1. Introduction

The theory of $L$-functions has played a central role in number theory from its origins in Dirichlet’s proof of the density of primes in arithmetic progressions to the present day. In this paper we consider the so-called triple product $L$-function. We have $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where each $\pi_i$ is an automorphic representation of $GL_2(F)$ for a number field $F$, and we consider the central value $L(\frac{1}{2}, \Pi)$. (See Section 4.1 for the definition of $L(s, \Pi)$.)

Understanding the value $L(\frac{1}{2}, \Pi)$ began with work of Garrett[9] and was furthered by Gross-Kudla[11], Kudla-Harris[14],[13] and others. An especially beautiful formula was given by Watson[29] for many choices of $\pi_i$. Recently, Ichino [15] gave a very general formula extending these results. Although his result is more general than Watson’s, it is less explicit. The results of the present paper can be interpreted as making explicit the formula of Ichino. We do this by computing certain local trilinear forms. As a consequence, we also obtain a generalization of Watson’s formula.

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Another important application of the results of this paper, and the initial motivation for this work, is to prove a conjecture of Venkatesh [28, Hyp 11.1] which has application toward subconvexity for the triple product \( L \)-function. It is often the case that controlling the growth of the special values over a family of \( L \)-functions has number theoretic applications. The general principle of convexity gives bounds on these values. An improvement on this bound is often required for number theoretic applications.

Bernstein and Reznikov, in [2], apply Watson’s formula to establish subconvexity in the “eigenvalue aspect.” They fix two Maass forms and vary the eigenvalue of the third form. Venkatesh [28] dealt with subconvexity of the triple product \( L \)-function in “level aspect.” Using ergodic theory, he establishes bounds for the period integral

\[
J(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \int_{\text{PGL}_2(F)\backslash \text{PGL}_2(A_F)} \varphi_1(g)\varphi_2(g)\varphi_3(g)dg
\]

for varying choices of \( \varphi_i \in \pi_i \). In particular, he proved the following.

**Theorem 1.1** (Venkatesh). Let \( F \) be a number field and \( \pi_1, \pi_2 \) automorphic cuspidal representations of \( \text{PGL}_2(A_F) \) with (finite) conductors \( n_1, n_2 \) respectively. Let \( \pi_3 \) be another such representation with prime conductor \( p \nmid n_1n_2 \). Let \( \varpi \) be a uniformizer of \( F_p \). For given \( \varphi_i \in \pi_i \) \((i = 1, 2)\) fixed by \( \text{GL}_2(O_F) \), we put \( \varphi = \varphi_1 \otimes \rho((\varpi^{-1}))\varphi_2 \otimes \varphi_3 \) where \( \varphi_3 \in \pi_3 \) and \( \rho \) is the right regular action. Then

\[
|J(\varphi)| \ll c_{\pi_1, \pi_2} \|\varphi_1\|_{L^4}\|\varphi_2\|_{L^4}\|\varphi_3\|_{L^2}N(p)^{-\frac{(1-4\alpha)(1-2\alpha)}{4(3-4\alpha)}}
\]

where \( \|f\|_{L^p} \) is the standard \( L^p \)-norm, and \( \alpha \) is any bound towards Ramanujan for \( \text{GL}_2 \) over \( F \). (We can take \( \alpha = 1/9 \) by Kim-Shahidi[19].)

Venkatesh conjectured an upper bound for the central \( L \)-value in terms of \( |J(\varphi)|^2 \) which would then give a subconvexity bound for the central \( L \)-value \( L(\frac{1}{2}, \Pi) \). (See Corollary 1.5.) To describe it properly, however, we need an analogy of the period integral \( J(\varphi) \) for a quaternion algebra \( B \) over \( F \).

### 1.1. Statement of results

We begin by setting notations. Let \( \pi_{i,v} \) denote irreducible admissible representations of \( \text{GL}_2(F_v) \) for \( i = 1, 2, 3 \). Write \( \pi^B_{i,v} \) for the corresponding representation of \( B_v \), the division quaternion algebra over \( F_v \), via Jacquet-Langlands. (This is zero if none exists.) Put \( \Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v} \). Globally, we let \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \) where each \( \pi_i = \bigotimes_v \pi_{i,v} \) is a cuspidal automorphic representation, and if \( B \) is a quaternion algebra over \( F \), we define \( \Pi^B \) to be the corresponding automorphic representation of \( B^\times \times B^\times \times B^\times \).

For an algebraic group \( G \) over \( F \) we denote \([G] = Z(G)(A)G(F) \backslash G(A)\). Recall that a quaternion algebra \( B/F \) is determined by an even set of places \( \Sigma = \Sigma_B \) where \( v \in \Sigma \) if and only if \( B(F_v) \) is a division algebra. We call \( v \in \Sigma \) **ramified**.

Given a global (or local) representation \( \Pi = \bigotimes_v \Pi_v \) (or \( \Pi_v \)) as above, there exist certain holomorphic functions \( \varepsilon_v(s, \Pi, \psi_v) \) depending on a choice of additive character \( \psi_v : F \to \mathbb{C}^\times \). See [26] for details of their definition and properties. In particular, in the global case, if \( \psi = \bigotimes \psi_v \) is trivial on \( F \), the function \( \varepsilon(s, \Pi) := \prod_v \varepsilon_v(s, \Pi, \psi) \) is independent of \( \psi \), and the \( L \)-function satisfies the functional equation

\[
L(1-s, \Pi) = \varepsilon(s, \Pi)L(s, \Pi).
\]
We refer to Section 4.1 for the definition of the triple product $L$-function $L(s, \Pi)$. Moreover, if the central character of $\Pi_\infty$ is trivial (which we will always assume) then $\epsilon_v(\frac{1}{2}, \Pi) = \pm 1$ is independent of $\psi_v$. Denote the set $\{ v \mid \epsilon_v(\frac{1}{2}, \Pi) = -1 \}$ by $\Sigma(\Pi)$. This is a finite set. Notice that $J : \Pi \to \mathbb{C}$ as given in (1.2) is a $GL_2(\mathbb{A})$-invariant linear form. Prasad showed that the existence of a $GL_2(\mathbb{A})$-invariant form on $\Pi$ is determined by the local epsilon factors. In [25] he proved the theorem below in almost all cases. Loke [22] completed the remaining cases.

**Theorem 1.2 (Prasad, Loke).** With the notation as above, the following is true.

1. $\dim \text{Hom}_{GL_2(F_v)}(\Pi_v, \mathbb{C}) + \dim \text{Hom}_{B_v}(\Pi_v^{B_v}, \mathbb{C}) = 1$.  
2. $\dim \text{Hom}_{GL_2(F_v)}(\Pi_v, \mathbb{C}) = 1$ if and only if $\epsilon(\frac{1}{2}, \Pi_v) = 1$.  
3. When $\epsilon(\frac{1}{2}, \Pi) = 1$ the global quaternion algebra $B = B_\Pi/F$ whose ramification set is $\Sigma(\Pi)$ is the unique quaternion algebra for which $\text{Hom}_{B \times (\mathbb{A})}(\Pi^B, \mathbb{C}) \neq 0$.  
4. When $\epsilon(\frac{1}{2}, \Pi) = -1$ one has $\text{Hom}_{B \times (\mathbb{A})}(\Pi^B, \mathbb{C}) = 0$ for all quaternion algebras $B$ over $F$.

We will prove the following in this paper.

**Theorem 1.3.** Fix $\pi_1, \pi_2$ cuspidal automorphic representations of $GL_2$ over a number field $F$ with conductors $n_1, n_2$ respectively. Fix an ideal $n$. Let $\pi_3$ be cuspidal automorphic with conductor $np$ for any fixed ideal $n$ and a prime $p \nmid n_1n_2n$. Let $\varpi$ be a uniformizer of $F_p$. Let $S_f = \{ q \mid \gcd(n_1, n_2, n) \}$, and $S_\infty = \{ v \text{ real} \}$. Then there is a quaternion algebra $B$ over $F$ such that $\Sigma_B \subset S_f \cup S_\infty$ and there is a finite set of vectors $\mathbf{F}_i^B \subset \pi_i^B$ for $i = 1, 2$ such that

$$L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\epsilon, F, R} N(p)^{1+\epsilon} \left| \int_{[B^+]} \varphi_1(b)\varphi_2(b(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}))\varphi_3(b)db \right|^2$$

as $N(p) \to \infty$ and the Langlands parameters of $\pi_{3,\infty}$ remain bounded by $R$, for some $\varphi_i \in \mathbf{F}_i^B$ ($i = 1, 2$) and $\varphi_3 \in \pi_3^B$ a new vector.

We also obtain a generalization of Watson’s formula to totally real number fields. In particular, we prove the following. (See Theorem 4.7 for the more general statement.)

**Theorem 1.4 (Watson when $F = \mathbb{Q}$).** Suppose that $F$ is a totally real number field and let $\Pi$ be a cuspidal automorphic representation as above of squarefree level $N$. Moreover, we assume that for each $v \mid \infty$, $\pi_{iv}$ are discrete series representations such that the largest weight is the sum of the two smaller weights. Let $B$ be any quaternion algebra such that $\Pi^B \neq 0$, so $\mathbf{d}_B = \prod_{p \in \Sigma_B} p$ divides $N$. Let $\epsilon_v = \epsilon_v(\frac{1}{2}, \Pi_v)$, and for finite $v$, and let $q_v$ be the order of the residue field of $F_v$. Then

$$\frac{\left| \int_{[B^+]} \varphi(b)db \right|^2}{\prod_{i=1}^3 \int_{[B^+]} |\varphi_i(b)|^2db} = \frac{\zeta(2)}{2^4} \frac{L(\frac{1}{2}, \Pi)}{L(1, \Pi, \text{Ad})} \prod_v C_v$$
where \( db \) is the Tamagawa measure on \([B^\times]\) (defined in Section 2.2), \( \varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \) with \( \varphi_1 \) a new vector of \( \pi_i^B \) (chosen so that at each real place the sum of the weights is zero), and

\[
C_v = \begin{cases} 
1 & \text{if } v \nmid \infty \mathfrak{N} \\
\frac{1}{q_v} (1 - \frac{1}{q_v}) & \text{if } v \mid \mathfrak{d}_B, \ v \nmid \infty \\
\frac{1}{q_v} (1 + \frac{1}{q_v}) & \text{if } v \mid \mathfrak{N}, \ v \nmid \infty \\
2 & \text{if } v \mid \infty.
\end{cases}
\]

The formula as stated in [29] appears slightly different only because it is presented in the language of classical modular forms. Note that this confirms Prasad’s theorem in this special case.

1.2. Application to subconvexity. From henceforth we will assume without loss of generality that \( \varepsilon(\frac{1}{2}, \Pi) = 1 \). Under this assumption, Theorems 1.1 and 1.3 imply the following.

**Corollary 1.5.** Let \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \) where \( \pi_i \) are cuspidal automorphic representations of \( \text{PGL}_2 \) over \( F \) with \( \pi_1, \pi_2 \) fixed and \( \pi_3 \) has prime conductor \( p \). For all \( \pi_3 \) such that \( \Sigma(\Pi) = \emptyset \), one has the following subconvexity bound.

\[
L\left(\frac{1}{2}, \Pi\right) \ll_{\pi_3, \infty} N(p)^{1 - \frac{1}{2}}
\]

We describe the limitations of this corollary. We first remark that if \( v \) is a nonarchimedean place then \( \varepsilon_v(\frac{1}{2}, \Pi_v) = +1 \) whenever any one of the representations is unramified. In particular, this implies that \( \varepsilon_v(\frac{1}{2}, \pi_v) = +1 \) for all but finitely many \( v \). Analogously, if \( v \) is archimedean then \( \varepsilon_v(\frac{1}{2}, \Pi_v) = +1 \) if any one of \( \pi_i,v \) is not a discrete series. If all three are discrete series of weights \( k, l \) and \( m \) where \( k \) is the largest weight then \( \varepsilon(\frac{1}{2}, \Pi_v) = -1 \) if and only if \( k < l + m \). If this is the case, we say that \( \Pi_v \) is **balanced**. Otherwise, we say \( \Pi_v \) is **unbalanced**.

Because Theorem 1.1 requires that \( n_1 n_2 \) be relatively prime to the conductor of \( \pi_3 \), which is required to be a prime \( p \), we see that in this case the quaternion algebra \( B \) for which (1.3) is satisfied ramifies exactly at real primes \( v \) such that \( \pi_{1,v}, \pi_{2,v} \) and \( \pi_{3,v} \) are discrete series and for which \( \Pi_v \) is balanced. In other words, the only restriction on \( \Pi \) is that at each real place \( v \) either

- at least one of the representations is a principal series, or
- if all three representations are discrete series, the weights are unbalanced.

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If Theorem 1.1 could be generalized to arbitrary quaternion algebras, our Theorem 1.3, would be enough to make Corollary 1.5 unconditional.

**Remark.** Michel and Venkatesh in [23] give a triple product formula when one of the representations is an Eisenstein series. It is possible that this formula could be adapted to the case that all three are cuspidal, and that it too would have applications to subconvexity.

