

# Twistor Geometry and the Standard Model in Euclidean Space

*Draft version*

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## 1 Introduction

The structure of relativistic quantum field theory is largely determined by the 10-dimensional Poincaré group  $\mathbf{R}^{3,1} \ltimes Spin(3,1)$  of space-time symmetries of

Minkowski space. The behavior of such quantum field theories is often best studied as an analytic continuation (“Wick rotation”) from the theory defined in Euclidean space, which has symmetry group the Euclidean group  $\mathbf{R}^4 \times Spin(4)$ . For massless particles or for the short-distance behavior in the massive case, a larger possible space-time symmetry group extending the Poincaré group is the 16-dimensional conformal group  $Spin(4, 2) = SU(2, 2)$ , which is related by analytic continuation to the conformal group  $Spin(5, 1) = SL(2, \mathbf{H})$  extending the Euclidean group ( $\mathbf{H}$  are the quaternions).

A compelling way to relate by analytic continuation the Euclidean and Minkowski conformally invariant theories of spinor fields is Penrose’s twistor theory, in which conformally compactified Euclidean and Minkowski spaces are two real slices of the same complexified space: the Grassmanian  $G_{2,4}(\mathbf{C})$  of complex 2-planes in the twistor space  $T = \mathbf{C}^4$ . A point in space-time is a complex 2-plane, providing tautologically the space of spinor degrees of freedom at the point. Instead of formulating a theory in space-time, it can be formulated in the twistor space  $T$ , or its projective version,  $PT = \mathbf{C}P^3$ , the space of complex lines in  $T$ . If one identifies twistor space  $T$  with not just  $\mathbf{C}^4$  but  $\mathbf{H}^2$ , then one can identify the compactified Euclidean real slice  $S^4$  with  $\mathbf{H}P^1$ , the space of quaternionic lines in twistor space  $\mathbf{H}^2$ .  $PT$  is then fibered over  $S^4$ , with fiber above a point the space  $\mathbf{C}P^1 = S^2$  of complex lines lying in the quaternionic line that defines the point of  $S^4$ .

Taking together the twistor point of view on space-time and the Euclidean space version of quantum field theory as fundamental, it is a remarkable fact that the specific internal symmetry groups and degrees of freedom of the Standard Model appear naturally:

- Projective twistor space  $PT$  can be thought of as

$$\mathbf{C}P^3 = \frac{SU(4)}{U(1) \times SU(3)}$$

or as

$$\frac{Sp(2)}{U(1) \times SU(2)}$$

So there are  $U(1), SU(2)$  and  $SU(3)$  internal symmetry groups at each point in projective twistor space.

- In Euclidean space-time quantization, the definition of the space of states requires singling out a specific direction in Euclidean space that will be the time direction. Lifting the choice of a tangent vector in the time direction from Euclidean space-time to  $PT$ , the internal  $U(1) \times SU(2)$  acts on this degree of freedom in the same way the Standard Model electroweak symmetry acts on the Higgs field.
- The degrees of freedom of a spinor on  $PT$  transform under internal and space-time symmetries like a generation of Standard Model fermions.

This implies a picture of fundamental physics in which the Higgs field picks out the direction of imaginary time and thus determines the state space of the theory. The electroweak  $SU(2)$  is spontaneously broken in the sense that changing the direction of the Higgs field changes the state space.

For much older embryonic work on this topic, see [22]. For more about Euclidean space quantization see [24], and for more about the fundamental role of the Dirac operator in quantization see [25].

## 2 Twistor geometry

Twistor geometry is a 1967 proposal [15] due to Roger Penrose for a very different way of formulating four-dimensional space-time geometry. For a detailed expository treatment of the subject, see [18] (for a version aimed at physicists and applications in amplitude calculations, see [1]). Fundamental to twistor geometry is the twistor space  $T = \mathbf{C}^4$ , as well as its projective version, the space  $PT = \mathbf{C}P^3$  of complex lines in  $T$ . The relation to space-time is that complexified and compactified space-time is identified with the Grassmanian  $M = G_{2,4}(\mathbf{C})$  of complex two-dimensional linear subspaces in  $T$ . A space-time point is thus a  $\mathbf{C}^2$  in  $\mathbf{C}^4$  which tautologically provides the spinor degree of freedom at that point. The spinor bundle  $S$  is the tautological two-dimensional complex vector bundle over  $M$  whose fiber  $S_m$  at a point  $m \in M$  is the  $\mathbf{C}^2$  that defines the point.

The group  $SL(4, \mathbf{C})$  acts on  $T$  and transitively on the spaces  $PT$  and  $M$  of its complex subspaces. Points in the Grassmanian  $M$  can be represented as elements

$$\omega = (v_1 \otimes v_2 - v_2 \otimes v_1) \in \Lambda^2(\mathbf{C}^4)$$

by taking two vectors  $v_1, v_2$  spanning the subspace.  $\Lambda^2(\mathbf{C}^4)$  is six complex dimensional and scalar multiples of  $\omega$  gives the same point in  $M$ , so  $\omega$  identifies  $M$  with a subspace of  $P(\Lambda^2(\mathbf{C}^4)) = \mathbf{C}P^5$ . Such  $\omega$  satisfy the equation

$$\omega \wedge \omega = 0 \tag{1}$$

which identifies (the ‘‘Klein correspondence’’)  $M$  with a submanifold of  $\mathbf{C}P^5$  given by a non-degenerate quadratic form. Twistors are spinors in six dimensions, with the action of  $SL(4, \mathbf{C})$  on  $\Lambda^2(\mathbf{C}^4) = \mathbf{C}^6$  preserving the quadratic form 1, and giving the spin double cover homomorphism

$$SL(4, \mathbf{C}) = Spin(6, \mathbf{C}) \rightarrow SO(6, \mathbf{C})$$

To get the tangent bundle of  $M$ , one needs not just the spinor bundle  $S$ , but also another two complex-dimensional vector bundle, the quotient bundle  $S^\perp$  with fiber  $S_m^\perp = \mathbf{C}^4/S_m$ . Then the tangent bundle is

$$TM = Hom(S, S^\perp) = S^* \otimes \perp$$

with the tangent space  $T_m M$  a four complex dimensional vector space given by the  $Hom(S_m, S_m^\perp)$ , the linear maps from  $S_m$  to  $S_m^\perp$ .

