$\mathcal{O}_C \cong (1 + p\mathcal{O}_C)^*.$$ 

So the cohomology groups are isomorphic:

$$H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \cong H^1(\bar{C}/W(k)_{\text{cris}}, (1 + p\mathcal{O}_C)^*).$$
We have the long exact sequence

\[ H^1(\bar{C}/W(k)_{\text{cris}}, \mathcal{O}_C) \longrightarrow H^1(\bar{C}/W(k)_{\text{cris}}, \mathcal{O}_C^*) \xrightarrow{\pi} \] 
\[ H^1(\bar{C}/W(k)_{\text{cris}}, (\mathcal{O}_C/p)^*) \cong \text{Pic} \left( \bar{C}/k_{\text{cris}} \right) \longrightarrow H^2(\bar{C}/W(k)_{\text{cris}}, \mathcal{O}_C) \] .

Let \( C \) be any lifting of \( \bar{C} \) to \( W(k) \). By [3] theorem 6.6, the category of crystals on \( \bar{C} \) is equivalent to the category of vector bundles with connections on \( C \). Hence \( \text{Pic} \left( \bar{C}/W(k)_{\text{cris}} \right) \) is isomorphic to the group of line bundles with connections on \( C \). And \( \pi \) is the pull back of such line bundle from \( C \) to \( \bar{C} \). Therefore \( \text{im} \pi \subset \text{Pic}^0(C/k_{\text{cris}}) \).

Since the obstruction of deforming the line bundle from \( \bar{C} \) to \( C \) vanishes and the deformation preserves the degree, \( \text{Pic}^0(C) \longrightarrow \text{Pic}^0(\bar{C}) \) is surjective. Another argument is that for any degree 0 line bundle \( L \) on \( C \), it corresponds to a divisor \( \sum n_i p_i \), with each \( p_i \) a \( k \)-point. Then by Hensel’s lemma, each \( p_i \) lifts to a \( W(k) \)-point \( \bar{p}_i \) (not uniquely). Let \( \sum n_i \bar{p}_i = \bar{L} \in \text{Pic}^0(C) \) and then \( \bar{L} \) reduces to \( L \).

For the connection, for any \( (L, \nabla) \in \text{Pic}^0(C/k_{\text{cris}}) \), choose a lifting \( \bar{L} \in \text{Pic}^0(C) \) of \( L \) and a connection \( \nabla \) on \( \bar{L} \). Let \( \nabla' \) be the reduction of \( \nabla \), then \( \nabla' - \nabla = f \in \Gamma(\omega_C) \). Choose \( \tilde{f} \in \Gamma(\omega_C) \) such that \( \tilde{f} \) reduces to \( f \). Then \( \nabla - \tilde{f} \) reduces to \( \nabla \). And \( \pi(\bar{L}, \nabla - \tilde{f}) = (L, \nabla) \).

Hence \( \text{im} \pi = \text{Pic}^0(C/k_{\text{cris}}) \). So we have the following sequence:

\[ H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \longrightarrow H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^*) \longrightarrow \text{Pic}^0(C/k_{\text{cris}}) \longrightarrow 0 \]

By lemma [7.5] and [7.6], \( \sigma^* - \text{Id} \) induces surjective endomorphisms on \( H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \) and \( \text{Pic}^0(C/k_{\text{cris}}) \). Therefore (\( \sigma^* - \text{Id} \)) maps \( H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \) surjectively on itself. \( \square \)

**Remark 7.8.** In the proof, we use the convergence of exponential and logarithm, which are true if and only if the characteristic \( p > 2 \).

8. **The Dieudonné Crystal \( \mathcal{V} \) and the Unit Crystal \( \mathcal{T} \)**

Now we can choose \( L'' \in \text{Pic} (C/W(k)_{\text{cris}}) \) such that \( (\sigma^* - \text{Id})(L'') = L^{-1}_1 \). Then \( \phi_1 \) induces isomorphism

\[ \gamma : \mathcal{V}_1^\sigma \otimes L'' \otimes B(k) \longrightarrow \mathcal{V}_1 \otimes L \otimes B(k). \]