Theorems 1.3 and 1.4 are obtained via the formula of Ichino and the explicit calculation of certain local trilinear forms. The nonvanishing of the said forms was already known by the work of Gross-Prasad [12], but our contribution is the exact evaluation of each. We discuss Ichino’s result in Section 2. The local forms are constructed first via matrix coefficients, which are computed for particular choices of vectors in Section 3, and then finally we combine these results in Section 4 to evaluate the forms. Theorem 1.4 and further generalizations are discussed in Section 4.4. In the final section we complete the proof of Theorem 1.3.
2. Global trilinear form

In this section, $F$ is a number field with ring of adeles $\mathcal{A} = \mathbb{A}_F$, $v$ a place of $F$ and $F_v$ the corresponding completion. Let $G = \text{GL}_2$ with center $Z$ and $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where each $\pi_i$ is a unitary cuspidal automorphic representation of $G$ over $F$. If $\omega_i$ is the central character of $\pi_i$ then we require that $\omega_1 \omega_2 \omega_3$ be trivial\footnote{In the sequel we will make the stronger assumption that each $\omega_i$ is trivial.}. We let $L(s, \Pi)$ denote the triple product $L$-function, $L(s, \Pi, \text{Ad}) = \prod_{i=1}^{3} L(s, \pi_i, \text{Ad})$ with $L(s, \pi_i, \text{Ad})$ the adjoint $L$-function attached to $\pi_i$ and $\zeta_F(s)$ the zeta function of the number field $F$. See Section 4.1 for precise definitions. We use the subscript $v$ to represent the analogous local $L$ and zeta functions.

2.1. Ichino’s formula. The central identity which we will use to relate the period on the right side of (1.3) to the $L$-value on the left is due to Ichino\cite{15} and arises when studying the space $\text{Hom}_{B^x(\mathcal{A})}(\Pi B, \mathbb{C})$ of $B^x(\mathcal{A})$-invariant linear forms where $B^x$ is embedded diagonally into $B^x \times B^x \times B^x$. These are the so-called trilinear forms. Theorem 1.2 tells us that this space is at most 1-dimensional.

It is more convenient to work with bilinear forms:

\begin{equation}
(2.1) \quad \text{Hom}_{B^x(\mathcal{A}) \times B^x(\mathcal{A})}(\Pi B \times \tilde{\Pi} B, \mathbb{C})
\end{equation}

This space is again at most 1-dimensional, and there is an obvious choice of invariant form:

\begin{equation}
(2.2) \quad I(\varphi \otimes \tilde{\varphi}) = \int_{[B^x]} \varphi_1 \varphi_2 \varphi_3(b) db \int_{[B^x]} \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3(b) db.
\end{equation}

This uniqueness follows from the local fact that $\dim \text{Hom}_{B^x(\mathcal{A})}(\Pi B \otimes \tilde{\Pi} B, \mathbb{C}) \leq 1$. In the local case there is also an obvious choice of $B^x(F_v)$-invariant form. Let

$\langle \cdot, \cdot \rangle_v : \pi'_i, v \times \pi'_i, v \rightarrow \mathbb{C}$

be the canonical $B^x(F_v)$-invariant pairings. For $\varphi_v = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v} \in \Pi^B_v$ and $\tilde{\varphi}_v \in \tilde{\Pi}^B_v$ defined similarly, this gives a matrix coefficient

$\langle \Pi^B_v(g_v) \varphi_v, \tilde{\varphi}_v \rangle_v = \langle \pi^B_{1,v}(g_v) \varphi_{1,v}, \tilde{\varphi}_{1,v} \rangle_v \langle \pi^B_{2,v}(g_v) \varphi_{2,v}, \tilde{\varphi}_{2,v} \rangle_v \langle \pi^B_{3,v}(g_v) \varphi_{3,v}, \tilde{\varphi}_{3,v} \rangle_v$

such that

$I'_v(\varphi_v \otimes \tilde{\varphi}_v) = \int_{F_v \setminus B^x(F_v)} \langle \Pi^B_v(g_v) \varphi_v, \tilde{\varphi}_v \rangle_v dg_v$

is $B^x(F_v)$-invariant. The product of these local forms

\begin{equation}
(2.3) \quad \prod_v I_v(\varphi_v \otimes \tilde{\varphi}_v) \quad \text{where} \quad I_v(\varphi_v \otimes \tilde{\varphi}_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \text{Ad})}{L_v(\frac{1}{2}, \Pi_v)} I'_v(\varphi_v \otimes \tilde{\varphi}_v)
\end{equation}

is well defined because the normalization is chosen precisely so that for all but finitely many $v$, $I_v = 1$. This product, like (2.2), is an element of (2.1), and since the space of such forms is at most 1-dimensional, they must differ by a constant. Ichino calculates this constant.

**Theorem 2.1** (Ichino). Let $I$ and $I_v$ be defined as above. Then

\begin{equation}
(2.4) \quad \frac{I(\varphi \otimes \tilde{\varphi})}{\prod_{j=1}^{3} \int_{[B^x]} \varphi_j(b) \tilde{\varphi}_j(b) db} = C^{\frac{1}{23}} \cdot \zeta_F(2)^2 \cdot \frac{L(\frac{1}{2}, \Pi)}{L(1, \Pi, \text{Ad})} \prod_v I_v(\varphi_v \otimes \tilde{\varphi}_v) \quad \langle \varphi_v, \tilde{\varphi}_v \rangle_v
\end{equation}
whenever the denominators are nonzero for a constant $C$ which depends only on the choice of Haar measure. (For the choice given in Section 2.2 below, $C = \zeta_F(2)^{-1}$.)

Note that in the case that the central characters are trivial, $\Pi$ is self dual. Using the composition of the injection $\Pi \hookrightarrow \Pi \times \Pi$ given by $\varphi \mapsto (\varphi, \overline{\varphi})$ with (2.2) yields a quadratic form, namely

$$I(\varphi) = I(\varphi \otimes \overline{\varphi}) = \left| \int_{[G]} \varphi_1(g) \varphi_2(g) \varphi_3(g) dg \right|^2,$$

such that $I(\varphi) = |J(\varphi)|^2$ in the notation of Theorem 1.1. Hence Ichino's result provides us with the necessary tools to prove Theorem 1.3. Indeed, we will derive exact formulas for $I_v(\varphi) := I_v(\varphi \otimes \overline{\varphi})$ from which sufficient lower bounds will be obtained. (See, for example, Corollary 4.2.)

As described in the introduction, the functional equation implies that $L(\frac{1}{2}, \Pi)$ must be zero unless $\varepsilon(\frac{1}{2}, \Pi) = 1$. Thus Theorem 1.2, together with (2.4), implies one direction of the fact conjectured by Jacquet and proved by Harris and Kudla in [13] and [14]. We record their result here as it will be used in Section 5.

**Theorem 2.2** (Harris-Kudla). The central value $L(\frac{1}{2}, \Pi) \neq 0$ if and only if there exists some $B$ and some $\varphi \in \Pi^B$ such that $\int_{[B^*]} \varphi(b) db \neq 0$.

By Theorem 1.2, the quaternion algebra $B$ is that for which $\Sigma_B = \Sigma(\Pi)$.

2.2. Measures. Let $F_v$ be a $p$-adic field with $O_v$ and $q$ as above, and let $B_v$ be a quaternion algebra over $F_v$. If $B_v^\times = \text{GL}_2(F_v)$, we choose the (multiplicative) Haar measure $db_v$ (or $dg_v$) to be that for which the maximal compact subgroup $K_v = \text{GL}_2(O_v)$ has volume 1. We abuse notation and also denote by $db_v$ (or by $dg_v$) the measure on $\text{PGL}_2(F_v)$ consistent with the choice above and the exact sequence

$$1 \to F_v^\times \to \text{GL}_2(F_v) \to \text{PGL}_2(F_v) \to 1.$$  

This means that the image of $K$ in $\text{PGL}_2(F_v)$ again has volume 1.

If $B_v$ is division then it contains a unique maximal order $R_v$. We denote by $db_v$ the Haar measure on $B_v^\times$ for which $R_v^\times$ has measure $(q - 1)^{-1}$. Again, we write $db_v$ for the measure on $PB_v^\times$ compatible with the analogous exact sequence to (2.6). We remark that $R_v^\times$ has index 2 in $F_v^\times \setminus B_v^\times$, and so $\text{vol}(F_v^\times \setminus B_v^\times) = \frac{2}{q-1}$.

In the real case, let $dx$ be the standard measure on $\mathbb{R}$ such that the volume of $[0, 1] = 1$. Define the subgroups

$$N_\infty = \{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \}, \quad A_\infty = \{ a(y) = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^\times \}, \quad K_\infty = \{ \kappa_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi) \}$$

of $\text{GL}_2(\mathbb{R})$ or $\text{PGL}_2(\mathbb{R})$. Then we define the Haar measure via

$$\frac{1}{2\pi} \frac{dydx}{|y|^2} d\theta, \quad \frac{1}{4\pi} (1 - y^{-2}) d\theta_1 dy d\theta_2$$

in the $N_\infty A_\infty K_\infty$ and $K_\infty A_\infty K_\infty$ coordinate systems respectively. Note that in $K_\infty A_\infty K_\infty$ coordinates we take $a(y)$ only for $|y| \geq 1$. 6
Although at first glance these choices may seem arbitrary, they are in fact consistent if one defines them in terms of the additive measure. Moreover, for these choices of local measures\textsuperscript{2}, if $B$ is a global quaternion algebra over a number field $F$ then $db = \zeta_F(2)^{-1} \prod_v db_v$ is the Tamagawa measure on $A_F^\times \backslash B^\times (A_F)$.

3. Matrix coefficients: nonarchimedean places

We fix the following notation for this section. Let $F$ be a $p$-adic field with ring of integers $\mathcal{O}_F$ and uniformizer $\varpi$. Let $q$ be the order of the residue field $\mathcal{O}_F/(\varpi)$. Let $v = \text{ord}_\varpi$ be the additive valuation, and define $|x| = q^{-v(x)}$. Let

$$G = \text{GL}_2(F), \quad K = \text{GL}_2(\mathcal{O}_F), \quad Z = \{(z, z) \in G\},$$

$$P = NA \quad \text{where} \quad N = \{(1, b) \in G\}, \quad A = \{(a_1 a_2) \in G\}.$$ 

Let $\sigma_n = (\varpi^n 1), \quad e = (1 1), \quad w = (1 1)$.

We assume that $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ are unramified characters, meaning that $\chi_i|_{\mathcal{O}_F^\times} = 1$ for $i = 1, 2$. The character $\chi_1 \otimes \chi_2$ gives a representation of $P$. We consider $\pi = \pi(\chi_1, \chi_2)$ which is the (admissible) unitary induction from $P$ to $G$ of this representation. Hence $\pi$ is the right regular action of $G$ on the space of functions $f : G \to \mathbb{C}$ (i.e. $[\pi(g)f](h) = f(hg)$) such that

$$(3.1) \quad f((a_1 b a_2)g) = \chi_1(a_1)\chi_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2}f(g)$$

for all $g \in G$, and there exists a compact open subgroup $L \subset G$ which acts trivially on $f$. We will follow the common abuse of notation in using $\pi$ to denote both the space and the action. Note that $\pi$ need not be irreducible.

The contragradient $\tilde{\pi}$ of $\pi$ is equal to $\pi(\chi_1^{-1}, \chi_2^{-1})$. Let $dk$ be the Haar measure on $G$ for which $K$ has volume one. Then the pairing

$$(3.2) \quad \langle \cdot, \cdot \rangle : \pi \times \tilde{\pi} \to \mathbb{C} \quad \langle f, \tilde{f} \rangle = \int_K f(k)\tilde{f}(k)dk$$

is bilinear and $G$-invariant. (See [1], [5] for details.) Using this one defines the matrix coefficient associated to $f, \tilde{f}$:

$$\Phi_{f, \tilde{f}}(g) = \langle \pi(g)f, \tilde{f} \rangle.$$ 

Note that in this section we do not necessarily assume that $\pi$ is unitary.

We will compute explicitly these matrix coefficients for particular choices of $f$ and $\tilde{f}$—namely for those vectors that are fixed by

$$K_0 := K_0(\varpi) = \{(a b c) \in K \mid v(c) \geq 1\}.$$ 

This is a two dimensional subspace. To perform this calculation, we derive general formulas in Section 3.1 and then apply these results to the unramified and special representations in Sections 3.2 and 3.3 respectively.

First, we record some standard results regarding the decomposition of $G$ and volumes of certain subsets. Recall that our measure on $G$ (or on $Z/G$) is that for which $K$ (or its image) has volume 1.

\textsuperscript{2}If $F$ has complex places or if $B_v$ is definite we could define consistent measures as well, but since the explicit archimedean calculations of this paper involve only the real/indefinite case this is not necessary.
Lemma 3.1. Let $\Omega_0 = K$ and $\Omega_n = K\sigma_nK$ for $n \geq 1$. Then $Z \backslash G = \bigcup_{n \geq 0} \Omega_n$ and for $n \geq 1$,
\[ K\sigma_nK = K_0\sigma_nK_0 \bigcup K_0w\sigma_nK_0 \bigcup K_0\sigma_nwK_0 \bigcup K_0w\sigma_nwK_0. \]
Moreover, all of these unions are disjoint.