A choice of coordinate chart on  $M$  is given by picking a point  $m \in M$  and identifying  $S_m^\perp$  with a complex two plane transverse to  $S_m$ . The point  $m$  will be the origin of our coordinate system, so we will denote  $S_m$  by  $S_0$  and  $S_m^\perp$  by  $S_0^\perp$ . Now  $T = S_0 \oplus S_0^\perp$  and one can choose basis elements  $\mathbf{e}_1, \mathbf{e}_2 \in S_0$ ,  $\mathbf{e}_3, \mathbf{e}_4 \in S_0^\perp$  for  $T$ . The coordinate of the two-plane spanned by the columns of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{01} & z_{01} \\ z_{10} & z_{11} \end{pmatrix}$$

will be the 2 by 2 complex matrix

$$Z = \begin{pmatrix} z_{01} & z_{01} \\ z_{10} & z_{11} \end{pmatrix}$$

This coordinate chart does not include all of  $M$ , since it misses those points in  $M$  corresponding to complex two-planes that are not transverse to  $S_0^\perp$ . Our interest however will ultimately be not in the global structure of  $M$ , but in its local structure near the chosen point  $m$ , which we will study using the 2 by 2 complex matrix  $Z$  as coordinates. When we discuss  $M$  we will sometimes not distinguish between  $M$  and its local version as a complex four-dimensional vector space with origin of coordinates at  $m$ .

Writing elements of  $T$  as

$$\begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix}$$

an element of  $T$  will be in the complex two plane with coordinate  $Z$  when

$$\begin{pmatrix} s_1^\perp \\ s_2^\perp \end{pmatrix} = Z \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad (2)$$

This incidence equation characterizes in coordinates the relation between lines (elements of  $PT$ ) and planes (elements of  $M$ ) in twistor space  $T$ . We'll sometimes also write this as

$$s^\perp = Zs$$

An  $SL(4, \mathbf{C})$  determinant 1 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

acts on  $T$  by

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix} \rightarrow \begin{pmatrix} As + Bs^\perp \\ Cs + Ds^\perp \end{pmatrix}$$

On lines in the plane  $Z$  this is

$$\begin{bmatrix} s \\ Zs \end{bmatrix} \rightarrow \begin{bmatrix} As + BZs \\ Cs + DZs \end{bmatrix} = \begin{bmatrix} (A + BZ)s \\ (C + DZ)(A + BZ)^{-1}(A + BZ)s \end{bmatrix}$$

so the corresponding action on  $M$  will be given by

$$Z \rightarrow (C + DZ)(A + BZ)^{-1}$$

Since  $\Lambda^2(S_0) = \Lambda^2(S_0^\perp) = \mathbf{C}$ ,  $S_0$  and  $S_0^\perp$  have (up to scalars) unique choices  $\epsilon_{S_0}$  and  $\epsilon_{S_0^\perp}$  of non-degenerate antisymmetric bilinear form, and corresponding choices of  $SL(2, \mathbf{C}) \subset GL(2, \mathbf{C})$  acting on  $S_0$  and  $S_0^\perp$ . These give (again, up to scalars), a unique choice of a non-degenerate symmetric form on  $Hom(S_0, S_0^\perp)$ , such that

$$\langle Z, Z \rangle = \det Z$$

The subgroup

$$Spin(4, \mathbf{C}) = SL(2, \mathbf{C}) \times SL(2, \mathbf{C}) \subset SL(4, \mathbf{C})$$

of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with

$$\det A = \det D = 1$$

acts on  $M$  in coordinates by

$$Z \rightarrow DZA^{-1}$$

preserving  $\langle Z, Z \rangle$ .

Besides the spaces  $PT$  and  $M$  of complex lines and planes in  $T$ , it is also useful to consider the correspondence space whose elements are complex lines inside a complex plane in  $T$ . This space can also be thought of as  $P(S)$ , the projective spinor bundle over  $M$ . There is a diagram of maps

$$\begin{array}{ccc} & P(S) & \\ \mu \swarrow & & \searrow \nu \\ PT & & M \end{array}$$

where  $\nu$  is the projection map for the bundle  $P(S)$  and  $\mu$  is the identification of a complex line in  $S$  as a complex line in  $T$ .  $\mu$  and  $\nu$  give a correspondence between geometric objects in  $PT$  and  $M$ . One can easily see that  $\mu(\nu^{-1}(m))$  is the complex projective line in  $PT$  corresponding to a point  $m \in M$  (a complex two plane in  $T$  is a complex projective line in  $PT$ ). In the other direction,  $\nu(\mu^{-1}(p))$  takes a point  $p$  in  $PT$  to  $\alpha(p)$ , a copy of  $\mathbf{C}P^2$  in  $M$ , called the “ $\alpha$ -plane” corresponding to  $p$ .

In our chosen coordinate chart, this diagram of maps is given by

$$\begin{array}{ccc} & (Z, s) \in P(S) & \\ \mu \swarrow & & \searrow \nu \\ \begin{bmatrix} s \\ Zs \end{bmatrix} \in PT & & Z \in M \end{array}$$

The incidence equation 2 relating  $PT$  and  $M$  implies that an  $\alpha$ -plane is a null plane in the metric discussed above. Given two points  $Z_1, Z_2$  in  $M$  corresponding to the same point in  $PT$ , their difference satisfies

$$s^\perp = (Z_1 - Z_2)s = 0$$

$Z_1 - Z_2$  is not an invertible matrix, so has determinant 0 and is a null vector.

### 3 The Penrose-Ward transform

The Penrose transform relates solutions of conformally-invariant wave equations on  $M$  to sheaf cohomology groups, identifying

- Solutions to a helicity  $\frac{k}{2}$  holomorphic massless wave equation on  $U$ .
- The sheaf cohomology group

$$H^1(\widehat{U}, \mathcal{O}(-k-2))$$

Here  $U \subset M$  and  $\widehat{U} \subset PT$  are open sets related by the twistor correspondence, i.e.

$$\widehat{U} = \mu(\nu^{-1}(U))$$

We will be interested in cases where  $U$  and  $\widehat{U}$  are orbits in  $M$  and  $PT$  for a real form of  $SL(4, \mathbf{C})$ . Here  $\mathcal{O}(-k-2)$  is the sheaf of holomorphic sections of the line bundle  $L^{\otimes(-k-2)}$  where  $L$  is the tautological line bundle over  $PT$ . For a detailed discussion, see for instance chapter 7 of [18].

The Penrose-Ward transform is a generalization of the above, introducing a coupling to gauge fields. One aspect of this is the Ward correspondence, an isomorphism between

- Holomorphic anti-self-dual  $GL(n, \mathbf{C})$  connections  $A$  on  $U \subset M$ .
- Holomorphic rank  $n$  vector bundles  $E$  over  $\widehat{U} \subset PT$ .

Here “anti-self-dual” means the curvature of the connection satisfies

$$*F_A = -F_A$$

where  $*$  is the Hodge dual. There are some restrictions on the open set  $U$ , and  $E$  needs to be trivial on the complex projective lines corresponding to points  $m \in U$ .

In one direction, the above isomorphism is due to the fact that the curvature  $F_A$  is anti-self-dual exactly when the connection  $A$  is integrable on the intersection of an  $\alpha$ -plane with  $U$ . One can then construct the fiber  $E_p$  of  $E$  at  $p$  as the covariantly constant sections of the bundle with connection on the corresponding  $\alpha$ -plane in  $M$ . In the other direction, one can construct a vector bundle  $\widetilde{E}$  on  $U$  by taking as fiber at  $m \in U$  the holomorphic sections of  $E$

on the corresponding complex projective line in  $PT$ . Parallel transport in this vector bundle can be defined using the fact that two points  $m_1, m_2$  in  $U$  on the same  $\alpha$ -plane correspond to intersecting projective lines in  $PT$ . For details, see chapter 8 of [18] and chapter 10 of [13].