Let \( \mathcal{V} = \mathcal{V}_1 \otimes \mathcal{L} \). Similarly, we have the isomorphism

\[ \beta : \mathcal{V}_2^\sigma \otimes \mathcal{V}_3^\sigma \otimes \cdots \otimes \mathcal{V}_{m+1}^\sigma \otimes (L''')^\sigma \otimes B(k) \longrightarrow \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \cdots \otimes \mathcal{V}_{m+1} \otimes L''' \otimes B(k). \]

Denote \( \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \cdots \otimes \mathcal{V}_{m+1} \otimes L''' \) as \( \mathcal{T} \). Hence as crystals,

\[ \mathcal{E} \cong \mathcal{V} \otimes \mathcal{T} \]

and as a morphism between isocrystals

\[ F = \gamma \otimes \beta, \quad V = pF^{-1} = p\gamma^{-1} \otimes \beta^{-1}. \]

There exists \( k \in \mathbb{Z} \) such that \( p^k \gamma : \mathcal{V}^\sigma \longrightarrow \mathcal{V} \) as a morphism between crystals and \( p^k \gamma \neq 0 \) (mod \( p \)). Since \( F = p^k \gamma \otimes p^{-k} \beta : \mathcal{E}^\sigma \longrightarrow \mathcal{E} \) is a morphism between crystals, \( p^{-k} \beta : \mathcal{T} \longrightarrow \mathcal{T} \) is also a morphism between crystals. Note since \( F \neq 0 \) (mod \( p \)), we have \( p^{-k} \beta \neq 0 \) (mod \( p \)).

Similarly, there exists \( k' \) such that the Verschiebung operator

\[ V = p^{k+1} \gamma^{-1} \otimes p^{-k'} \beta^{-1} \]
such that $p^{-k'} \beta \neq 0 \mod p$.

**Lemma 8.1.** $k + k' = 0$

**Proof.** By the isomorphism \[ \alpha = (V_1 / L \otimes L')^\sigma \otimes T^\sigma. \] Over $k$, as in the proof of proposition 6.9 the Frobenius

\[ F = p^k \gamma \otimes p^{-k} \beta : \mathcal{E}^\sigma \rightarrow \mathcal{E} \]

induces an injection

\[ \alpha^{(p)} = (V_1 / L \otimes L')^{(p)} \otimes T^{(p)} \rightarrow \mathcal{E}. \]

Hence $p^{-k} \beta$ is injective on $\mathcal{T}$. Since $F$ is a bundle map, $p^{-k} \beta$ is isomorphic.

Note the fact that for any $W(k)$-algebra $R$ and any $r \in R$, if the image $\bar{r} \in \bar{R}$ over $k$ is a unit, then $r$ is a unit in $R$. By checking on the stalk, it is easy to show $p^{-k} \beta : T^\sigma \rightarrow T$ is also isomorphic.

Similarly, $p^{-k'} \beta^{-1}$ is also isomorphic. So $p^k \gamma : \mathcal{T}^\sigma \rightarrow \mathcal{T}$ is isomorphic. Then necessarily $k = -k'$.

Then $p^k \gamma \circ p^{-k'} \beta^{-1} = p \cdot \text{Id}$. Combining with the fact $p^{-k'} \beta^{-1}$ and $p^{-k} \beta$ are isomorphisms, we have

**Corollary 8.2.**

\[ (\mathcal{V}, F_\mathcal{V} = p^k \gamma, V_\mathcal{V} = p^{-k'} \beta^{-1}) \]

is a Dieudonné crystal and

\[ (\mathcal{T}, F_\mathcal{T} = p^{-k} \beta) \]

is a unit crystal. And

\[ (\mathcal{E}, F) \cong (\mathcal{V}, F_\mathcal{V}) \otimes (\mathcal{T}, F_\mathcal{T}). \]

Before considering the BT group, let us show another easy corollary.

Let the filtration $\text{Fil}_V$ be $L \otimes L' \subset V$ and $\text{Fil}_T$ be the trivial filtration. (For $L$, see the isomorphism (4); for $L'$, refer to isomorphism (5)) Note the Hodge filtration of $\mathcal{E}$ comes from a filtration $L \subset L_1$:

\[ L \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1} \subset \mathcal{E} = V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1}. \]

Since $L \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1} = (L \otimes L') \otimes (V_2 \otimes V_3 \cdots \otimes V_{m+1} \otimes L') = (L \otimes L') \otimes T$, we have

**Corollary 8.3.** The Hodge filtration of $\mathcal{E}$ comes from a sub line bundle $L \otimes L'$ of $V$. In other words, the Hodge filtration on $\mathcal{E}$ is isomorphic to $\text{Fil}_V \otimes \text{Fil}_T$.