Let $X_n = \{(a \ b \ c \ d) \in K_0 \mid v(c) = n\}$ and $Y_n = \{(a \ b \ c \ d) \in K_0 \mid v(b) = n\}$. The following table is valid.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$K_0$</th>
<th>$K_0w\sigma_nK_0$</th>
<th>$K_0\sigma_nwK_0$</th>
<th>$K_0\sigma_nK$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{vol}(X)$</td>
<td>$(1 + q)^{-1}$</td>
<td>$q^n(1 + \frac{1}{q})$</td>
<td>$q^{n-1}$</td>
<td>$\frac{q^n-1}{1+\frac{1}{q}}$</td>
</tr>
<tr>
<td>$X_n$</td>
<td>$\frac{q^n-2}{1+\frac{1}{q}}$</td>
<td>$\frac{q^n}{1+\frac{1}{q}}$</td>
<td>$\frac{q^{n-1}}{1+\frac{1}{q}}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Volumes of various subsets of $GL_2(F)$, $F$ a $p$-adic field

3.1. Matrix coefficient associated to Iwahori fixed vectors. Recall that $\pi = \pi(\chi_1, \chi_2)$ is the induced representation defined in (3.1). The Iwasawa decomposition $G = NAK$ implies that a vector $f \in \pi$ is uniquely determined by its restriction to $K$. With this in mind we define $f_0$ and $f_1$ to be the vectors whose restrictions to $K$ are the characteristic function on $K_0$ and $K_0wK_0$ respectively. Let $\tilde{f}_0$ and $\tilde{f}_1$ be the analogously defined vectors in $\tilde{\pi}$.

We call the space $\pi^{K_0}$ of vectors fixed by $K_0$ the space of Iwahori fixed vectors. Note that every such vector is a linear combination of $f_0$ and $f_1$ because $K = K_0 \cup K_0wK_0$ and $K_0wK_0 = N_KwK_0$. Moreover, if $f \in \pi^{K_0}$ then $f = af_0 + bf_1$ where $a = f(e)$ and $b = f(w)$.

Using the coset decompositions of Lemma 3.1 it is immediate that
\[
(3.3) \quad \Phi_{i,j}(g) := \Phi_{f_i\tilde{f}_j}(g) = \begin{cases} 
\int_{K_0} f_i(kg)dk & \text{if } j = 0 \\
q\int_{K_0} f_i(kwg) & \text{if } j = 1.
\end{cases}
\]

Our goal is to determine $\Phi_{i,j}(g)$ for all $g \in G$. By the $G$-invariance of the inner form if $f \in \pi^{K_0}$ and $\tilde{f} \in \pi^{K_0}$ and $k \in K_0$ then
\[
\Phi_{f,\tilde{f}}(kg) = \langle \pi(kg)f, \tilde{f} \rangle = \langle \pi(g)f, \pi^{-1}(k)\tilde{f} \rangle = \Phi_{f,\tilde{f}}(g) \quad \text{and}
\]
\[
\Phi_{f,\tilde{f}}(gk) = \langle \pi(g)\pi(k)f, \tilde{f} \rangle = \Phi(g).
\]

Hence $\Phi_{i,j}(g)$ depends only on the double coset $K_0gK_0$. So by Lemma 3.1 we need to perform the calculation for each of only four $K_0$-double coset representatives.

Proposition 3.2. The values of $\Phi_{i,j}$ are given by $2^{-n/2}$ times the values in the following table. (Note that the values for $\Phi_{i,j}(w\sigma_n)$ hold only when $n > 0$. Otherwise, the table is valid for all $n \geq 0$.)

8
where (3.4) depend only on the equivalence class of $g$. Since the Iwahori fixed vectors are, by definition, invariant by $K_0$, $f_i(g)$ depends only on the equivalence class of $g$.

**Proof.** We prove the formula in the case that $g \in K_0\sigma_nK_0$. Let $\alpha_i = \chi_i(\varpi)$.

Let $g, g' \in G$. We define an equivalence relation such that $g \sim g'$ if there exists $k \in K_0$ such that $g = g'k$. Since the Iwahori fixed vectors are, by definition, invariant by $K_0$, $f_i(g)$ depends only on the equivalence class of $g$. Realizing that $K_0$ acts on $G$ by column operations it is easy to see that for $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in K_0$,

$$
(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \sigma_n = (\begin{smallmatrix} a\varpi^n & b \\ c\varpi^n & d \end{smallmatrix}) \sim (\begin{smallmatrix} \varpi^n & b' \\ 0 & 1 \end{smallmatrix})
$$

where $b$ and $b'$ have the same valuation. Applying this to (3.3) (for $j = 0$) yields

$$
\Phi_{0,0}(\sigma_n) = \int_{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in K_0} f_0((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \sigma_n) dk = \int_{K_0} f_0((\begin{smallmatrix} \varpi^n & b \\ 0 & 1 \end{smallmatrix})) dk = vol(K_0) \alpha_1^n q^{-n} = \frac{q^{-n/2}}{1 + \frac{1}{q} \alpha_1^n} \cdot \frac{1}{q},
$$

and

$$
\Phi_{1,0}(\sigma_n) = \int_{K_0} f_1(k\sigma_n) dk = \int_{K_0} f_1((\begin{smallmatrix} \varpi^n & b \\ 0 & 1 \end{smallmatrix})) dk = 0.
$$

The equivalence class of an element of $G$ of the form $w \cdot (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \sigma_n$ with $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in K_0$. Using the relation $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix})$, it is not hard to show that

$$
w \cdot (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \sigma_n \sim \begin{cases} w\sigma_n w, & v(b) \geq n \\
(\begin{smallmatrix} 0 & 1 \\ b & \varpi^n \end{smallmatrix}) & \in \left(\begin{smallmatrix} 1 \\ \varpi^n \end{smallmatrix}\right) w K_0 \\
(\begin{smallmatrix} \varpi^{-m} & \varpi \\ b \varpi^{-m} \end{smallmatrix}) & \in \left(\begin{smallmatrix} \varpi^{-m} & \varpi \\ 0 \varpi^{-m} \end{smallmatrix}\right) K_0 \\
m = v(b) < n.
\end{cases}
$$

\[\text{In the calculations below we will abuse notation by writing } b \text{ instead of } b' \text{ which differs from } b \text{ by a unit (and similarly for } c, c'). \text{ Since our character } \chi \text{ is unramified, this is harmless.}\]
Hence, (using (3.3) in the case \( j = 1 \)) we have

\[
\Phi_{0,1}(\sigma_n) = q \int_{K_0} f_0(wk\sigma_n)dk
\]

\[
= q \sum_{m=0}^{n-1} \text{vol}(Y_m)f_0( (w^{n-m}w^m ) )
\]

\[
= \frac{1 - \frac{1}{q}}{1 + \frac{1}{q}} \sum_{m=0}^{n-1} q^{-m} \alpha_1^{-m}\alpha_2^m q^{m-n/2}
\]

\[
= \frac{q^{-n/2}}{1 + \frac{1}{q}} (1 - \frac{1}{q}) \alpha_1^n \sum_{m=0}^{n-1} (\alpha_1^{-1}\alpha_2)^m = \frac{q^{-n/2}}{1 + \frac{1}{q}} \frac{\alpha_1^n - \alpha_2^n}{1 - \alpha_1^{-1}\alpha_2} (1 - \frac{1}{q}),
\]

and

\[
\Phi_{1,1}(\sigma_n) = q \int_{K_0} f_1(wk\sigma_n)dk
\]

\[
= q \sum_{m=n}^{\infty} \text{vol}(Y_m)f_1( (w^1w^n ) ) w
\]

\[
= \frac{1 - \frac{1}{q}}{1 + \frac{1}{q}} \sum_{m=n}^{\infty} \alpha_2^m q^{-m} q^{n/2} = \frac{q^{-n/2}}{1 + \frac{1}{q}} \alpha_2^n.
\]

These results give the values in the first column of Table 2. The calculations for the other columns are similar. The only nuances are that the decompositions analogous to (3.4) and (3.5) may depend instead on \( v(c) \) and take a slightly different shape. We leave the details to the reader. \( \square \)

3.2. Application to unramified representations. Strictly speaking, the calculation of the local trilinear form could be carried out at this point using the results of Proposition 3.2 with respect to the basis \( \{ f_0, f_1 \} \) of Iwahori fixed vectors. However, it turns out that the calculations are drastically simplified by using a different basis. In this section we describe this basis and the corresponding matrix coefficients in the case that \( \chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1} \). This implies that \( \pi = \pi(\chi_1, \chi_2) \) is irreducible. It is called an unramified principal series.

The vector \( \phi_0 = f_0 + f_1 \), which we call the normalized new vector, is obviously fixed by \( K \) and, in fact, \( \pi^K = \mathbb{C}\phi_0 \). Note that \( \phi_0(e) = 1 \). Using bilinearity, \( \Phi_{\phi_0,\phi_0} = \sum_{i,j} \Phi_{i,j} \). It is easy to see that \( \Phi_{\phi_0,\phi_0} \) is \( K \)-biinvariant, so using only the first (or any other) column of Table 2 above, we obtain the well-known formula of Macdonald.

**Proposition 3.3.** Let \( \pi = \pi(\chi_1, \chi_2) \) be an unramified admissable representation of \( \text{GL}_2(F) \). If \( \phi_0, \tilde{\phi}_0 \) are the normalized new vectors of \( \pi \) and \( \tilde{\pi} \) respectively then the function \( \Phi_{\phi_0,\tilde{\phi}_0} \) is \( K \)-biinvariant and

\[
\Phi_{\phi_0,\tilde{\phi}_0}(\sigma_n) = \frac{q^{-n/2}}{1 + \frac{1}{q}} \left( \alpha_1^n \frac{1 - \alpha_1^{-1}\alpha_2}{q} + \alpha_2^n \frac{1 - \alpha_1\alpha_2^{-1}}{1 - \alpha_1^{-1}\alpha_2} \right).
\]

for \( n \geq 0 \) and \( \alpha_i = \chi_i(\varpi) \).
Recall that $\pi = \pi(\chi_1, \chi_2)$. For $\pi' = \pi(\chi_2, \chi_1)$, there exists an intertwining operator

(3.6) \hspace{1cm} M : \pi \rightarrow \pi'.

The map $M$ is given by the formula

(3.7) \hspace{1cm} (Mf)(g) = \int_F f(w(1,x))g dx

whenever the integral converges. It is defined elsewhere by analytic continuation.

**Lemma 3.4.** Let $\pi = \pi(\chi_1, \chi_2)$, and $\pi' = \pi(\chi_2, \chi_1)$. Let $\phi_0$ (respectively $\phi'_0$) be the normalized new vector of $\pi$ (resp. $\pi'$) as above. Let $\phi_1 = f_0 - \frac{1}{q} f_1$, and let $\phi'_1$ be the similarly defined vector in $\pi'_i$. Then

(3.8) \hspace{1cm} M\phi_0 = \frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} \phi'_0 \quad \text{and} \quad M\phi_1 = -\frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} \phi'_1.

Given this result we abuse notation and call $\phi_0, \phi_1$ eigenvectors of the $M$.

**Proof.** As was remarked at the beginning of Section 3.1, if $f \in V$ then $Mf = af_0 + bf_1$ where $a = Mf(e)$ and $b = Mf(w)$. Using the identity $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} -x^{-1} & 1 \\ 1 & 0 \end{array} \right)$ one deduces that

(3.9) \hspace{1cm} f_0(w(0,1,x)) = \begin{cases} 0 & \text{if } x \in \mathcal{O}_F \\
 \chi_1^{-1}(-x)\chi_2(x) |x|^{-1} & \text{if } x \notin \mathcal{O}_F, \end{cases}

and

(3.10) \hspace{1cm} f_0(w(1,x)) = \begin{cases} 1 & \text{if } x \in \varpi \mathcal{O}_F \\
 0 & \text{if } x \notin \varpi \mathcal{O}_F. \end{cases}

Therefore, by combining (3.7) and (3.9) we have

\[
Mf_0(e) = \int_F f_0(w(0,1,x)) = \sum_{n=-1}^{\infty} \int_{\varpi^n \mathcal{O}_F} f_0(w(1,x)) dx \\
= \sum_{n=-1}^{\infty} \text{vol}(\varpi^n \mathcal{O}_F)(\alpha_1^{-1} \alpha_2)^n \\
= (1 - \frac{1}{q}) \sum_{n=1}^{\infty} (\alpha_1^{-1} \alpha_2)^n = (1 - \frac{1}{q}) \frac{\alpha_1 \alpha_2^{-1}}{1 - \alpha_1 \alpha_2}.
\]

Similarly, (3.7) and (3.10) combine to give $Mf_0(w) = \text{vol}(\varpi \mathcal{O}_F) = \frac{1}{q}$. So

(3.11) \hspace{1cm} Mf_0 = (1 - \frac{1}{q}) \frac{\alpha_1 \alpha_2^{-1}}{1 - \alpha_1 \alpha_2} f_0 + \frac{1}{q} f'_1.

\footnote{In all of these calculations, we see that the integral converges if and only if $|\alpha_1 \alpha_2^{-1}| < 1$. However, via analytic continuation, the results are the same for all $|\chi_1 \chi_2^{-1}| \neq 1$.}
A similar calculation for $f_1$ shows that

$$Mf_1 = f_0' + \frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} f_1' .$$

Using the linearity of $M$, the result follows. \qed

We have now completed all of the computations necessary to prove the following extension of Proposition 3.3 using the basis $\{\phi_0, \phi_1\}$ of eigenvectors of $M$.