Given an anti-self-dual gauge field as above, the Penrose transform can be generalized to a Penrose-Ward transform, relating

- Solutions to a helicity  $k$  holomorphic massless wave equation on  $U$ , coupled to a vector bundle  $\tilde{E}$  with anti-self-dual connection  $A$ .
- The sheaf cohomology group

$$H^1(\hat{U}, \mathcal{O}(E)(-k-2))$$

For more about this generalization, see [7].

## 4 Twistor geometry and real forms

So far we have only considered complex twistor geometry, in which the relation to space-time geometry is that  $M$  is a complexified version of a four real dimensional space-time. From the point of view of group symmetry, the Lie algebra of  $SL(4, \mathbf{C})$  is the complexification

$$\mathfrak{sl}(4, \mathbf{C}) = \mathfrak{g} \otimes \mathbf{C}$$

for several different real Lie algebras  $\mathfrak{g}$ , which are the real forms of  $\mathfrak{sl}(4, \mathbf{C})$ . To organize the possibilities, recall that  $SL(4, \mathbf{C})$  is  $Spin(6, \mathbf{C})$ , the spin group for orthogonal linear transformations in six complex dimensions, so  $\mathfrak{sl}(4, \mathbf{C}) = \mathfrak{so}(6, \mathbf{C})$ . If we instead consider orthogonal linear transformations in six real dimensions, there are different possible signatures of the inner product to consider, all of which become equivalent after complexification. This corresponds to the possible real forms

$$\mathfrak{g} = \mathfrak{so}(3, 3), \mathfrak{so}(4, 2), \mathfrak{so}(5, 1), \text{ and } \mathfrak{so}(6)$$

which we will discuss (there's another real form,  $\mathfrak{su}(3, 1)$ , which we won't consider). For more about real methods in twistor theory, see [26].

### 4.1 $Spin(3, 3) = SL(4, \mathbf{R})$

The simplest way to get a real version of twistor geometry is to take the discussion of section 2 and replace complex numbers by real numbers. Equivalently, one can look at subspaces invariant under the usual conjugation, given by the map  $\sigma$

$$\sigma \begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix} = \begin{pmatrix} \overline{s_1} \\ \overline{s_2} \\ \overline{s_1^\perp} \\ \overline{s_2^\perp} \end{pmatrix}$$

which acts not just on  $T$  but on  $PT$  and  $M$ . The fixed point set of the action on  $M$  is  $M^{2,2} = G_{2,4}(\mathbf{R})$ , the Grassmanian of real two-planes in  $\mathbf{R}^4$ . As a manifold,  $G_{2,4}(\mathbf{R})$  is  $S^2 \times S^2$ , quotiented by a  $\mathbf{Z}_2$ .  $M^{2,2}$  is acted on by the group  $Spin(3,3) = SL(4, \mathbf{R})$  of conformal transformations.  $\sigma$  acting on  $PT$  acts on the  $\mathbf{C}P^1$  corresponding to a point in  $M^{2,2}$  with an action whose fixed points form an equatorial circle.

Coordinates can be chosen as in the complex case, but with everything real. A point in  $M^{2,2}$  is given by a real 2 by 2 matrix, which can be written in the form

$$Z = \begin{pmatrix} x_0 + x_3 & x_1 - x_2 \\ x_1 + x_2 & x_0 - x_3 \end{pmatrix}$$

for real numbers  $x_0, x_1, x_2, x_3$ .  $M^{2,2}$  is acted on by the group  $Spin(3,3) = SL(4, \mathbf{R})$  of conformal transformations as in the complex case by

$$Z \rightarrow (C + DZ)(A + BZ)^{-1}$$

with the subgroup of rotations

$$Z \rightarrow DZA^{-1}$$

for  $A, D \in SL(2, \mathbf{R})$  given by

$$Spin(3,3) = SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$$

This subgroup preserves

$$\langle Z, Z \rangle = \det Z = x_0^2 - x_3^2 - x_1^2 + x_2^2$$

For the Penrose transform in this case, see Atiyah's account in section 6.5 of [2]. For the Ward correspondence, see section 10.5 of [13].

## 4.2 $Spin(4,2) = SU(2,2)$

The real case of twistor geometry most often studied (a good reference is [18]) is that where the real space-time is the physical Minkowski space of special relativity. The conformal compactification of Minkowski space is a real submanifold of  $M$  which we'll call  $M^{3,1}$ . It is acted upon transitively by the conformal group  $Spin(4,2) = SU(2,2)$ . This conformal group action on  $M^{3,1}$  is most naturally understood using twistor space, as the action on complex planes in  $T$  coming from the action of the real form  $SU(2,2) \subset SL(4, \mathbf{C})$  on  $T$ .

$SU(2,2)$  is the subgroup of  $SL(4, \mathbf{C})$  preserving a real Hermitian form  $\Phi$  of signature (2,2) on  $T = \mathbf{C}^4$ . In our coordinates for  $T$ , a standard choice for  $\Phi$  is given by

$$\Phi \left( \begin{pmatrix} s \\ s^\perp \end{pmatrix}, \begin{pmatrix} s' \\ (s^\perp)^\perp \end{pmatrix} \right) = \begin{pmatrix} \bar{s} & \overline{s^\perp} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s' \\ (s^\perp)^\perp \end{pmatrix} = s^\dagger (s^\perp)^\perp + (s^\perp)^\dagger s' \quad (3)$$



Minkowski space is given by complex planes on which  $\Phi = 0$ , so

$$\Phi \left( \begin{pmatrix} s \\ Zs \end{pmatrix}, \begin{pmatrix} s \\ Zs \end{pmatrix} \right) = s^\dagger (Z + Z^\dagger) s = 0$$

Thus coordinates of points on Minkowski space are anti-Hermitian matrices  $Z$ , which can be written in the form

$$Z = -i \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = -i(x_0 \mathbf{1} + \mathbf{x} \cdot \boldsymbol{\sigma})$$

where  $\sigma_j$  are the Pauli matrices. The metric is the usual Minkowski metric, since

$$\langle Z, Z \rangle = \det Z = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

One can identify compactified Minkowski space  $M^{3,1}$  as a manifold with the Lie group  $U(2)$  which is diffeomorphic to  $(S^3 \times S^1)/\mathbf{Z}_2$ . The identification of the tangent space with anti-Hermitian matrices reflects the usual identification of the tangent space of  $U(2)$  at the identity with the Lie algebra of anti-Hermitian matrices.