9. **The BT Groups Corresponding to $\mathcal{V}$, $\mathcal{T}$ and $\mathcal{E}$**

From (6), we know that over a smooth curve $C/k$, the category of finite locally free Dieudonné crystals on $C$ is equivalent to the category of BT groups on $C$. Obviously $(\mathcal{E}, F, V)$ corresponds to $X_{\infty}$. Let $G$ be the BT group corresponding to $(\mathcal{V}, F_V, V_\mathcal{V})$.

Since $\mathcal{T}$ is a unit crystal, from (2, 2.4.10), $\mathcal{T}$ comes from an etale BT group $\mathcal{H}$ over $C$. In particular, $\mathbb{D}(H[n]) \cong T/p^n$ and each truncated $T/p^n$ comes from a local system (4, theorem 2.2)

\[ \rho_n : \pi_1(C, c) \rightarrow GL(4, \mathbb{Z}/p^n). \]
Then there exists a finite etale covering \( f_n : \tilde{C}' \rightarrow C \) such that \( \pi_1(\tilde{C}', c) \cong \ker \rho_n \). Hence we have \( f_n^*(T/p^n) \cong \mathcal{O}_{\tilde{C}' \backslash W_n}^\text{sm} \) as unit root \( F \)-crystals. From ([2], 2.4.1), over smooth curve \( \tilde{C} \), the category of finite locally free etale group schemes is equivalent to the category of \( p \)-torsion unit crystals. Thus

\[
f_n^*(\tilde{H}[n]) \cong (\mathbb{Z}/p^n)^{\otimes m}.
\]

**Definition 9.1.** Define the binary operation between two BT groups:

\[
G \odot H := \text{colim}_n (G[n] \otimes_{\mathbb{Z}} H[n]).
\]

**Remark 9.2.** The inductive system \( (G[n] \otimes_{\mathbb{Z}} H[n]) \) will be explained in the appendix 9.A after this section. Note in general \( G \odot H \) is just an abelian sheaf rather than a group scheme. But in our case, \( H \) is etale and we will show in the appendix 9.A right that \( G \odot H \) is indeed a BT group and \((G \odot H)[n] = G[n] \otimes_{\mathbb{Z}} H[n]\).

**Proposition 9.3.**

\[
\tilde{X}[n] \cong \tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n].
\]

**Proof.** We will show that \( \mathbb{D}(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n]) = \mathcal{V} \otimes T/p^n = \mathcal{E}/p^n \) (corollary 8.2) as Dieudonné crystals. Over \( \tilde{C}' \),

\[
^{(*)} \mathbb{D}_{\tilde{C}'}(f_n^*(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n])) \cong f_n^*(\mathcal{V}/p^n)^{\otimes m} \cong f_n^*(\mathcal{V}/p^n \otimes_{\mathcal{O}_{\tilde{C}'}} f^*T/p^n)
\]

as Dieudonné crystals. Both sides have effective descent datum with respect to \( \tilde{C}' \rightarrow \tilde{C} \). For any \( g \in \text{Aut}(\tilde{C}'/\tilde{C}) \), \( g^* \) acts on both of \( f^*(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n]) \) and \( f_n^*(\mathcal{V}/p^n \otimes_{\mathcal{O}_{\tilde{C}'}} f^*T/p^n) \) which is compatible with the functor \( \mathbb{D}_{\tilde{C}'} \):

\[
\begin{align*}
\mathbb{D}_{\tilde{C}'}(f_n^*(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n])) & \xrightarrow{g^*} f_n^*(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n]) \\
\mathbb{D}_{\tilde{C}'}(f_n^*(\mathcal{V}/p^n \otimes_{\mathcal{O}_{\tilde{C}'}} f^*T/p^n)) & \xrightarrow{g^*} f_n^*(\mathcal{V}/p^n \otimes_{\mathcal{O}_{\tilde{C}'}} f^*T/p^n)
\end{align*}
\]

is commutative (we leave the details leave to the reader). Hence the isomorphism \(^{*}\) between effective descent datum also descends to \( \tilde{C} \).