**Proposition 3.5.** Let $\pi = \pi(\chi_1, \chi_2)$ such that $\chi_1 \chi_2^{-1} \neq \pm |\cdot|$. Let $\phi_0, \phi_1$ be the eigenvectors in the sense of Lemma 3.4, and $\phi_0, \phi_1$ the analogous vectors in $\tilde{\pi} = \pi(\chi_1^{-1}, \chi_2^{-1})$. The values of the matrix coefficients $\Psi_{i,j}(g) = \Phi_{\phi_i, \phi_j}(g)$ are $\frac{q^{-n/2}}{1 + \frac{1}{q}}$ times the values given by Table 3 in which $A = \frac{1 - \alpha_1 \alpha_2}{1 - \alpha_1 \alpha_2}$ is the eigenvalue of $\phi_0$ and $-B$ is the eigenvalue of $\phi_1$.

<table>
<thead>
<tr>
<th>$(i,j)$</th>
<th>$\Phi_{\phi_i, \phi_j}(\sigma_n)$</th>
<th>$\Phi_{\phi_i, \phi_j}(w\sigma_n)$</th>
<th>$\Phi_{\phi_i, \phi_j}(\sigma_n w)$</th>
<th>$\Phi_{\phi_i, \phi_j}(w\sigma_n w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,1)$</td>
<td>$\frac{1}{q} (\alpha_1^n - \alpha_2^n) A$</td>
<td>$- (\alpha_1^n - \alpha_2^n) A$</td>
<td>$\frac{1}{q} (\alpha_1^n - \alpha_2^n) A$</td>
<td>$-(\alpha_1^n - \alpha_2^n) A$</td>
</tr>
<tr>
<td>$(1,0)$</td>
<td>$(\alpha_1^n - \alpha_2^n) B$</td>
<td>$(\alpha_1^n - \alpha_2^n) B$</td>
<td>$- \frac{1}{q} (\alpha_1^n - \alpha_2^n) B$</td>
<td>$- \frac{1}{q} (\alpha_1^n - \alpha_2^n) B$</td>
</tr>
<tr>
<td>$(0,1)$</td>
<td>$\frac{1}{q} (\alpha_1^n A + \alpha_2^n B)$</td>
<td>$- (\alpha_1^n A + \alpha_2^n B)$</td>
<td>$- \frac{1}{q} (\alpha_1^n A + \alpha_2^n B)$</td>
<td>$\frac{1}{q} (\alpha_1^n A + \alpha_2^n B)$</td>
</tr>
<tr>
<td>$(0,0)$</td>
<td>$\alpha_1^n B + \alpha_2^n A$</td>
<td>$\alpha_1^n B + \alpha_2^n A$</td>
<td>$\alpha_1^n B + \alpha_2^n A$</td>
<td>$\alpha_1^n B + \alpha_2^n A$</td>
</tr>
</tbody>
</table>

**Table 3.** Values of matrix coefficients of Iwahori fixed vectors $\Pi$

**Proof.** In the case of $\Psi_{1,0} = \Phi_{\phi_1, \phi_0}$, we see that

$$\Psi_{1,0}(kg) = \langle \pi(kg) \phi_1, \tilde{\phi}_0 \rangle = \langle \pi(g) \phi_1, \tilde{\pi}(k^{-1}) \tilde{\phi}_0 \rangle = \langle \pi(g) \phi_1, \tilde{\phi}_0 \rangle = \Psi_{1,0}(g)$$

for all $k \in K$. So $\Phi(\sigma_n) = \Phi(w\sigma_n)$ and $\Phi(w\sigma_n w) = \Phi(\sigma_n w)$. Now we use the bilinearity of matrix coefficients to write

$$\Phi_{\phi_1, \phi_0} = \Phi_{0,0} + \Phi_{0,1} - \frac{1}{q} (\Phi_{1,0} + \Phi_{1,1}),$$

and calculate the value on $\sigma_n$ using Table 2:

$$\Psi_{1,0}(\sigma_n) = \Phi_{0,0}(\sigma_n) + \Phi_{0,1}(\sigma_n) - \frac{1}{q} (\Phi_{1,0}(\sigma_n) + \Phi_{1,1}(\sigma_n))$$

$$= \alpha_1^n \cdot \frac{1}{q} + \frac{\alpha_1^n - \alpha_2^n}{1 - \alpha_1^{-1} \alpha_2} (1 - \frac{1}{q}) - \frac{1}{q} (0 + \alpha_2^n)$$

$$= (\alpha_1^n - \alpha_2^n) \left( \frac{1}{q} + \frac{1 - \frac{1}{q}}{1 - \alpha_1^{-1} \alpha_2} \right)$$

$$= (\alpha_1^n - \alpha_2^n) \left( \frac{\frac{q}{q} - \alpha_1^{-1} \alpha_2}{1 - \alpha_1^{-1} \alpha_2} \right) = (\alpha_1^n - \alpha_2^n) B.$$
For $w\sigma_n w$:

$$
\Psi_{1,0}(w\sigma_n w) = \Phi_{0,0}(w\sigma_n w) + \Phi_{0,1}(w\sigma_n w) - \frac{1}{q}(\Phi_{1,0}(w\sigma_n w) + \Phi_{1,1}(w\sigma_n w))
$$

$$
= \alpha_2^n \cdot \frac{1}{q} + 0 - \frac{1}{q} \left( \frac{\alpha_2^n - \alpha_1^n}{1 - \alpha_1 \alpha_2} (1 - \frac{1}{q}) + \alpha_1^n \right)
$$

$$
= -\frac{1}{q} (\alpha_2^n - \alpha_1^n) \left( 1 - \frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} \right) = -\frac{1}{q} (\alpha_2^n - \alpha_1^n) B.
$$

The computation of $\Psi_{1,1}(\sigma_n) = \Phi_{\phi_1,\phi_1}(\sigma_n)$ is similar:

$$
\Psi_{1,1}(\sigma_n) = \Phi_{0,0}(\sigma_n) - \frac{1}{q}(\Phi_{0,1}(\sigma_n) + \Phi_{1,0}(\sigma_n)) = \frac{1}{q^2} \Phi_{1,1}(\sigma_n)
$$

$$
= \frac{1}{q} \alpha_1^n - \frac{1}{q} \left( \frac{\alpha_2^n - \alpha_1^n}{1 - \alpha_1 \alpha_2} + \frac{1}{q} \alpha_2^n \right)
$$

$$
= \frac{1}{q} \alpha_1^n \left( 1 - \frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} \right) + \alpha_2^n \left( \frac{1}{q} + \frac{1 - \frac{1}{q}}{1 - \alpha_1 \alpha_2} \right)
$$

$$
= \frac{1}{q} (\alpha_1^n A + \alpha_2^n B).
$$

The formulas for $\Psi_{1,1}(w\sigma_n)$, $\Psi_{1,1}(\sigma_n w)$, $\Psi_{1,1}(w\sigma_n w)$ and for $\Phi_{\phi_0,\phi_1}$ are derived in the same fashion. The final row is just a restatement of Proposition 3.3.

In order to prove Theorems 1.3 and 4.7 we will need to know something about the matrix coefficient $\Psi_{2,2} = \Phi_{\phi_2,\phi_2}$ where $\phi_2 = (\varpi^{-1}) \phi_0$. (In particular, see Corollary 1.5.) By way of the following lemma and a messy calculation, one could give a formula for $\Psi_{2,2}$ similar to those of Proposition 3.5.

**Lemma 3.6.** Let $\pi = \pi(\chi, \chi^{-1})$, and $\alpha = \chi(\varpi)$. Let $\phi_2 = (\varpi^{-1}) \phi_0 \in \pi$ where $\phi_0$ is as above. Then $\phi_2 \in \pi K_0$. More precisely,

$$
\phi_2 = \frac{1}{1 + \frac{1}{q}} \left( (\alpha + \alpha^{-1}) q^{-1/2} \phi_0 + (\alpha^{-1} q^{1/2} - \alpha q^{-1/2}) \phi_1 \right)
$$

**Proof.** Let $k = (a \ b \ c \ d) \in K_0$. Then

$$
\phi_2(gk) = \phi_0(g \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left( \varpi^{-1} 1 \right)) = \phi_0(g \left( \begin{smallmatrix} a \varpi^{-1} & b \\ c \varpi^{-1} & d \end{smallmatrix} \right))
$$

$$
\phi_0(g \left( \begin{smallmatrix} \varpi^{-1} & 0 \\ 0 & \varpi^{-1} \end{smallmatrix} \right)) = \pi \left( \begin{smallmatrix} \varpi^{-1} & 0 \\ 0 & \varpi^{-1} \end{smallmatrix} \right) \phi_{p,0}(g) = \phi_{p,2}(g)
$$

since $\left( \begin{smallmatrix} a \ b \\ c \varpi^{-1} & d \end{smallmatrix} \right) \in K$ and $\phi_{p,0}$ is fixed by $K$.

So $\phi_2 = \phi_2(e)f_0 + \phi_2(w)f_1$. Writing this in terms of the basis $\{\phi_0, \phi_1\}$ and simplifying leads to the desired result. \hfill \Box

### 3.3. Application to the special representations.

If $\chi_1 \chi_2^{-1} = |\cdot|^{\pm 1}$ then the resulting induced representation is reducible. Indeed, writing

$$
\pi = \pi(\chi |\cdot|^{1/2}, \chi |\cdot|^{-1/2}) \quad \text{and} \quad \tilde{\pi} = \pi(\chi^{-1} |\cdot|^{-1/2}, \chi^{-1} |\cdot|^{1/2}),
$$

this is reflected in the exact sequences

$$
\begin{align*}
(3.12) \quad & 0 \longrightarrow \sigma_\chi \longrightarrow \pi \longrightarrow \mathbb{C}_\chi \longrightarrow 0, \\
(3.13) \quad & 0 \longrightarrow \mathbb{C}_\chi^{-1} \longrightarrow \tilde{\pi} \longrightarrow \sigma_{\chi^{-1}} \longrightarrow 0
\end{align*}
$$
for any given these models of true. Namely if (3.15) as a corollary to Lemma 3.7, we have that Lemma 3.7.

Result.

In order to denote elements of it is better to consider (3.14).

Note that (3.13). That is to say, if (3.14) we rewrite Table 2 including the factor $\frac{q^{-n/2}}{1+\frac{n}{2}}$ as Table 4.

Our standard model of $\sigma_\chi$ will be as a subset of $\pi$ as in (3.12). We will see that $\tilde{\sigma}_\chi \simeq \sigma_{\chi^{-1}}$ which could be considered as a subset of $\pi(\chi^{-1} | \cdot |^{1/2}, \chi^{-1} | \cdot |^{-1/2})$. However, for the purposes of computing the inner form (3.14) $\langle \cdot, \cdot \rangle : \sigma_\chi \times \tilde{\sigma}_\chi \to \mathbb{C}$

it is better to consider $\tilde{\sigma}_\chi$ as the quotient $\tilde{\pi}/\mathbb{C}_{\chi^{-1}}$ as in (3.13). As a matter of notation, given these models of $\sigma_\chi$ and $\tilde{\sigma}_{\chi^{-1}}$ we use letters $f, \phi$ to denote elements of $\pi$ (or $\sigma_\chi$), $\tilde{f}, \tilde{\phi}$ to denote elements of $\tilde{\pi}$, and $f, \phi$ to denote elements of $\sigma_{\chi^{-1}}$. The following is a well-known result.

**Lemma 3.7.** The pairing on $\pi \times \tilde{\pi}$ defined in (3.2) descends to a well defined pairing on $\sigma_\chi \times \sigma_{\chi^{-1}}$. That is to say, if $f \in \sigma_\chi$ and $\tilde{f} \in \sigma_{\chi^{-1}}$ then

$$\langle f, \tilde{f} \rangle = \int_K f(k)\tilde{f}(k)dk$$

for any $\tilde{f} \in \tilde{\pi}$ whose image in $\sigma_{\chi^{-1}}$ is $\tilde{f}$.

Note that $\tilde{\phi}_0 = \tilde{f}_0 + \tilde{f}_1 \in \tilde{\pi}$ is $K$-invariant so it’s image in $\sigma_{\chi^{-1}}$ must be zero. Therefore, as a corollary to Lemma 3.7, we have that

$$\int_K f(k)dk = \int_K f\tilde{\phi}_0(k)dk = \langle f, 0 \rangle = 0$$

for all $f \in \sigma_\chi$. In other words, if $f \in \sigma_\chi$ then $\int_K f(k)dk = 0$. In fact, the converse is also true. Namely if $f \in \pi$ is such that $\int_K f(k)dk = 0$ then $f \in \sigma_\chi$.