$SL(4, \mathbf{C})$  matrices are in  $SU(2, 2)$  when they satisfy

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Poincaré subgroup  $P$  of  $SU(2, 2)$  is given by elements of  $SU(2, 2)$  of the form

$$\begin{pmatrix} A & 0 \\ C & (A^\dagger)^{-1} \end{pmatrix}$$

where  $A \in SL(2, \mathbf{C})$  and  $A^\dagger C = -C^\dagger A$ . These act on Minkowski space by

$$Z \rightarrow (C + (A^\dagger)^{-1}Z)A^{-1} = (A^\dagger)^{-1}ZA^{-1} + CA^{-1}$$

One can show that  $CA^{-1}$  is anti-Hermitian and gives arbitrary translations on Minkowski space. The Lorentz subgroup is  $Spin(3, 1) = SL(2, \mathbf{C})$  acting by

$$Z \rightarrow (A^\dagger)^{-1}ZA^{-1}$$

Here  $SL(2, \mathbf{C})$  is acting by the standard representation on  $S_0$ , and by the conjugate-dual representation on  $S_0^\perp$ .

Note that, for the action of the Lorentz  $SL(2, \mathbf{C})$  subgroup, twistors written as elements of  $S_0 \oplus S_0^\perp$  behave like usual Dirac spinors (direct sums of a standard  $SL(2, \mathbf{C})$  spinor and one in the conjugate-dual representation), with the usual Dirac adjoint, in which the  $SL(2, \mathbf{C})$ -invariant inner product is given by the signature (2, 2) Hermitian form

$$\langle \psi_1, \psi_2 \rangle = \psi_1^\dagger \gamma_0 \psi_2$$

Twistors, with their  $SU(2, 2)$  conformal group action and incidence relation to space-time points, are however something different than Dirac spinors.

The  $SU(2, 2)$  action on  $M$  has six orbits:  $M_{++}, M_{--}, M_{+0}, M_{-0}, M_{00}$ , where the subscript indicates the signature of  $\Phi$  restricted to planes corresponding to points in the orbit. The last of these is a closed orbit  $M^{3,1}$ , compactified Minkowski space. Acting on projective twistor space  $PT$ , there are three orbits:  $PT_+, PT_-, PT_0$ , where the subscript indicates the sign of  $\Phi$  restricted to the line in  $T$  corresponding to a point in the orbit. The first two are open orbits with six real dimensions, the last a closed orbit with five real dimensions. The points in compactified Minkowski space  $M_{00} = M^{3,1}$  correspond to projective lines in  $PT$  that lie in the five dimensional space  $PT_0$ . Points in  $M_{++}$  and  $M_{--}$  correspond to projective lines in  $PT_+$  or  $PT_-$  respectively.

One can construct infinite dimensional irreducible unitary representations of  $SU(2, 2)$  using holomorphic geometry on  $PT_+$  or  $M_{++}$ , with the Penrose transform relating the two constructions [8]. For  $\overline{PT}_+$  the closure of the orbit  $PT_+$ , the Penrose transform identifies the sheaf cohomology groups  $H^1(\overline{PT}_+, \mathcal{O}(-k-2))$  for  $k > 0$  with holomorphic solutions to the helicity  $\frac{k}{2}$  wave equation on  $M_{++}$ . Taking boundary values on  $M^{3,1}$ , these will be real-analytic solutions to the helicity  $\frac{k}{2}$  wave equation on compactified Minkowski space. If one instead considers the sheaf cohomology  $H^1(PT_+, \mathcal{O}(-k-2))$  for the open orbit  $PT_+$  and takes boundary values on  $M^{3,1}$  of solutions on  $M_{++}$ , the solutions will be hyperfunctions, see [19].

The Ward correspondence relates holomorphic vector bundles on  $PT_+$  with anti-self-dual  $GL(n, \mathbf{C})$  gauge fields on  $M_{++}$ . However, in this Minkowski signature case, all solutions to the anti-self-duality equations as boundary values of such gauge fields are complex, so one does not get anti-self-dual gauge fields for compact gauge groups like  $SU(n)$ .

### 4.3 $Spin(5, 1) = SL(2, \mathbf{H})$

Changing from Minkowski space-time signature 3,1 to Euclidean space-time signature 4,0, the compactified space-time  $M^4 = S^4$  is again a real submanifold of  $M$ . To understand the conformal group and how twistors work in this case, it is best to work with quaternions instead of complex numbers, identifying  $T = \mathbf{H}^2$ . When working with quaternions, one can often instead use corresponding 2 by 2 matrices, with a standard choice

$$q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \leftrightarrow q_0 - i(q_1 \sigma_1 + q_2 \sigma_2 + q_3 \sigma_3)$$

For more details of the quaternionic geometry that appears here, see [2] or [17]

The relevant conformal group acting on  $S^4$  is  $Spin(5, 1) = SL(2, \mathbf{H})$ , again best understood in terms of twistors and the linear action of  $SL(2, \mathbf{H})$  on  $T = \mathbf{H}^2$ . The group  $SL(2, \mathbf{H})$  is the group of quaternionic 2 by 2 matrices satisfying a single condition that one can think of as setting the determinant to one, although the usual determinant does not make sense in the quaternionic case. Here one can interpret the determinant using the isomorphism with complex matrices, or, at the Lie algebra level,  $\mathfrak{sl}(2, \mathbf{H})$  is the Lie algebra of 2 by 2 quaternionic matrices with purely imaginary trace.

While one can continue to think of points in  $S^4 \subset M$  as complex two planes, one can also identify these complex two planes as quaternionic lines and  $S^4$  as  $\mathbf{H}P^1$ , the projective space of quaternionic lines in  $\mathbf{H}^2$ . The conventional choice of identification between  $\mathbf{C}^2$  and  $\mathbf{H}$  is

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \leftrightarrow s = s_1 + s_2 \mathbf{j}$$

One can then think of the quaternionic structure as providing an alternate notion of conjugation than the usual one, given instead by left multiplying by  $j \in \mathbf{H}$ . Using  $jzj = -\bar{z}$  one can show that

$$\sigma \begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix} = \begin{pmatrix} -\bar{s}_2 \\ \bar{s}_1 \\ -s_2^\perp \\ s_1^\perp \end{pmatrix} \quad (4)$$

$\sigma$  satisfies  $\sigma^2 = -1$  on  $T$ , so  $\sigma^2 = 1$  on  $PT$ . We will see later that while  $\sigma$  has no fixed points on  $PT$ , it does fix complex projective lines.