Then we have

\[
\mathbb{D}_{\tilde{C}'}(\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n]) = (\mathcal{V} \otimes T/p^n, F_\mathcal{V} \otimes F_T, V_\mathcal{V} \otimes F_T^{-1}).
\]

Since \( \tilde{C} \) is smooth over an algebraically closed field \( k \), it has locally \( p \)-basis. Hence we can apply ([2], 4.1.1), the Dieudonné functor is fully faithful, so

\[
\tilde{G}[n] \otimes_{\mathbb{Z}} \tilde{H}[n] \cong \tilde{X}[n].
\]

\[\square\]

**Corollary 9.4.** \( \tilde{G} \odot \tilde{H} = \tilde{X}[\infty] \).

To complete the proof of theorem 1.2, it remains to show the isomorphism in proposition 9.3 lifts to \( C \).

From [6], the BT group \( \tilde{G} \) induces a filtration of \( \mathbb{D}(\tilde{G})_C = \mathcal{V}_C \):

\[
0 \rightarrow \omega_{\tilde{G}} \rightarrow \mathcal{V}_{\tilde{G}} \rightarrow t_{\tilde{G}} \rightarrow 0.
\]

**Lemma 9.5.** The above filtration coincides with the filtration \( \text{Fil}_\mathcal{V} \mod p \).
Remark 9.6. For the definition of $\text{Fil}^0_V$, refer to corollary 8.3.

Proof. From the proof of proposition 6.9 and corollary 8.2,

$$\ker \tilde{F}_V = \ker (p^{-k}\gamma) = \ker (\tilde{\varphi}_1) = L \otimes L' \otimes E.$$

By ([9], theorem 2.5.2 and remark 2.5.5), the subbundle of $V$ satisfying this condition is unique and $\omega_{\tilde{G}} \cong L \otimes L'$. □

Proposition 9.7. The curve $\tilde{C}$ is a versal deformation of the BT group $\tilde{G}$.

Proof. From ([12], theorem 0.9), any Shimura curve of Hodge type admits the maximal Higgs field. So $C$ and thus $\tilde{C}$ has maximal Higgs field:

$$\begin{array}{cccccc}
0 & \longrightarrow & \tilde{\omega} & \longrightarrow & \mathcal{E}_C \\
& & \downarrow \nabla & & \\
\mathcal{E}_C \otimes \Omega_C^1 & \longrightarrow & \tilde{\alpha} \otimes \Omega_C^1 & \longrightarrow & 0
\end{array}$$

the induce map $\theta : \tilde{\omega} \longrightarrow \tilde{\alpha} \otimes \Omega_C^1$, is an isomorphism.

By corollary 8.3, the filtration of $\mathcal{E}_C$ comes from that of $V_C$. So $\theta = \theta_{V_C} \otimes \text{Id}_T$ where $\theta_{V_C}$ is the Higgs field of $V_C$. Hence the Higgs field of $V_C/\tilde{C}$ is maximal and combining with lemma 9.5, it implies $\omega_{\tilde{G}} \cong t_{\tilde{G}} \otimes \Omega_C^1$. By ([9], A.2.3.6), $C$ is a versal deformation of the BT group $G$. □

From ([19], the main theorem), such a curve $\tilde{C}$ admits a lifting to $W(k)$ over which $\tilde{G}$ admits a lifting as a BT group.

Proposition 9.8. The lifting of $\tilde{C}$ coincides with $C$.

Proof. From ([11], theorem 1.6), the lifting of the BT group $\tilde{G}$ is equivalent to lifting the filtration $\omega_{\tilde{G}} \hookrightarrow V_{\tilde{C}}$. By lemma 9.5, it is equivalent to lifting $L \otimes L' \hookrightarrow V_C$ and hence the curve $C/W(k)$ admits a lifting of $\tilde{G}$. From [19], we know the lifting of $\tilde{G}$ is unique. □

Let $G$ be the lifting of $\tilde{G}$ on $C$ and $H$ be the lifting of $\tilde{H}$ on $C$. Since $\tilde{H}$ is etale, $H$ is etale and unique up to isomorphism. By proposition 9.4 in the appendix, $G \odot H$ is a BT group.