The space $\sigma_{\chi,0}^K$ is well known to be 1-dimensional. The vector $\phi = f_0 - \frac{1}{q} f_1 \in \pi$ meets the criterion above, hence is an element of $\sigma_\chi$. We call it the *normalized new vector*.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$\sigma_n$</th>
<th>$w\sigma_n$</th>
<th>$\sigma_n w$</th>
<th>$w\sigma_n w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_{0,0}(g)$</td>
<td>$\frac{\alpha^n q^{-n-1}}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n(q^{-1}-q^{-n})}{1+\frac{1}{q}}$</td>
<td>$0$</td>
<td>$\frac{\alpha^n q^{-1}}{1+\frac{1}{q}}$</td>
</tr>
<tr>
<td>$\Phi_{1,0}(g)$</td>
<td>$0$</td>
<td>$\frac{\alpha^n}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n q^{-n-1}}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n(1-q^{-n})}{1+\frac{1}{q}}$</td>
</tr>
<tr>
<td>$\Phi_{0,1}(g)$</td>
<td>$\frac{\alpha^n(q^{-1}-q^{-n-1})}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n q^{-n}}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n q^{-1}}{1+\frac{1}{q}}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\Phi_{1,1}(g)$</td>
<td>$\frac{\alpha^n}{1+\frac{1}{q}}$</td>
<td>$0$</td>
<td>$\frac{\alpha^n(1-q^{-n-1})}{1+\frac{1}{q}}$</td>
<td>$\frac{\alpha^n q^{-n}}{1+\frac{1}{q}}$</td>
</tr>
</tbody>
</table>

Table 4. Values of matrix coefficients for $\pi = \pi(\chi | \cdot |^{1/2}, \chi | \cdot |^{-1/2})$
which \( \sigma \) choose a new vector in \( \sigma_{\chi^{-1}} \) consistent with the choice of \( \phi \), note that by

\[
M : \tau = \pi(\chi^{-1} | \cdot |^{-1/2}, \chi^{-1} | \cdot |^{-1/2}) \to \pi(\chi^{-1} | \cdot |^{1/2}, \chi^{-1} | \cdot |^{-1/2}) = \pi',
\]

\( \sigma_{\chi^{-1}} \) is isomorphic to a subspace of \( \pi' \).

So we define the normalized new vector \( \tilde{\phi} \) of \( \sigma_{\chi^{-1}} \) to be the image of any vector \( \tilde{\phi} \) for which \( M(\tilde{\phi}) = f'_0 - \frac{1}{q} f'_1 \). By Lemma 3.4, \( M(f_0 - \frac{1}{q} f_1) = (1 + \frac{1}{q}) \tilde{\phi}_1 \). A simple calculation then shows that \( (1 + \frac{1}{q}) \tilde{\phi}_1 \) and \( \tilde{f}_0 \) have the same image in \( \sigma_{\chi^{-1}} \).

We now have all of the necessary ingredients to prove the generalization of Proposition 3.3 to the case of \( \pi \) a special representation.

**Proposition 3.8.** Let \( \chi \) be an unramified character of \( F \) and \( \alpha = \chi(\varpi) \). If \( \phi, \tilde{\phi} \) are new vectors of a special representation \( \sigma_\chi \) and its contragradient \( \sigma_{\chi^{-1}} \) respectively, the value of the matrix coefficient \( \Phi = \Phi_{\phi_1, \tilde{\phi}_1}/\Phi_{\phi_1, \tilde{\phi}_1}(e) \) is determined by the Table 5 in which \( n \geq 1 \).

<table>
<thead>
<tr>
<th>( g )</th>
<th>( w )</th>
<th>( \sigma_n )</th>
<th>( w\sigma_n )</th>
<th>( \sigma_n w )</th>
<th>( w\sigma_n w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi(g) )</td>
<td>( -q^{-1} )</td>
<td>( -\alpha^q q^{-n} )</td>
<td>( -\alpha^q q^{-n} )</td>
<td>( -\alpha^q q^{-n} )</td>
<td>( -\alpha^q q^{-n} )</td>
</tr>
</tbody>
</table>

**TABLE 5.** Values of matrix coefficient associated to the new vector of a special representation

**Proof.** We take \( \phi, \tilde{\phi} \) as above. Then \( \tilde{\phi} \) is the image of \( \tilde{f}_0 \). So, by Lemma 3.7,

\[
\Phi_{\phi_1, \tilde{\phi}_1}(g) = \langle \sigma_\chi(g)\phi, \tilde{\phi} \rangle = \int_K (f_0 - \frac{1}{q} f_1)(gj)(\tilde{f}_0(j))dk = \Phi_{0, 0}(g) - \frac{1}{q} \Phi_{1, 0}(g).
\]

The result can now be deduced from Table 4. \( \square \)

**Remark.** For all of our applications we are interested in determining \( \frac{I(\phi \phi \tilde{\phi})}{(\phi, \phi)} \). Since \( \langle \phi, \tilde{\phi} \rangle = \Phi_{\phi, \tilde{\phi}}(e) \), it is convenient to normalize \( \Phi \) in this way.

4. **Local trilinear forms**

In this section we compute the term \( I_v(\varphi_v) \) appearing on the right hand side of (2.4). To this end, we tabulate the local \( L \) and zeta factors that appear in Ichino’s formula in Section 4.1. In Section 4.2 we calculate the forms \( I'_v(\varphi_v) \) using our results above.

We now assume that all local representations are unitary and have trivial central character. In the unramified case this means we only consider \( \pi(\chi, \chi^{-1}) \) and that \( \chi(\varpi) \) is either complex with absolute value 1 or else equal to \( q^{-\lambda} \) for some real \( \lambda \) which we can assume to be in the interval \((0, \frac{1}{2})\). (The latter case are the so-called complementary series.) In the case of special representations \( \sigma_\chi \), we have \( \chi^2 = 1 \).

4.1. **Local \( L \)-factors.** Let \( F \) be a nonarchimedean local field with uniformizer \( \varpi \), and let \( q \) be the order of the residue field. We let \( \chi, \nu, \mu : F^\times \to \mathbb{C}^\times \) be unramified characters, and denote \( \gamma = \chi(\varpi) \), \( \beta = \nu(\varpi) \) and \( \alpha = \mu(\varpi) \). Then \( L_v(s, \chi) = (1 - \gamma q^{-s})^{-1} \), and the local zeta function is \( \zeta_F(s) = L_v(s, 1) \).

Corresponding to each irreducible admissible representation \( \pi \) of \( \text{GL}_2(F) \) there is a representation \( \rho : W_F \to \text{GL}_2(\mathbb{C}) \) of the Weil group. Any representation \( \rho \) of the \( W_F \) gives rise
to an $L$-factor as in [27]. Let $\rho_i$ correspond to $\pi_i$ in this manner. Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$. We define

$$L(s, \Pi) = L(s, \rho_1 \otimes \rho_2 \otimes \rho_3)$$

and

$$L(s, \Pi, \text{Ad}) = L(s, \oplus_i \text{Ad}(\rho_i)) = \prod_{i=1}^3 L(s, \text{Ad}(\rho_i)) = \prod_{i=1}^3 L(s, \pi_i, \text{Ad})$$

where $\rho_1 \otimes \rho_2 \otimes \rho_3 : W_F \to \text{GL}_8(\mathbb{C})$ is the standard tensor product representation and $\text{Ad}(\rho_i) : W_F \to \text{GL}_8(\mathbb{C})$ is the adjoint representation.

For the triple product, we are especially interested in the following three cases:

$$\Pi^i = \pi(\mu, \mu^{-1}) \otimes \pi(\nu, \nu^{-1}) \otimes \sigma_\chi,$$

$$\Pi^2 = \pi(\mu, \mu^{-1}) \otimes \sigma_\nu \otimes \sigma_\chi,$$

$$\Pi^3 = \sigma_\mu \otimes \sigma_\nu \otimes \sigma_\chi.$$

The triple product $L$-function in each of these cases is

$$L_v(s, \Pi^1) = \prod_{\epsilon, \delta \in \{\pm 1\}} L_v(s + \frac{1}{2}, \mu^\epsilon \nu^\delta \chi) = \prod_{\epsilon, \delta \in \{\pm 1\}} \frac{1}{1 - \alpha^\epsilon \beta^\delta q^{-s+1/2}},$$

$$L_v(s, \Pi^2) = \prod_{\epsilon \in \{\pm 1\}} L_v(s, \mu^\epsilon \nu^\chi) L_v(s + 1, \mu^\epsilon \nu^\chi) = \prod_{\epsilon \in \{\pm 1\}} \frac{1}{(1 - \alpha^\epsilon \beta^\gamma q^{-s})(1 - \alpha^\epsilon \beta^\gamma q^{-s-1})},$$

$$L_v(s, \Pi^3) = L_v(s + \frac{3}{2}, \mu \nu \chi) L_v(s + \frac{1}{2}, \mu \nu \chi^2) = \frac{1}{(1 - \alpha \beta \gamma q^{-s+3/2})(1 - \alpha \beta \gamma q^{-s+1/2})^2}.$$

The adjoint $L$-functions agree with that given by Gelbart-Jacquet [10]. For the cases at hand,

$$L_v(s, \pi(\chi, \chi^{-1}), \text{Ad}) = L_v(s, 1) \frac{1}{(1 - \gamma^2 q^{-s})(1 - \gamma^{-2} q^{-s})},$$

$$L_v(s, \sigma_\chi, \text{Ad}) = L_v(s + 1, 1) = \frac{1}{1 - q^{-s+1}}.$$

Note that this is equal to the symmetric square $L$-function, which is so precisely because we are assuming that $\pi$ has trivial central character.

For the discrete series representations $\pi^k_{\text{dis}}$ of $\text{GL}_2(\mathbb{R})$ of weight $k$ we have the following $L$-functions. First, let

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2), \quad \zeta_{\mathbb{C}}(s) = (2\pi)^{-s} \Gamma(s)$$

where $\Gamma(s)$ is the standard $\Gamma$-function. Then

$$L(s, \pi^k_{\text{dis}} \otimes \pi^k_{\text{dis}} \otimes \pi^k_{\text{dis}}) = \zeta_{\mathbb{C}}(s + 1/2) \zeta_{\mathbb{C}}(s + k_1 - 3/2) \zeta_{\mathbb{C}}(s + k_2 - 1/2) \zeta_{\mathbb{C}}(s + k_3 - 1/2),$$

$$L(s, \pi^k_{\text{dis}}, \text{Ad}) = \zeta_{\mathbb{C}}(s + k - 1) \zeta_{\mathbb{R}}(s + 1).$$

4.2. Explicit computations: $F$ nonarchimedean. Recall that we want to make Ichino’s formula (2.4) explicit, i.e. calculate $I_v(\varphi)$ which is a product of $L$-factors times

$$I_v'(\varphi) = \int_{Z(F) \backslash G(F)} \langle \Pi(g) \varphi, \varphi \rangle = \int_{Z(F) \backslash G(F)} \Phi_1(g) \Phi_2(g) \Phi_3(g) dg.$$

If all of the local data is unramified, Ichino calculated that the trilinear form $I(\varphi) = 1$—the normalization of (2.3) is chosen precisely to give this result. So in all of our computations at least one of the representations will be special.
We assume from now on that $\pi_3 = \sigma_\chi$, and we denote the normalized matrix coefficient associated to a new vector of $\sigma_\chi$ by $\Phi^3$ as in Proposition 3.8. So our object is to calculate

$$I'_\chi(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \int_{Z(F) \backslash G(F)} \Phi^1(g)\Phi^2(g)\Phi^3(g)dg$$

where $\Phi^i = \Phi_{\varphi_i, \bar{\sigma}_i}$ for various choices of $\varphi_i$.

From Tables 1 and 4 we read off that (for all $n \geq 0$)

$$\nu(K_0gK_0)\Phi(g) = \pm \frac{\gamma^n}{q(1 + \frac{1}{q})}$$

for $g \in \{\sigma_n, w\sigma_n, \sigma_nw, w\sigma_nw\}$ with the minus sign in the cases $\sigma_nw$ or $w\sigma_n$ and the positive sign otherwise.

All of the matrix coefficients we are considering are $K_0$-biinvariant, so we combine this with Lemma 3.1 to obtain the formula

\begin{equation}
\int_{Z(F) \backslash G(F)} \Phi_1(g)\Phi_2(g)\Phi_3(g)dg = \frac{\Phi_1(e)\Phi_2(e) - \Phi_1(w)\Phi_2(w)}{1 + q} + \sum_{n=1}^{\infty} \frac{\gamma^n}{1 + q} \left( \Phi_1\Phi_2(\sigma_n) - \Phi_1\Phi_2(w\sigma_n) - \Phi_1\Phi_2(\sigma_nw) + \Phi_1\Phi_2(w\sigma_nw) \right).
\end{equation}

In computing the forms, we will use (many times) the following identities. Let

\begin{equation}
A = \frac{1 - \frac{\alpha^2}{q}}{1 - \alpha^2} \quad \text{and} \quad B = \frac{1 - \frac{\alpha^{-2}}{q}}{1 - \alpha^{-2}}.
\end{equation}

Then

\begin{align*}
A + B &= \frac{1 - \frac{\alpha^2}{q}}{1 - \alpha^2} + \frac{1 - \frac{\alpha^{-2}}{q}}{1 - \alpha^{-2}} = \frac{-\alpha^{-1} + \frac{q}{\alpha} + \alpha - \frac{1}{\alpha}}{\alpha - \alpha^{-1}} = 1 + \frac{1}{q}, \\
A\alpha + B\alpha^{-1} &= \frac{1 - \frac{\alpha^2}{q}}{\alpha - \alpha^{-1}} + \frac{1 - \frac{\alpha^{-2}}{q}}{\alpha - \alpha^{-1}} = \frac{1 + \frac{\alpha^2}{q} + 1 - \frac{\alpha^{-2}}{q}}{\alpha - \alpha^{-1}} = \frac{1}{q}(\alpha + \alpha^{-1}), \\
A\alpha^{-1} + B\alpha &= \frac{\alpha^{-1} - \frac{q}{\alpha}}{1 - \alpha^2} + \frac{\alpha - \frac{1}{\alpha^{-1}}}{1 - \alpha^{-2}} = \frac{-\alpha^{-2} + \frac{1}{q} + \alpha^2 - \frac{1}{q}}{\alpha - \alpha^{-1}} = \alpha + \alpha^{-1} \\
A\alpha^2 + B\alpha^{-2} &= (A\alpha + B\alpha^{-1})(\alpha + \alpha^{-1}) - (A + B) = \frac{1}{q}(\alpha^2 + \alpha^{-2}) + \frac{1}{q} - 1 \\
A\alpha^{-2} + B\alpha^2 &= (A\alpha^{-1} + B\alpha)(\alpha + \alpha^{-1}) - (A + B) = \alpha^2 + \alpha^{-2} + 1 - \frac{1}{q}.
\end{align*}

\subsection{4.2.1. Only one of the representations is special.}

In this section we treat the case when two of the representations are unramified, and only one is special. For the application to subconvexity this is the case of most interest since it corresponds to the place $p$ in Corollary 1.5. Let $\pi_1 = \pi(\mu, \mu^{-1})$ and $\pi_2 = \pi(\nu, \nu^{-1})$ and $\pi = \sigma_\chi$. Set $\alpha = \mu(\varpi)$, $\beta = \nu(\varpi)$ and $\gamma = \chi(\varpi)$.