The same coordinates we used in the complex case can be used here, where now  $S_0^\perp$  is a quaternionic line transverse to  $S_0$ , so coordinates on  $T$  are the pair of quaternions

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix}$$

These are also homogeneous coordinates for points on  $S^4 = \mathbf{H}P^1$  and our choice of  $Z \in \mathbf{H}$  given by

$$\begin{pmatrix} s \\ Zs \end{pmatrix}$$

as the coordinate in a coordinate system with origin the point with homogeneous coordinates

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The point at  $\infty$  will be the one with homogeneous coordinates

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is the quaternionic version of the usual sort of choice of coordinates in the case of  $S^2 = \mathbf{C}P^1$ , replacing complex numbers by quaternions. The coordinate of a point on  $S^4$  with homogeneous coordinates

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix}$$

will be

$$s^\perp s^{-1} = \frac{(s_1^\perp + s_2^\perp \mathbf{j})(\bar{s}_1 - s_1 \mathbf{j})}{|s_1|^2 + |s_2|^2} = \frac{s_1^\perp \bar{s}_1 + s_2^\perp \bar{s}_2 + (-s_1^\perp s_2 + s_2^\perp s_1) \mathbf{j}}{|s_1|^2 + |s_2|^2} \quad (5)$$

A coordinate of a point will now be a quaternion  $Z = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$  corresponding to the 2 by 2 complex matrix

$$Z = x_0\mathbf{1} - i\mathbf{x} \cdot \sigma = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & x_0 + ix_3 \end{pmatrix}$$

The metric is the usual Euclidean metric, since

$$\langle Z, Z \rangle = \det Z = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

The conformal group  $SL(2, \mathbf{H})$  acts on  $T = \mathbf{H}^2$  by the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $A, B, C, D$  are now quaternions, satisfying together the determinant 1 condition. These act on the coordinate  $Z$  as in the complex case, by

$$Z \rightarrow (C + DZ)(A + BZ)^{-1}$$

The Euclidean group in four dimensions will be the subgroup of elements of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

such that  $A$  and  $D$  are independent unit quaternions, thus in the group  $Sp(1) = SU(2)$ , and  $C$  is an arbitrary quaternion. The Euclidean group acts by

$$Z \rightarrow DZA^{-1} + CA^{-1}$$

with the spin double cover of the rotational subgroup now  $Spin(4) = Sp(1) \times Sp(1)$ . Note that spinors behave quite differently than in Minkowski space: there are independent unitary  $SU(2)$  actions on  $S_0$  and  $S_0^\perp$  rather than a non-unitary  $SL(2, \mathbf{C})$  action on  $S_0$  that acts at the same time on  $S_0^\perp$  by the conjugate transpose representation.

The projective twistor space  $PT$  is fibered over  $S^4$  by complex projective lines

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & PT = \mathbf{C}P^3 \\ & & \downarrow \pi \\ & & S^4 = \mathbf{H}P^1 \end{array} \quad (6)$$

The projection map  $\pi$  is just the map that takes a complex line in  $T$  identified with  $\mathbf{H}^2$  to the corresponding quaternionic line it generates (multiplying elements by arbitrary quaternions). In this case the conjugation map  $\sigma$  of 4 has no fixed points on  $PT$ , but does fix the complex projective line fibers and thus the points in  $S^4 \subset M$ . The action of  $\sigma$  on a fiber takes a point on the sphere to the opposite point, so has no fixed points.

Note that the Euclidean case of twistor geometry is quite different and much simpler than the Minkowski one. The correspondence space  $P(S)$  (here the

complex lines in the quaternionic line specifying a point in  $M^4 = S^4$ ) is just  $PT$  itself, and the twistor correspondence between  $PT$  and  $S^4$  is just the projection  $\pi$ . Unlike the Minkowski case where the real form  $SU(2, 2)$  has a non-trivial orbit structure when acting on  $PT$ , in the Euclidean case the action of the real form  $SL(2, \mathbf{H})$  is transitive on  $PT$ .

In the Euclidean case, the projective twistor space has another interpretation, as the bundle of orientation preserving orthogonal complex structures on  $S^4$ . A complex structure on a real vector space  $V$  is a linear map  $J$  such that  $J^2 = -1$ , providing a way to give  $V$  the structure of a complex vector space (multiplication by  $i$  is multiplication by  $J$ ).  $J$  is orthogonal if it preserves an inner product on  $V$ . While on  $\mathbf{R}^2$  there is just one orientation-preserving orthogonal complex structure, on  $\mathbf{R}^4$  the possibilities can be parametrized by a sphere  $S^2$ . The fiber  $S^2 = \mathbf{C}P^1$  of 6 above a point on  $S^4$  can be interpreted as the space of orientation preserving orthogonal complex structures on the four real dimensional tangent space to  $S^4$  at that point.

One way of exhibiting these complex structures on  $\mathbf{R}^4$  is to identify  $\mathbf{R}^4 = \mathbf{H}$  and then note that, for any real numbers  $x_1, x_2, x_3$  such that  $x_1^2 + x_2^2 + x_3^2 = 1$ , one gets an orthogonal complex structure on  $\mathbf{R}^4$  by taking

$$J = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

Another way to see this is to note that the rotation group  $SO(4)$  acts on orthogonal complex structures, with a  $U(2)$  subgroup preserving the complex structure, so the space of these is  $SO(4)/U(2)$ , which can be identified with  $S^2$ .

More explicitly, in our choice of coordinates, the projection map is

$$\pi : \begin{bmatrix} s \\ s^\perp = Zs \end{bmatrix} \rightarrow Z = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & x_0 + ix_3 \end{pmatrix}$$

For any choice of  $s$  in the fiber above  $Z$ ,  $s^\perp$  associates to the four real coordinates specifying  $Z$  an element of  $\mathbf{C}^2$ . For instance, if  $s = (1, 0)$ , the identification of  $\mathbf{R}^4$  with  $\mathbf{C}^2$  is

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} x_0 - ix_3 \\ -ix_1 + x_2 \end{pmatrix}$$

The complex structure on  $\mathbf{R}^4$  one gets is not changed if  $s$  gets multiplied by a complex scalar, so it just depends on the point  $[s]$  in the  $\mathbf{C}P^1$  fiber.

For another point of view on this, one can see that for each point  $p \in PT$ , the corresponding  $\alpha$ -plane  $\nu(\mu^{-1}(p))$  in  $M$  intersects its conjugate  $\rho(\nu(\mu^{-1}(p)))$  in exactly one real point,  $\pi(p) \in M^4$ . The corresponding line in  $PT$  is the line determined by the two points  $p$  and  $\rho(p)$ . At the same time, this  $\alpha$ -plane provides an identification of the tangent space to  $M^4$  at  $\pi(p)$  with a complex two plane, the  $\alpha$ -plane itself. The  $\mathbf{C}P^1$  of  $\alpha$ -planes corresponding to a point in  $S^4$  are the different possible ways of identifying the tangent space at that point with a complex vector space. The situation in the Minkowski space case is quite

different: there if  $\mathbf{CP}^1 \subset PT_0$  corresponds to a point  $Z \in M^{3,1}$ , each point  $p$  in that  $\mathbf{CP}^1$  gives an  $\alpha$ -plane intersecting  $M^{3,1}$  in a null line, and the  $\mathbf{CP}^1$  can be identified with the “celestial sphere” of null lines through  $Z$ .