Proposition 9.9.

$$X[\infty] \cong G \odot H.$$

Proof. In proposition 9.3, we have shown that $X[\infty] \cong G \odot H$ as BT groups over $\tilde{C}$. Both sides are liftable to $C$ (proposition 9.8), induced by the same filtration $\omega_{G} \otimes T \hookrightarrow \mathcal{E}$ (lemma 9.5). Again by ([11], theorem 1.6), $X[\infty] \cong G \odot H$. □

Now we have finished the proof of theorem 1.2

9.A. The dot tensor product $G \odot H$. In this appendix, let $G$ and $H$ be BT groups over any connected smooth variety $X$. And $H$ is further an etale BT group with height $h$.

Lemma 9.B. For any two integers $n < m$, $G[n] \otimes_{\mathbb{Z}/p^n} H[m] = G[n] \otimes_{\mathbb{Z}/p^n} H[n]$. 

18
Proof. Consider the following sequence:

\[ H[m] \xrightarrow{p^n} H[m] \rightarrow H[n] \rightarrow 0. \]

Tensoring with \( G[n] \) yields

\[ H[m] \otimes_{\mathbb{Z}/p^n} G[n] \xrightarrow{p^n \otimes G[n] = 0} H[m] \otimes_{\mathbb{Z}/p^n} G[n] \rightarrow H[n] \otimes_{\mathbb{Z}/p^n} G[n] \rightarrow 0 \]

is exact. Therefore \( H[m] \otimes_{\mathbb{Z}/p^n} G[n] \cong H[n] \otimes_{\mathbb{Z}/p^n} G[n] \cong H[n] \otimes_{\mathbb{Z}/p^n} G[n] \).

Lemma 9.C. \( G[n] \otimes_{\mathbb{Z}} H[n] \) is a \( BT_n \) group scheme over \( X \).

Proof. From the arguments before (6), we know there exists an etale morphism \( f_n : X' \rightarrow X \) such that

\[ f_n^*(G[n] \otimes_{\mathbb{Z}} H[n]) \cong f_n^*(\mathcal{G}[n])^{\times h} \]

is a \( BT_n \) group. Further, it defines a descent datum with respect to the etale covering \( X' \rightarrow X \). Since \( \mathcal{G}[n] \) is a finite group scheme, \( f_n^*(\mathcal{G}[n])^{\times h} \rightarrow X' \) is an affine morphism. By (5, Chapter Descent, lemma 33.1, lemma 35.1), the group scheme representing \( f_n^*(\mathcal{G}[n])^{\times h} \) descents to \( C \) and represents \( G[n] \otimes_{\mathbb{Z}} H[n] \). Hence \( G[n] \otimes_{\mathbb{Z}} H[n] \) is a group scheme. Obviously it is \( p^n \)-torsion.

Furthermore, all the extra structure of \( BT_n \) group can also descent from \( f_n^*(\mathcal{G}[n])^{\times h} \) to \( G[n] \otimes_{\mathbb{Z}} H[n] \). Hence \( G[n] \otimes_{\mathbb{Z}} H[n] \) is a \( BT \) group.

Proposition 9.D. \( G \otimes H \) is a \( BT \) group over \( X \).

Proof. Let

\[ i_n : G[n] \rightarrow G[n + 1] \]

be the inclusions of truncated \( BT \) groups. Since each \( H[n + 1] \) is a flat \( \mathbb{Z}/p^{n+1} \) module, \( i_n \) induces an inclusion

\[ G[n] \otimes H[n + 1] \hookrightarrow G[n + 1] \otimes H[n + 1] \]


\[ G[n] \otimes H[n] \cong G[n] \otimes H[n + 1], G[1] \otimes H[n + 1] \cong G[1] \otimes H[1]. \]

Hence we have the short exact sequence of truncated \( BT \) groups

\[ 0 \rightarrow G[n] \otimes H[n] \rightarrow G[n + 1] \otimes H[n + 1] \rightarrow G[1] \otimes H[1] \rightarrow 0. \]

Similarly, one can show this sequence holds even replacing \( 1 \) by any positive integer \( k \).

Hence \( \{G[n] \otimes H[n], i_n\} \) forms a inductive system and taking the inductive limit gives a \( BT \) group. \( \square \)

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