Finally, let $\{\phi_0^i, \phi_1^i\}$ be the basis of eigenvectors of $\pi_i$ as in Proposition 3.5.

We combine the results of the previous sections to prove the following.

**Proposition 4.1.** Let $\Phi^3$ be the normalized matrix coefficient corresponding to a new vector $\phi^3 \in \sigma_\chi$ as in Proposition 3.8, and define $\Psi^i_{j,k} = \Phi_{\phi_j^i, \phi_k^i}$ for $i = 1, 2$ as in Proposition 3.5.
Then
\[ (4.8) \quad \int_{Z \setminus G} \Phi \, \Psi(x) \Phi(x) \, \Psi(x) \, \Phi^3(g) \, dg = L(1, \Pi_1) \cdot \frac{1}{\rho^2} (1 - \frac{1}{\rho^2}) \cdot C \]
where
\[ C = \begin{cases} 1 + \frac{1}{\rho^2} - \frac{\alpha^2 + \beta^2}{\rho^2}, & \text{if } j = k = 1, \\ 0, & \text{otherwise}. \end{cases} \]

**Proof.** To show directly that \( C = 0 \) when \( j \) and \( k \) are not both 1 is not hard using (4.1). However, we give a more conceptual proof. Suppose that \( C = 0 \) and \( \Phi \) is any \( G \)-invariant linear form. Note that if \( \phi \in \sigma \) then Lemma 3.7 implies \( \int_{G} \Phi(k) \phi \, dk = 0 \).

We claim that \( \ell(\phi_1^1, \phi_0^2, \phi) = 0 \) for all \( \phi \in \sigma \). Since \( \phi_1^1 \) and \( \phi_0^2 \) are \( K \)-invariant (and \( \ell \) is \( G \)-invariant),
\[ \ell(\phi_1^1, \phi_0^2, \pi(k) \phi) = \ell(\pi(k)^{-1} \phi_1^1, \pi(k)^{-1} \phi_0^2, \phi) = \ell(\phi_1^1, \phi_0^2, \phi) \]
for all \( k \in K \). Integrating over \( K \), we find that
\[ \ell(\phi_1^1, \phi_0^2, \phi) = \ell(\phi_1^1, \phi_0^2, \int_{K} \pi(k) \phi \, dk) = \ell(\phi_1^1, \phi_0^2, 0) = 0. \]

Since the matrix coefficients are bilinear forms, fixing the vectors in one of the coordinates gives a linear form in the other. For example, if we fix \( \vec{\phi} \)\( ^1 \otimes \vec{\phi} \otimes \vec{\phi} \in Z \) then
\[ \ell : \pi_1 \times \pi_2 \times \pi \to \mathbb{C} \quad \ell(\phi^1, \phi^2, \phi) = \int_{Z \setminus G} \Psi_1 \Psi_2 \Psi(g) \, dg \]
is a linear form on \( Z \). From this point of view, it is immediate that in each of the cases of Proposition 4.1 that are claimed to be zero, as a form on either \( G \) or \( Z \) (or both) the claim applies.

To complete the proof of Proposition 4.1, we need to do the calculation of (4.8). Let \( A_1, B_1 \) be the eigenvalues of the intertwining operator \( M \) on \( \pi_1 \). By (4.1), the integral (4.8) is equal to \( \frac{1}{q} \) times
\[ \frac{1}{q} + (1 + \frac{1}{q})^{-1} \sum_{n=1}^{\infty} \left[ \frac{\alpha \beta}{q} \right]^n \frac{B_1 A_2 + \left( \frac{\alpha^2}{q} \right)^n A_1 B_2}{1 - \frac{\alpha^2 \beta^2}{q}} + \frac{B_1 A_2 + \left( \frac{\beta^2}{q} \right)^n A_1 B_2}{1 - \frac{\alpha^2 \beta^2}{q}} + \frac{A_1 A_2}{1 - \frac{\alpha^2 \beta^2}{q}} + \frac{B_1 B_2}{1 - \frac{\alpha^2 \beta^2}{q}} \right] \]
Simplifying the expression inside the parentheses yields \( L(\frac{1}{2}, \Pi) \) times
\[ \frac{1}{q} (A_1 \alpha^{-1} + B_1 \alpha)(A_2 \beta + B_2 \beta^{-1}) - \frac{1}{q^4} (A_1 + B_1)(A_2 + B_2) \]
\[ + \frac{1}{q^3} [(A_2 + B_2)(A_1 \alpha^{-1} + B_1 \alpha)(\beta + \beta^{-1}) + (A_1 \alpha + B_1 \alpha^{-1})(A_2 \beta + B_2 \beta^{-1})] \]
\[ - \frac{1}{q^2} [(A_1 + B_1)(A_2 + B_2) + (A_1 + B_1)(A_2 \beta^2 + B_2 \beta^{-2}) + (A_2 + B_2)(A_1 \alpha^{-2} + B_1 \alpha^2)] \]

\[ 18 \]
Applying formulas (4.3)-(4.7), this becomes \( \frac{1}{q}(1 + \frac{1}{q})L(\frac{1}{2}, \Pi) \) times
\[
(4.9) \quad -\frac{1}{q}(1 + \frac{1}{q}) \gamma(\alpha + \alpha^{-1})(\beta + \beta^{-1}) (1 + \frac{1}{q}) \frac{1}{q^2} (\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2})
\]
So, combining the above, we find that (4.8) is equal to \( \frac{1}{q}L(\frac{1}{2}, \Pi) \) times
\[
L(\frac{1}{2}, \Pi)^{-1} + (4.9).
\]

Since \( L(\frac{1}{2}, \Pi)^{-1} \) is equal to
\[
(4.10) \quad (1 + \frac{1}{q})^2 - \frac{1}{q}(1 + \frac{1}{q}) \gamma(\alpha + \alpha^{-1})(\beta + \beta^{-1}) + \frac{1}{q^2}(\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2}),
\]
the value of \( C \) in the statement of the proposition follows.

**Corollary 4.2.** Let the notation be as in Proposition 4.1, and set \( \phi = \phi_0^1 \otimes \phi^2_2 \otimes \phi^3 \) where \( \phi^2_2 = (\varpi^{-1}) \phi^2_0 \) as in Lemma 3.6. Then
\[
(4.11) \quad \frac{I_v(\phi)}{\langle \phi, \phi \rangle} = \frac{1}{q}(1 + \frac{1}{q})^{-1}
\]
In particular, \( \frac{I_v(\phi)}{\langle \phi, \phi \rangle} \gg \frac{1}{q} \) where the implicit constant can be taken to be independent of \( \Pi \) and \( q \).

**Proof.** Apply Lemma 3.6 to write
\[
\phi^2_2 = a \phi^2_0 + b \phi^2_1 \quad \text{where} \quad a = \frac{\beta q^{-1/2} + \beta^{-1} q^{-1/2}}{1 + \frac{1}{q}}, \quad b = \frac{\beta^{-1} q^{1/2} - \beta^{-1} q^{-1/2}}{1 + \frac{1}{q}}.
\]

First, assume that \( \overline{\beta} = \beta^{-1} \). By the bilinearity of matrix coefficients,
\[
(4.12) \quad \Psi_{2,2} = \Phi_{a \phi_0 + b \phi_1, a \phi_0 + b \phi_1} = |a|^2 \Psi_{1,1} + a \overline{b} \Psi_{0,1} + \overline{a} b \Psi_{1,0} + |b|^2 \Psi_{1,1}.
\]

Recall that \( I_v = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi^1, \text{Ad})}{L_v(\Pi^1, \Pi^1)} I_v' \). Therefore, (4.8) implies that
\[
\frac{I_v(\phi)}{\langle \phi, \phi \rangle} = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi^1, \text{Ad})}{L_v(\frac{1}{2}, \Pi^1)} \int_{Z(G)} \Psi_{0,0} \Psi_{2,2} \Phi^3(g) dg
\]
\[
= \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi^1, \text{Ad})}{L_v(\frac{1}{2}, \Pi^1)} \left[ b^2 \int_{Z(G)} \Psi_{1,0} \Psi_{1,1} \Phi^3(g) dg \right] = \frac{1}{q^2}(1 + \frac{1}{q})^{-1}.
\]

In the final step we have used that
\[
\zeta_{F_v}(2)^{-2} L_v(1, \Pi^2, \text{Ad}) = \frac{1 + \frac{1}{q}}{(1 + \frac{1}{q})^2 - \frac{\alpha^2 + \alpha^{-2}}{q^2} - (1 + \frac{1}{q^2} - \frac{\beta^2 + \beta^{-2}}{q})^{-1}.
\]

Although the result is the same if \( \pi_2 \) is a complementary series (i.e. \( \beta = q^x \) for some \( \lambda \in (0, \frac{1}{2}) \)) the method is slightly different because the inner product is not the same. If \( \phi, \phi' \in \pi_2 \), then
\[
\langle \phi, \phi' \rangle = \int_K \phi(k)(M \phi')(k) dk.
\]
Then, Proposition 4.3. \( \sigma \) and \( \Psi \)

**Proof.** Appropriate normalizing factors as given in Section 4.1. Equation (4.14) is immediate. The value of (4.14) is obtained by a calculation analogous to the proof of (4.8).

Thus, given (4.13), the conclusion that \( \Pi^2 = \pi(\mu, \mu) \otimes \sigma_\nu \otimes \sigma_\chi \). The following result will be used in Theorem 4.7.

**Proposition 4.4.** Let \( \Pi^2 \) be as above, and denote by \( \Phi^2 \) and \( \Phi^3 \) the normalized matrix coefficients corresponding to the new vectors \( \phi^2 \in \sigma_\nu \) and \( \phi^3 \in \sigma_\chi \) as in Proposition 3.8. Let \( \Psi_{1,0} = \Phi_{\phi_0, \phi_0} \) be the matrix coefficient corresponding to \( \phi_0 \in \pi(\mu, \mu^{-1}) \) as in Proposition 3.5. Then

\[
\Psi_{1,0} = a^2 A_2 \Psi_{0,0} + ab A_2 \Psi_{1,0} - ab B_2 \Psi_{0,1} - b^2 B_2 \Psi_{1,1}.
\]

The remainder of proof goes through as above, except one replaces \( |b|^2 \) with \(-b^2 B_2 / A_2\). (We must divide by \( A_2 \) because \( \langle \phi_2, \overline{\phi}_2 \rangle = \Psi_{2,2}(\epsilon) = A_2 \).) As it turns out, this is exactly the same expression in terms of \( \beta \) as was \( |b|^2 \).

\( \square \)

4.2.2. Two of the representations are special. We now consider the case \( \Pi^2 = \pi(\mu, \mu) \otimes \sigma_\nu \otimes \sigma_\chi \). The following result will be used in Theorem 4.7.

**Proposition 4.3.** Let \( \Pi^2 \) be as above, and denote by \( \Phi^2 \) and \( \Phi^3 \) the normalized matrix coefficients corresponding to the new vectors \( \phi^2 \in \sigma_\nu \) and \( \phi^3 \in \sigma_\chi \) as in Proposition 3.8. Let \( \Psi_{1,0} = \Phi_{\phi_0, \phi_0} \) be the matrix coefficient corresponding to \( \phi_0 \in \pi(\mu, \mu^{-1}) \) as in Proposition 3.5. Then

\[
(4.13) \quad \int_{Z \setminus G} \Psi_{1,0} \Phi^2 \Phi^3(g) dg = \frac{1}{q} (1 - \frac{1}{q}) L(\frac{1}{2}, \Pi^2)(1 - \frac{\alpha^2}{q})(1 - \frac{\alpha^{-2}}{q}).
\]

Therefore, if we set \( \phi = \phi_0 \otimes \phi^2 \otimes \phi^3 \) then \( \frac{I_v(\phi)}{\langle \phi, \phi \rangle} = \frac{1}{q} \).

**Proof.** The evaluation of (4.13) is obtained by a calculation analogous to the proof of (4.8). Since the method is identical, we leave the details to the reader.

Note that

\[
\zeta_{F_i}(2) L_v(1, \Pi^2, \text{Ad}) = (1 - \frac{\alpha^2}{q})^{-1}(1 - \frac{\alpha^{-2}}{q})^{-1}(1 - \frac{1}{q})^{-1}.
\]

Thus, given (4.13), the conclusion that \( \frac{I_v(\phi)}{\langle \phi, \phi \rangle} = \frac{1}{q} \) is immediate.