In the Euclidean case, the Penrose transform will identify the sheaf cohomology group  $H^1(\pi^{-1}(U), \mathcal{O}(-k-2))$  for  $k > 0$  with solutions of helicity  $\frac{k}{2}$  linear field equations on an open set  $U \subset S^4$ . Unlike in the Minkowski space case, in Euclidean space there are  $U(n)$  bundles  $\tilde{E}$  with connections having non-trivial anti-self-dual curvature. The Ward correspondence between such connections and holomorphic bundles  $E$  on  $PT$  for  $U = S^4$  has been the object of intensive study, see for example Atiyah’s survey [2]. The Penrose-Ward transform identifies

- Solutions to a field equation on  $U$  for sections  $\Gamma(S^k \otimes \tilde{E})$ , with covariant derivative given by an anti-self-dual connection  $A$ , where  $S^k$  is the  $k$ ’th symmetric power of the spinor bundle.
- The sheaf cohomology group

$$H^1(\hat{U}, \mathcal{O}(E)(-k-2))$$

where  $\hat{U} = \pi^{-1}U$ .

For the details of the Penrose-Ward transform in this case, see [10].

#### 4.4 $Spin(6) = SU(4)$

If one picks a positive definite Hermitian inner product on  $T$ , this determines a subgroup  $SU(4) = Spin(6)$  that acts on  $T$ , and thus on  $PT, M$  and  $P(S)$ . One has

$$PT = \frac{SU(4)}{U(3)}, \quad M = \frac{SU(4)}{S(U(2) \times U(2))}, \quad P(S) = \frac{SU(4)}{S(U(1) \times U(2))}$$

and the  $SU(4)$  action is transitive on these three spaces. There is no four real dimensional orbit in  $M$  that could be interpreted as a real space-time that would give  $M$  after complexification.

In this case the Borel-Weil-Bott theorem relates sheaf-cohomology groups of equivariant holomorphic vector-bundles on  $PT, M$  and  $P(S)$ , giving them explicitly as certain finite dimensional irreducible representations of  $SU(4)$ . For more details of the relation between the Penrose transform and Borel-Weil-Bott, see [3]. The Borel-Weil-Bott theorem [4] can be recast in terms of index theory, replacing the use of sheaf-cohomology with the Dirac equation [5]. For a more general discussion of the relation of representation theory and the Dirac operator, see [11] and [25].

## 5 Twistor geometry and the Standard Model

Conventional attempts to relate twistor geometry to fundamental physics have concentrated on the Minkowski signature real form, with the Penrose trans-

form providing an alternative treatment of conformally invariant massless linear field equations on Minkowski space in terms of sheaf cohomology of powers of the tautological bundle on  $PT$ . Since there are no non-trivial solutions of the Minkowski space  $SU(n)$  anti-self-duality equations, there is no role for the Penrose-Ward transform to play. One can develop perturbation theory for the Standard Model, building it out of the twistor formulation of massless spinor fields, the Maxwell field and linearized Yang-Mills fields. This has allowed a wealth of new sorts of perturbative calculations, but provided no insight into unification. In Minkowski space, while twistor geometry provides a different perspective on space-time symmetries, it appears to have little relevance to the internal symmetries of the Standard Model. In Euclidean space however, space-time symmetries behave differently and we will see that twistor geometry provides an intriguing realization of the degrees of freedom and symmetries of the Standard Model.

## 5.1 Twistor theory and quantum field theory in Euclidean space-time

Fundamental to a relativistic quantum field theory of multi-particle states is identification of the space  $\mathcal{H}_1$  of positive energy single-particle states. The full state space of the theory can then be built using the Fock space on  $\mathcal{H}_1$ . The Poincaré group acts on  $\mathcal{H}_1$  and thus on the Fock space. For the matter fields of the Standard Model,  $\mathcal{H}_1$  will be the space of positive energy solutions to a Dirac equation.

Since the time dependence of states is given by  $e^{-itE}$ , if one takes time to be a complex variable  $z = t + i\tau$ , then the positive energy condition will correspond to a holomorphicity condition in the  $\tau < 0$  lower-half  $z$ -plane. Even for the simplest free field theories, one finds that, as a function of  $z$ , the theory has good behavior for  $\tau < 0$ . The behavior at the real time axis however needs to be thought of in terms of limits of boundary values of holomorphic functions as  $\tau \rightarrow 0^-$ . For scalar field theories, on the real time axis expectation values of fields are the Wightman distributions. These are boundary values of holomorphic functions, with singular behavior that reflects the non-commutativity of Minkowski space quantum fields. If one instead “Wick rotates” and considers the theory for  $t = 0, \tau \neq 0$ , the analytically continued Wightman distributions become the Schwinger functions. These are now much less singular, given by actual functions, which can be interpreted as expectation values of commuting fields.

Such a Euclidean quantum field theory has quite different space-time symmetry behavior than the Minkowski version. While one can define Euclidean quantum fields whose expectation values are the Schwinger functions, and the Euclidean group acts on these in the expected manner, the definition of the physical state space breaks the Euclidean symmetry.  $\tau$ -translations only act in one direction on the states, and while spatial rotations act on states, the Euclideanized version of boosts don't. The very definition of the physical state space requires picking a time direction in  $\mathbf{R}^4$ , with the inner product defined

using reflection in that direction, and Osterwalder-Schrader positivity corresponding to unitarity.

The situation for spinor fields is even more confusing, since the spinor representations are quite different in the Minkowski ( $SL(2, \mathbf{C})$ ) and Euclidean ( $SU(2) \times SU(2)$ ) cases. The standard account of how to deal with this is that of Osterwalder-Schrader [14], and involves doubling the number of degrees of freedom. Twistor geometry provides a way of naturally understanding this issue, allowing one to move the question of the relation between theories defined on two different real slices of complexified Minkowski space  $M$  to questions about the behavior of the theory on projective twistor space  $PT$ .

One way to characterize the single-particle state space  $\mathcal{H}_1$  for a spinor field is in terms of the initial data at  $t = 0$  for a solution to a Dirac equation. This has the disadvantage of obscuring the Poincaré group action on  $\mathcal{H}_1$ , but the advantage that one can identify the spacelike  $t = 0$  subspace of Minkowski space  $M^{3,1}$  (which will be a 3-sphere that we'll call  $M^3$ ) with the  $\tau = 0$  equator  $M_0^4$  in Euclidean space  $M^4 = S^4$  that divides the space into upper ( $\tau > 0$ ) and lower ( $\tau < 0$ ) hemispheres  $M_+^4$  and  $M_-^4$ .