\( \square \)

4.2.3. All three representations are special. This is the case that corresponds to the primes dividing \( \mathfrak{m} \) in Theorem 1.4. We denote

\[
\epsilon = \epsilon(\frac{1}{2}, \sigma_\chi \otimes \sigma_\mu \otimes \sigma_\nu) = -(\chi \mu \nu)(\varpi) = -\alpha \beta \gamma.
\]

**Proposition 4.4.** Let \( \pi_1 = \sigma_\nu \) and \( \pi_2 = \sigma_\chi \). Denote the matrix coefficient associated to the new vector \( \phi^i \in \pi_i \) as in Proposition 3.8 by \( \Phi^i \) and set \( \phi = \phi^1 \otimes \phi^2 \otimes \phi^3 \). Then

\[
(4.14) \quad \int_{Z \setminus G} \Phi^1 \Phi^2 \Phi^3(g) dg = (1 + \epsilon) L_v(\frac{1}{2}, \Pi^3) \frac{1}{q} (1 - \frac{1}{q})(1 + \frac{\epsilon}{q})^2,
\]

and \( \frac{I_v(\phi)}{\langle \phi, \phi \rangle} = \frac{1 - \epsilon}{q} (1 + \frac{1}{q}) \).

**Proof.** We use Proposition 3.8 and apply (4.1):

\[
1 - \frac{1}{q^2} + \frac{(1 - q^2)(1 - \frac{1}{q^2})}{1 + q} \sum_{n=1}^{\infty} (-\frac{\epsilon}{q^2})^n = \frac{1 - \frac{1}{q^2}}{1 + q} \left( 1 - \frac{(1 - q^2)(\frac{\epsilon}{q^2})}{1 + \frac{1}{q^2}} \right)
\]

Equation (4.14) is immediate. The value of \( I_v(\phi) \) now follows by multiplying by the appropriate normalizing factors as given in Section 4.1.

\( \square \)
Proposition 4.5. Let $\Pi = \pi_\mu \otimes \pi_\nu \otimes \pi_\chi$ and $\Pi^B$ the admissible representation of $B$ associated to $\Pi$ via Jacquet-Langlands where $B$ is the unique quaternion division algebra over $F$. Let $\varepsilon = - (\mu \nu \chi)(\varpi)$. If $\phi \in \Pi^B$ and $\tilde{\phi} \in \tilde{\Pi}^B$ then

$$
(4.15) \quad \frac{I_v(\phi)}{\langle \phi, \tilde{\phi} \rangle} = (1 - \varepsilon)^1 \frac{1}{q} (1 - \frac{1}{q}).
$$

Proof. The Jacquet-Langlands lift of $\sigma_\chi$ is the character $\chi_B : B^* \to \mathbb{C}^*$ given by $\chi_B(\beta) = \chi \circ N_B(\beta)$ where $N_B$ is the reduced norm. Hence $\Pi^B \simeq \eta_B$ where $\eta_B = \mu_B \nu_B \chi_B$, and

$$
\int_{F^x \setminus B^x} \Phi_{\phi, \bar{\phi}}(\beta) d\beta = \int_{F^x \setminus B^x} \langle \Pi^B(\beta) \phi, \bar{\phi} \rangle d\beta
$$

$$
= \int_{F^x \setminus B^x} \eta_B(\beta) \langle \phi, \bar{\phi} \rangle d\beta
$$

$$
= \langle \phi, \bar{\phi} \rangle \left\{ \begin{array}{ll}
\text{vol}(F^x \setminus B^x) & \text{if } \eta_B \text{ is trivial,} \\
0 & \text{otherwise.}
\end{array} \right.
$$

Recall (see Section 2.2) that $\text{vol}(F^x \setminus B^x) = \frac{2}{q^1}$. To obtain $I_v$, one multiplies this by the factor $\zeta_F(2) \frac{L_0(1, \Pi_{\text{Ad}})}{L_0(2, \Pi_\nu)}$. Using the values given in Section 4.1 and simplifying, the result follows. \qed

Our factor $\varepsilon$ is indeed the local factor $\varepsilon(\frac{1}{2}, \Pi)$. So Propositions 4.4 and 4.5 provide an explicit realization of Prasad’s result. More precisely, the integrated matrix coefficient provides a trilinear form on $\Pi$ (respectively $\Pi^B$) that is invariant by the diagonal action of $G$ (resp. $B^x$.) It is nonzero on $B^x$ if and only if $\varepsilon = -1$. On $G$, $\phi = \phi_1^1 \otimes \phi_2^2 \otimes \phi_3^3$ is a test vector if and only if $\varepsilon = +1$.

4.3. Explicit computations: $F = \mathbb{R}$. Let $K = \{ \kappa_\theta = (\cos \theta, \sin \theta) \mid \theta \in [0, 2\pi) \}$, $N = \{ n(x) = (1, x) \mid x \in \mathbb{R} \}$ and $A = \{ a(y) = (y^1) \mid |y| \geq 1 \}$. Then $\text{PGL}_2(\mathbb{R})$ can be given by the coordinates $KAK$. Recall (2.7) gives the Haar measure in this case.

It is well known that for the weight $k$ discrete series, which we denote by $\pi^k_{\text{dis}}$, the matrix coefficient corresponding to a new vector $\phi$ is given by

$$
(4.16) \quad \Phi(g) = \frac{(2\sqrt{\det g})^k}{(a + d + i(b - c))^k} g = (a \ b \ c \ d), \text{ det } g > 0
$$

and $\Phi(g) = 0$ if $\text{det } g < 0$. In the $KAK$ coordinates (and $y \geq 1$) this is

$$
(4.17) \quad \Phi(\kappa_\theta, a(y) \kappa_{\theta_1}) = 2^k e^{2\pi i k(\theta_2 - \theta_1)} \frac{g^{k/2}}{y^{1+k}}.
$$

As a matter of notation, let $\phi^- = \pi((1, -1)) \phi$. In particular, if $\phi \in \pi^k_{\text{dis}}$ has weight $m$ then $\phi^-$ has weight $-m$. Moreover, if $\phi$ is a new vector then the matrix coefficient associated to $\phi^-$ is $\Phi$.

Proposition 4.6. For $i = 1, 2, 3$, let $\pi_i = \pi^k_{\text{dis}}$ be such that $k = k_1 + k_2 + k_3$, and let $\phi_i \in \pi_i$ be a new vector. Set $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \in \Pi_\infty = \pi_1^k \otimes \pi_2^k \otimes \pi_3^k$. Then

$$
\int_{\text{PGL}_2(\mathbb{R})} \Phi_{\phi, \phi}(g) dg = \frac{2\pi}{k - 1}.
$$
and $I_v(\phi) = 2$.

**Proof.** Using the description above and the $KAK$ coordinates, we have

\[
\int_{\text{PGL}_2(\mathbb{R})} \Phi_{v,y}(g) dg = \frac{2k_1 + k_2 + k_3 - 2}{\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{y^{k_1 + k_2 + k_3}(1 - y^{-2})}{(1 + y)^{k_1 + k_2 + k_3}} d\theta_1 d\theta_2 dy
\]

\[
= 2^{2k} \pi \int_1^{\infty} y^k - y^{-2k} dy = 2^{2k} \pi \left[ -\frac{y^{k-1}}{(k-1)(1+y)^{2k-2}} \right]_1^{\infty} = 2\pi / (k-1).
\]

Since $\langle \phi, \phi \rangle = \Phi(e) = 1$, to get $I_v(\phi)$ one multiplies this result by $\zeta_B(2)^{-2} L_v(1, \Pi_\infty, Ad)$ which is easily seen to be equal to $(k-1)/\pi$. Hence, $I_v(\phi) = 2$. \hfill \Box

### 4.4. Generalizing Watson’s formula.

We now work globally. Let $F$ be a totally real number field, and $\pi_1, \pi_2, \pi_3$ cuspidal automorphic representations of $\text{PGL}_2(F)$. It is clear from the description in Section 4.3 that if $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ for some $\varphi_i = \boxtimes_v \varphi_{i,v} \in \pi_i$ then $I(\varphi) = 0$ unless at each infinite place the weights $k_i,v$ of $\varphi_i$ sum to zero.

We say that $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ is almost balanced if at each real place $v$, the largest weight is the sum of the two smaller weights.

**Theorem 4.7.** Let $\pi_1, \pi_2, \pi_3$ cuspidal automorphic representations of $\text{PGL}_2$ over a totally real number field $F$ such that $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ is almost balanced. Assume that the conductor $\mathfrak{N}_i$ of $\pi_i$ is squarefree for each $i = 1, 2, 3$. Let $\mathfrak{N} = \text{gcd}(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3)$. For $\{j, k, l\} = \{1, 2, 3\}$, write

\[
\mathfrak{N}_j = \mathfrak{N} n_j n_{jk} n_{jl} \quad \text{where} \quad n_{jk} = n_{kj} = \text{gcd}(\mathfrak{N}_j, \mathfrak{N}_k) / \mathfrak{N}.
\]

Let $\mathfrak{B} = \mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{N}_3$. Let $B$ be the global quaternion algebra such that $\dim \text{Hom}_B(\Pi^B, \mathbb{C}) = 1$. (So the discriminant of $B$ divides $\mathfrak{N}$.)

Define the vector $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \Pi^B$ as follows. For $v \mid \infty$, let $k_v$ be the maximum weight, and let $\varphi_{i,v}$ be the normalized new vectors such that their weights add to zero as in Proposition 4.6. At a finite prime $p \mid n_j$, let $\varphi_{i,p}$ be the normalized new vector for $i = j, k$, and the twists of the normalized new vector by $\left( \frac{-1}{p} \right)$ as in Lemma 3.6 (where it is denoted as $\varphi_2$.) In all other cases we take $\varphi_{i,v}$ to be the normalized new vector. Let $\varphi_i = \boxtimes_v \varphi_{i,v}$.

Under this choice of $\varphi$ and for $db$ the Tamagawa measure,

\[
\left( \frac{1}{\prod_{i=1}^{3} \int_{B^x} \varphi_i(b) db} \right)^2 = \zeta_F(2) \frac{L(\frac{1}{2}, \Pi)}{2^3 L(1, \Pi, Ad)} \prod_{v \mid \infty \mathfrak{N}} C_v
\]

where, for finite places,

\[
C_p = \begin{cases} 
\frac{2}{N(p)} (1 + \frac{\varepsilon_p}{N(p)}) & \text{if } p \mid \mathfrak{N} \\
\frac{1}{N(p)} (1 + \frac{\varepsilon_p}{N(p)})^{-1} & \text{if } p \mid n_j \\
\frac{1}{N(p)} & \text{if } p \mid n_{jk}
\end{cases}
\]

and for infinite places, $C_v = 2$ if all three representations $\pi_{i,v}$ are discrete series and $C_v = 1$ otherwise. Here $\varepsilon_p = \varepsilon_p(\frac{1}{2}, \Pi_p)$. 


Proof. This is proven by combining Ichino’s formula with local calculations above. The constant for \( p \mid \mathfrak{N} \) is immediate given Propositions 4.4 and 4.5. The constants for \( p \mid n_j \) or \( n_{ijk} \) follow from Corollary 4.2 and Proposition 4.3 respectively.

In the real case, Proposition 4.6 gives the constant if all three local representations are discrete series. The remaining real cases follow from [15, Proposition 5.1.] together with the archimedean calculations of [29, Theorem 2].

\[ \square \]

Remark. Watson’s archimedean calculations of [29, Theorem 2] in the case that all three representations are principal series are taken directly from [17].

5. PROOF OF THEOREM 1.3

We return to the notation of Section 2 in which \( F \) is any fixed number field and \( \mathbb{A} = \mathbb{A}_F \) its ring of adeles. Let \( \pi_1, \pi_2 \) be cuspidal automorphic representations of \( \text{GL}_2 \) over \( F \) with trivial central character and fixed conductors \( n_1 \) and \( n_2 \) respectively. Let \( \pi_3 \) be a cuspidal automorphic representation of \( \text{GL}_2 \) over \( F \) with trivial central character and conductor \( np \) where \( p \) is prime and does not divide \( nn_1n_2 \). We denote \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \). Let \( \varpi \) be a normalizer of \( F_p \). Note that the Langlands parameters of \( \pi_{3,\infty} \) are allowed to vary (within a bounded set.) Moreover, although \( n \) is fixed, the local components \( \pi_{3,q} \) for \( q \mid n \) need not be.

At the real places \( i,v \) corresponds to either \( \pi_{dis}^k \), a weight \( k \)-discrete series with \( k \geq 2 \) even, or an irreducible (weight zero) principal series \( \pi_{B}^k = \pi(\cdot^k, |\cdot|^{-s}) \) defined in the same way as (3.1). A complex place \( i,v \) is an irreducible principal series \( \pi_{C}^{s,k} \).

Each of these is a \((g,K)\)-module where \( g \) is the complexified Lie algebra of \( G_v = \text{GL}_2(F_v) \) and \( K = O(2) \) or \( U(2) \) depending on whether \( v \) is real or complex respectively. The irreducible representations of \( K \) are called weights. In the real case these are integers and, as is well-known they are given by the characters \( \kappa_{\theta} = e^{in\theta} \). In the complex case, the weights are nonnegative integers \( k \) which correspond to representations of dimension \( k + 1 \). In the notation above for \( \pi_{C}^{s,k} \) and \( \pi_{B}^k \), the minimal weight is encoded by the parameter \( k \). The parameter \( s \) is a complex number. The assumption that \( \pi_{\infty} \) be restricted to a bounded set means that the \( s \) and \( k \) are bounded for each \( v \mid \infty \).

We will show that for the quaternion algebra \( B \) such that \( \Sigma_B = \Sigma(\Pi) \) there exists a finite collection of vectors \( \mathcal{F}_1^B \subset \pi_1^B \) and \( \mathcal{F}_2^B \subset \pi_2^B \) such that for \( \varphi_3 \) a new vector,

\[
L(\frac{1}{2}, \Pi) \ll \epsilon, F, \pi_{3,\infty,n} N(p)^{1+\epsilon} \left| \int_{[B^\times]} \varphi_1(b)\varphi_2(b)\left(\frac{w^{-1}}{1}\right)\varphi_3(b)db \right|^2
\]

for some \( \varphi_i \in \mathcal{F}_i^B \). As a first step, we prove the following.

Proposition 5.1. Let \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \) be as in Theorem 1.3. Let \( B \) be the quaternion algebra such that \( \Sigma_B = \Sigma(\Pi) \). There exist vectors \( \varphi_i \in \pi_i^B \) such that for \( \varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \),

\[
L(\frac{1}{2}, \Pi) \ll N(p)^{1+\epsilon} \prod_{v \mid n_1n_2n_\infty} \left( I_v(\varphi_v) \right)^{-1} \left| \int_{[B^\times]} \varphi(b)db \right|^2
\]

where the implied constant is dependant on \( \epsilon, \Pi_{\infty}, N(n_1n_2n) \) and \( \varphi_{i,v} \) where \( i = 1, 2 \) and \( v \mid \infty n_1n_2n \).

\[
\text{Proof. We apply Theorem 2.1. Write } \pi_i^B = \otimes_v \pi_{i,v}. \text{ Clearly, we may assume that } L(\frac{1}{2}, \Pi) \neq 0. \text{ Hence Theorem 2.2 guarantees that we may choose some } \varphi_i = \otimes_v \varphi_{i,v} \text{ for } i = 1, 2, 3 \text{ such}
\]
that, writing \( \varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \), \( I(\varphi) \neq 0 \) and \( I_v(\varphi_v) \neq 0 \) for all \( v \). In particular, we may let \( \varphi_{1,v} \) be the (normalized) new vector for all \( v \). For such places \( v \), \( I_v(\varphi_v) = 1 \). Note that twisting \( \varphi_2 \) by \( \begin{pmatrix} \pi \end{pmatrix} \) means that \( I_p(\varphi_p) \) is as in Corollary 4.2, hence \( I_p(\varphi_p) \sim N(p)^{-1} \).

To deal with the term

\[
\prod_{j=1}^{3} \int_{[B \times]} |\varphi_j(b)|^2 \, db = \prod_{j=1}^{3} \frac{L(1, \pi_j, Ad)}{\int_{[G]} |\varphi(g)|^2 \, dg}
\]

in (2.4), we use the bound of Iwaniec[18] which says that if \( \varphi \in \pi \) is a new vector then

\[
(5.2) \quad \frac{L(1, \pi_i, Ad)}{\int_{[G]} |\varphi(g)|^2 \, dg} \ll N(\pi_i)^c
\]

as \( N(\pi_i) \to \infty \). Here, \( N(\pi_i) \) denotes the conductor of \( \pi_i \) and the implied constant depends continuously on the Langlands parameters of \( \pi_{1,\infty} \).

After solving for \( L(\frac{1}{2}, \Pi) \) in (2.4) and applying (5.2), (5.1) follows. Since we are fixing \( n_1, n_2 \) and \( n \), all of the terms in \( [N(\pi_1)N(\pi_2)N(\pi_3)]^c \) except for \( N(p)^c \) can be absorbed into the implied constant. \( \square \)

As a matter of terminology, we call \( \varphi_v \) a test vector if \( I_v(\varphi_v) \). Given test vectors for all \( v \) we can construct the global vector \( \varphi = \otimes_v \varphi_v \) such that \( I(\varphi) \neq 0 \). Therefore, given Proposition 5.1, Theorem 1.3 will follow provided we can show the following.

- Show that for each \( v \mid n_1, n_2, n_\infty \) there exists a finite sets of vectors such that as \( \pi_{3,\infty} \) and \( \pi_{3,p} \) vary a test vector can be chosen from the said finite sets.
- Show that the values of the corresponding linear forms is uniformly bounded.

We begin by bounding the terms \( I_v(\varphi_v)/\langle \varphi_v, \varphi_v \rangle \) for archimedean primes \( v \).

**Lemma 5.2.** Let \( \pi_{i,\infty} = \prod_v \pi_{i,v} \) be the archimedean parts of automorphic representations \( \pi_i \) for \( i = 1, 2, 3 \) such that \( \pi_1 \) and \( \pi_2 \) are fixed, and \( \pi_{3,\infty} \) is bounded. Assuming that \( L(\frac{1}{2}, \Pi) \neq 0 \), let \( B \) be the quaternion algebra such that \( \Sigma_B = \Sigma(\Pi) \). Then there exists a finite collection of vectors \( F_i^B \) in \( \pi_{i,\infty}^B \) for \( i = 1, 2 \) such that if \( \varphi_{3,\infty} \in \pi_{3,\infty}^B \) is a new vector then

\[
\prod_{v|\infty} I_v(\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{2,v}) \geq \delta
\]

for some choice of \( \varphi_{i,\infty} \in F_i \) and some \( \delta > 0 \).

**Proof.** If \( v \) is real, for each of the finitely many choices \( k \) for which \( \pi_{3,v} \) could be \( \pi_{3,v}^k \), Theorem 1.2 guarantees we may chose vectors \( \varphi_{1,v} \in \pi_{1,v}^B \) and \( \varphi_{2,v} \in \pi_{2,v}^B \) such that \( \varphi = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v} \) is a test vector.

Now assume that \( \pi_{3,v} = \pi_{3,v}^k \) is a principal series. Loke[22] gives a choice of vectors \( \varphi_{i,v} \in \pi_{i,v} \) (\( i = 1, 2 \)) independent of \( s \) such that for \( \varphi_{3,v} \) equal to the normalized new vector, the unique \((g, K)\)-invariant linear form, \( \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v} \) is a test vector, hence the matrix coefficient is nonzero.

Because the matrix coefficient attached to \( \varphi_{3,v} \) is a continuous function with respect to its Langlands parameter\(^5\), under the boundedness condition there must be a lower bound on

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\(^5\)The local \( L \)-factors are also continuous since they are products of gamma functions.
the values of the matrix coefficient. More explicitly, given Loke’s choice of $\varphi_{1,v}, \varphi_{2,v}$ we have a nonzero continuous map

$$\lambda : \Omega \rightarrow \mathbb{C}^\times \quad s \mapsto I_v(\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}^s)$$

where $\varphi_{3,v}^s \in \pi_v^s$ is the normalized new vector and $s \in \Omega \subset \mathbb{C}$. By the boundedness assumption implies we may assume $\Omega$ is compact, hence its image is uniformly bounded away from zero.

If $v$ is a complex place, Loke[22] again proved that there must be a test vector independent of $s$. The continuity argument of above again applies with $\Omega$ replaced by $\Omega \times \{k_1, \ldots, k_n\}$ where $k_i$ are the distinct nonnegative integers corresponding the possible weights. In this case, Loke does not give a test vector independent of $s$. However, a test vector for a given value $s$, by continuity, will also be a test vector for some open subset $U_s \subset \Omega$. Since $\Omega$ is compact, there exist finitely many values $s_i$ such that $\bigcup_i U_s_i = \Omega$.

The finite sets $\mathcal{F}_{i,\infty}$ is obtained by taking the union of all of the vectors obtained above. 

In each of Corollary 4.2, Proposition 4.3, Proposition 4.4 and Proposition 4.5 the value of $I_v(\varphi_v)/\langle \varphi_v, \varphi_v \rangle$ is constant even if one of the representations is varied. That is to say, suppose $v = q$ is a finite prime such that $q^2 \nmid n$, for $i = 1, 2$ and $q^2 \nmid n$. In particular, $\pi_{3,q} = \pi(\mu, \mu^{-1})$ or $\sigma_{\mu}$ for some unramified character $\mu$ if $v$ is finite. Hence there is a choice of $\varphi_v$ such that $I_v(\varphi_v)/\langle \varphi_v, \varphi_v \rangle$ is nonzero and independent of $\mu$. Hence, this completes the proof of Theorem 1.3 in the case that $n_1, n_2, n$ are squarefree.

Although we believe that this phenomenon—the existence a test vector such that $I_v(\varphi_v)$ depends only on the level of $\pi_{3,v}$—should hold more generally, it is not a priori evident. However, the following lemma allows us to conclude that $n_1, n_2, n$ may be taken arbitrarily.

**Lemma 5.3.** Fix irreducible admissible representations $\pi_{1,v}, \pi_{2,v}$ of $\text{GL}_2(F_v)$ that are local components of global automorphic representations. There exist finite collections $\mathcal{F}_{1,v} \subset \pi_{1,v}$ and $\mathcal{F}_{2,v} \subset \pi_{2,v}$ and a constant $\delta > 0$ such that if $\pi_{3,v}$ is any other such representation of fixed level such that $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$ admits a $\text{GL}_2(F_v)$-invariant linear form,

$$I_v(\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}) \geq \delta$$

for $\varphi_{3,v}$ the new vector and some $\varphi_{1,v} \in \mathcal{F}_{1,v}$.

**Proof.** Let $\omega_i$ be the central character associated to $\pi_i,v$. A necessary condition for the existence of an invariant form is that $\omega_1 \omega_2 \omega_3$ be trivial, hence we may assume without loss of generality that $\omega_i,v$ has trivial central character for each $i = 1, 2, 3$. Moreover, since $\pi_i,v$ is the local component of an automorphic representation it is unitary.

The level of a principal series representation $\pi(\mu, \mu^{-1})$ is twice the level of the character $\mu$. We conclude that principal series representations of fixed level and trivial central character are parametrized by characters $\chi : F_v^\times \rightarrow \mathbb{C}$ with fixed level.

Let $\varpi$ be a uniformizer, $q_v$ the order of the residue field, and let

$$U(i) = \begin{cases} \{1 + u\varpi^i \mid u \in \mathcal{O}^\times \} & \text{if } i \geq 1 \\ \mathcal{O}^\times & \text{if } i = 0. \end{cases}$$

As is well known, for $\chi$ to be a character of level $n$ means that

$$\chi(u \varpi^k) = \varpi^{nk} \chi(u)$$
where $\tilde{\chi}$ is a character of the finite group $O^\times/U^{(n)}$. If $\pi(\chi,\chi^{-1})$ comes from an automorphic representations, we may assume $s = s_\chi$ is a complex number with Re$(s) \in [0, \lambda]$ where $\lambda$ is any bound towards Ramanujan. Note we may also assume that Im$(s)$ is bounded. In other words, $s$ is restricted to a compact set.

Let $\pi_{3,v} \simeq \pi(\chi_0, \chi_0^{-1})$ be such that $I_v \neq 0$, and denote the new vector by $\varphi_{3,v}$. Given $\varphi_{1,v}, \varphi_{2,v}$ such that $\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ is a test vector, as in the proof of Lemma 5.2, $\bar{\omega}_{1,v} \otimes \bar{\omega}_{2,v} \otimes \varphi_{3,v}$ is also a test vector for $\pi_{3,v} \simeq \pi(\mu, \mu^{-1})$ where $\bar{\mu} = \bar{\chi}$ and $s_\mu$ is in some open set containing $s_\chi$. Since the set of possible $s_\chi$ is compact, and the number of characters $\bar{\chi}$ is finite, we find that a finite choice of choices $\varphi_{1,v}, \varphi_{2,v}$ suffice to give a set of test vectors for $\pi_{3,v}$ any principal series representation.

A completely analogous argument gives a finite set of choices for $\varphi_{1,v}, \varphi_{2,v}$ in the case that $\pi_{3,v} \simeq \sigma_\chi$. In this case, however, the argument is simplified by the fact that $\chi$ must be unitary, i.e. Re$(s_\chi) = 0$.

Finally, if $\pi_{3,v}$ is a supercuspidal representation then (by [7]) its Kirillov model associated to a nontrivial additive character $\psi$ is completely determined by the level of $\pi_{3,v}$ and the epsilon factor $\epsilon(\frac{1}{2}, \pi_{3,v}, \psi)$. The assumption that $\omega_1$ is trivial forces the epsilon factor to be $\pm 1$. So, at worst, two more choices for each of $\varphi_{1,v}$ and $\varphi_{2,v}$ will suffice.

The existence of the constant $\delta$ is established in exactly the same fashion as in Lemma 5.2. Namely, values assumed by $I_v$ on the given set of test vectors is the image of a compact set (corresponding to the finite number of characters of $O^\times/U^{(n)}$ and the admissible values of $s_\chi$).

This completes the proof of Theorem 1.3. \hfill $\square$

Remark. In [8] explicit test vectors are given for all cases except that where all three representations $\pi_{1,v}$ are supercuspidals. This can be used to make the choices of $\mathcal{F}_{1,v}, \mathcal{F}_{2,v}$ in Lemma 5.3 explicit. However, note that the argument above is still needed to ensure the existence of $\delta$.

References

[16] Atsushi Ichino and Tamotsu Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture.

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