Taking the Euclidean point of view as starting point, recall from section 4.3 that, after choosing an identification of  $\mathbf{H}^2$  with  $\mathbf{C}^4$ , we have a fibration of  $PT$  over  $S^4 = M^4$ . In the coordinates for  $S^4$  of equation 5, setting  $\tau = x_0 = 0$  corresponds to the condition that the real part of the numerator vanish, so

$$s_1^\perp \overline{s_1} + s_2^\perp \overline{s_2} + \overline{s_1^\perp} s_1 + \overline{s_2^\perp} s_2 = 0$$

Note that (by equation 3), this is exactly the condition

$$\Phi(s, s) = 0$$

that describes the five-dimensional subspace  $N = PT_0$  of  $PT$  which contains the complex lines corresponding to Minkowski space  $M^{3,1}$ . We have the fibration

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & N = PT_0 \longleftarrow & PT = \mathbf{C}P^3 \\ & & \downarrow & \downarrow \pi \\ & & M^3 & \longleftarrow & S^4 = \mathbf{H}P^1 \end{array}$$

as well as

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & PT_\pm \longrightarrow & PT = \mathbf{C}P^3 \\ & & \downarrow & \downarrow \pi \\ & & M_\pm^4 & \longrightarrow & S^4 = \mathbf{H}P^1 \end{array}$$

Instead of relating solutions of the massless Dirac equation by analytic continuation between  $M^4$  and  $M^{3,1}$ , one can instead use the Euclidean and Minkowski Penrose transforms to relate both to holomorphic objects on  $PT$ , in particular to hyperfunctions on  $PT_0$  that are differences of holomorphic sections on  $PT_+$  and  $PT_-$ .



## 5.2 Twistor geometry and the Standard Model

Our best understanding of fundamental physics is that one can describe it with quantum field theories that depend on the distance scale, with all experimental evidence now implying that the Standard Model quantum field theory gives the effective theory up to the TeV scale, and all theoretical evidence implying that it can be consistently extrapolated up to vastly higher energy scales. A natural conjecture is that there is a fundamental theory that describes arbitrarily short distances, with corresponding effective theory at the TeV scale given by the Standard Model, and that such a theory is closely related to the Standard Model, for instance exhibiting the same symmetries and degrees of freedom. The proposal here is essentially that twistor geometry provides the correct context for such a theory, exhibiting conformal invariance at arbitrarily short distances, and, in Euclidean space, the internal symmetries of the Standard Model. The next sections examine how these internal symmetries appear.

### 5.2.1 Breaking of $SO(4)$ invariance

As discussed above, given a quantum field theory defined in Euclidean space time, the four-dimensional rotational symmetry needs to be broken by a choice of time direction in order to define the states of the theory. A choice of time direction has been made by our choice of coordinates on  $M^4$  (equation 5): the real direction in the quaternionic coordinate. This choice could be changing by changing coordinates, for instance by an action of  $Spin(4)$ ,

$$s^\perp s^{-1} \rightarrow q_1 s^\perp s^{-1} q_2^{-1}$$

for  $(q_1, q_2)$  a pair of unit quaternions. The subgroup  $q_1 = q_2$  is a  $Spin(3)$  subgroup which changes the coordinates while leaving the time direction invariant. This will correspond to spatial rotations, and these transformations will act on the states of the theory.

### 5.2.2 $U(2)$ electroweak symmetry

As discussed in section 4.3, the fibration 6 of  $PT$  over  $M^4$  can be identified with the projective spinor bundle  $P(S)$ . The fiber above each point of  $M^4$  is the space of orthogonal complex structures on the tangent space at the point, so a copy of  $SO(4)/U(2)$ . To each element  $s$  of the fiber  $S_0$ , one gets an identification of the real tangent space at 0 with maps from  $s$  to elements of  $S_0^\perp$ , which has a complex vector space structure. The corresponding complex structure this puts on the real tangent space at 0 only depends on the complex line generated by  $s$ , so the point it determines in  $P(S_0)$ .

One thus has for each point in  $PT = P(S)$  a  $U(2)$  group that leaves that

point invariant. These together give a principal bundle

$$\begin{array}{ccc} U(2) & \longrightarrow & Sp(2) \\ & & \downarrow \\ & & PT = Sp(2)/U(2) \end{array}$$

over  $PT$ .

The choice of a time direction is given by the choice of vector in the tangent space of  $M^4$ . For each point in the fiber of 6 this tangent space gets identified with  $\mathbf{C}^2$  and a tangent vector in the time direction transforms under  $U(2)$  as the usual representation on  $\mathbf{C}^2$ . Note that this is the way the Higgs field in the Standard Model transforms under the electroweak  $U(2)$ . This indicates that the Higgs field of the Standard Model has a space-time geometrical significance, as a vector pointing in the (Euclidean) time direction, with the necessary breaking of symmetry needed to define the space of states corresponding to electroweak symmetry breaking.

### 5.2.3 Spinors on $PT$

Taking as fundamental the space  $PT$  with its fibration to  $M^4$ , one can ask what holomorphic vector bundle on  $PT$  corresponds to the Standard Model matter fields. It turns out that the spinor bundle on  $PT$  has the correct properties to describe a generation of leptons. At a point  $p \in PT$ , the complex tangent space splits into a sum

$$T_p = V_p \oplus H_p$$

of

- a complex one-dimensional vertical subspace  $V_p$ , tangent to the  $\mathbf{C}P^1$  fiber.
- a complex two-dimensional horizontal subspace  $H_p$ , which is the real four-dimensional tangent space to  $M^4$  at  $\pi(p)$ , given the complex structure corresponding to the point  $p$  in the fiber above  $\pi(p)$ .

For details about the relation between spinors and the complex exterior algebra, see chapter 31 of [23], in particular section 31.5 about the case of spinors in four dimensions.

The way spinors work, spinors for the sum  $V_p \oplus H_p$  will be given by a tensor product of spinors for  $V_p$  and those for  $H_p$ . Spinors for  $V_p$  give the usual spinor fiber  $S_{\pi(p)}$ , those for  $H_p$  are given by  $\Lambda^*(H_p) \otimes \mathbf{C}_p$ , where  $\mathbf{C}_p$  is the complex line in the fiber  $S_{\pi(p)}$  corresponding to the point  $p$ . Elements of  $\Lambda^*(H_p)$  transform  $U(2)$  like a generation of leptons:

- $\Lambda^1(H_p)$  is complex two-dimensional, has the correct transformation properties to describe a left-handed neutrino and electron.
- $\Lambda^2(H_p)$  is complex one-dimensional, has the correct transformation property (weak hypercharge  $-2$ ) to describe a right-handed electron.

- $\Lambda^0(H_p)$  is complex one-dimensional, has the correct transformation properties (zero electroweak charges) to describe a conjectural right-handed neutrino.

#### 5.2.4 $SU(3)$ symmetry

So far we have just been using aspects of twistor geometry that at a point  $p \in PT$  involve the fiber  $L_p \subset \mathbf{C}^4$  of the tautological line bundle  $L$  over  $PT$ , as well as the fibration  $6$  to  $M^4$ . Just as in the case of the Grassmanian  $M$ , where we could define not just a tautological bundle  $S$ , but also a quotient bundle  $S^\perp$ , over  $PT$  one has not just  $L$ , but also a quotient bundle  $L^\perp$ . This quotient bundle will have a complex 3-dimensional fiber at  $p$  given by  $L_p^\perp = \mathbf{C}^4/L_p$ . One can think of  $PT$  as

$$PT = \frac{U(4)}{U(1) \times U(3)} = \frac{SU(4)}{S(U(1) \times U(3))} = \frac{SU(4)}{U(3)}$$

where the  $U(1)$  factor acts as unitary transformations on the fiber  $L_p$ , while the  $U(3)$  acts as unitary transformations on the fiber  $\mathbf{C}^4/L_p$ . The  $SU(3) \subset U(3)$  subgroup provides a possible origin for the color gauge group of the Standard Model, with fermion fields taking values in  $L_p^\perp$  giving the quarks.

In the case of the  $U(2)$  electroweak symmetry, to a point  $p \in PT$  we associated not just the line  $L_p$ , but also the spinor space  $S_{\pi(p)}$ , with  $L_p \subset S_{\pi(p)}$ . The internal electroweak  $SU(2)$  acts on  $S_{\pi(p)}^\perp$ , while the color  $SU(3)$  acts on  $L_p^\perp$ . One needs to avoid defining these spaces as subspaces of the same  $\mathbf{C}^4$  in order to ensure that the two group actions commute as needed by the Standard Model.

#### 5.2.5 Twistor space and Standard Model symmetries

Twistor geometry inherently is based on a different picture of space-time symmetries than the usual one. In particular it is chiral-asymmetric, with a point in space-time identified with a chiral spinor. These spinors are the fundamental geometrical quantities, with tangent vectors and more general tensors built out of them. While this chiral asymmetry causes problems with using twistors in conventional geometry associated with general relativity, the possible connection to the chiral nature of electroweak interactions has sometimes been noted. The argument here is that by going to Euclidean signature, effectively one of the two  $SU(2)$  factors in  $Spin(4)$  takes on aspects of an internal symmetry from the point of view of physical Minkowski space. The breaking of this symmetry by the ground state of the theory is inherent in defining a state space in Euclidean signature.

In the conventional definition of the Standard Model, internal symmetry groups are attached to each point in space-time, giving a gauge symmetry when treated independently at each point in space-time. In the twistor space setting described here, internal symmetry groups are attached instead to each point in

PT. Recall that, from the Minkowski space point of view, such a point corresponds to a null-line, a light ray, so gauge degrees of freedom live not at points but on light rays. From the Euclidean point of view, each point in  $PT$  projects to a single point in  $M^4$ , but this is true for an entire sphere of points in  $PT$ . So, for a Euclidean space-time point one has not a single gauge degree of freedom, but a sphere's worth of them.

From the above, it should be clear that the proposal to think about the Standard Model on  $PT$  rather than on Minkowski space involves a dramatic reconfiguration of the degrees of freedom and symmetry principles governing the theory.

## 6 Conclusions

The main conclusion of this work is that twistor geometry and conformal invariance provide a compelling picture of short-distance fundamental physics, integrating internal and space-time symmetries, as long as one treats together its Euclidean and Minkowski aspects, related through the projective twistor space  $PT$ . The Euclidean aspect is crucial for understanding the origin of the Standard Model internal symmetries and the breaking of electroweak symmetry, which is inherent in the Euclidean space definition of physical states.

Much remains though to be done in order to realize a fundamental theory based on this sort of geometry. Our usual quantum field theory formalism is based upon fields on space-time, but in twistor geometry it is not space-time, but projective twistor space  $PT$  that plays the fundamental role. It is not clear how the quantum field theory formalism should be implemented on  $PT$ , other than that one wants the Penrose transforms to  $M^4$  and  $M^{3,1}$  to in some sense be related by analytic continuation on  $M$ . For some possibly relevant discussion of a theory formulated on  $PT$ , see section 4 of Witten's paper[21] on the twistor string. One should note that from the point of view of geometric quantization and representation theory, the relevant case here of the orbits of  $SU(2, 2)$  on  $PT$  is an exceptionally challenging one. It is a fundamental example of a "minimal" orbit, for which geometric quantization runs into difficult technical problems due to the lack of an appropriate invariant polarization. For a 1982 history of work on this specific case, see appendix A of [16]. Perhaps what is needed involves the ideas about Dirac cohomology and quantization discussed in [25]. For some discussion of the relation of the Dirac operator on a manifold such as  $M^4$  to the Dolbeault operator on the the projective twistor space, see [6].

More speculatively, it is possible that the fundamental theory involves not just the usual twistor geometry of  $PT$ , but should be formulated on the seven-sphere  $S^7$ , which is a circle bundle over  $PT$ .  $S^7$  is a remarkably unusual geometric structure, exhibiting a wide range of different symmetry groups, since one has

$$S^7 = Spin(8)/Spin(7) = Spin(7)/G_2 = Spin(6)/SU(3) = Spin(5)/Sp(1)$$

as well as algebraic structures arising from identifying  $S^7$  with the unit octo-

nions. Our discussion has exploited the last two geometries on  $S^7$ , not the first two.

While we have identified a proposed source of the global symmetries of the Standard Model, an appropriate formalism for exploiting the gauged version of these symmetries and understanding the dynamics of gauge fields remains to be developed. The Penrose-Ward transform intriguingly relates anti-self-dual gauge fields on Euclidean space-time to holomorphic vector bundles on  $PT$ , but this is far from what one needs, a fully quantum theory of such gauge fields. As another speculative comment, note that the anti-self-duality equation can be formulated as the vanishing of a moment map, perhaps indicating a relation to the invariant piece of a representation of the gauge group. The anti-self-duality equation also appears in Witten's original twisted  $N = 2$  supersymmetric Yang-Mills topological quantum field theory [20], which localizes on solutions to the equation.

The discussion of twistor geometry here has purely dealt with its flat and highly symmetric version, concentrating on the role of spinors and Standard Model internal symmetries. A large part of the twistor program has been the effort to extend these ideas to more general manifolds. In particular, Penrose's nonlinear graviton construction provides a gravitational version of the Ward correspondence, associating a deformation of  $PT$  with a space-time of anti-self-dual curvature.

There is a long history of study of gravity theories formulated in terms of the spin connection, with one motivation unification with the Standard Model which is a theory based on connections. It is well-known that such gravity theories can be written in a formalism involving just the self-dual or anti-self-dual part of the spin connection. This corresponds in twistor geometry to the fact that one just needs a connection in the spinor bundle  $S$ . For more about "self-dual" quantum gravity theories, see for instance [12]. For discussion of the relation of this to twistors, see [9]. Perhaps the new ingredients proposed here (emphasis on the Euclidean picture and the identification of electroweak symmetry breaking with a sort of Lorentz symmetry breaking) may provide a new pathway towards a successful quantization of the space-time degrees of freedom themselves.

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