

Lagrangian quantization versus representation-theoretical approach to QFT

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Abstract

This is a pedagogical attempt to elucidate the conceptual and mathematical differences between Lagrangian quantization, causal perturbation theory guided by the setting of algebraic QFT and some recent developments of nonperturbative intrinsic constructions of QFT based on the conceptual setting of AQFT.

1 The aim

From looking at the lively discussions taking place on various weblogs within the quadrangle whose corners are defined by such catchwords as:

- Lagrangian quantization (canonical quantization, Euclidean functional integral representation)
- Algebraic quantum field theory (Local quantum physics)
- Loop gravity
- String theory

one realizes that often people use these words without paying much attention to their actual content. This sometimes leads to misunderstandings and confusions even in cases of Lagrangian QFT versus AQFT where the concepts are actually well-defined and many tested or testable physical results are available. For this reason the comparison will be limited to Lagrangian quantization versus AQFT. Since the notion of an “action” i.e. a Lagrangian description is a prerequisite for the functional integral approach no conceptual distinction will be made between the two.

For a critical evaluation it is very helpful to start from an earlier attempt in this direction. As such a reference point we will use an interesting article by David Wallace *In defence of naivité: The conceptual status of Lagrangian quantum field theory* quant-ph/0112148. Some of the points the author touched upon within his viewpoint could have been questioned already at the time of

writing. But perhaps some important changes (which probably would lead to some modifications of the authors actual point of view) happened during recent years. I am referring to the successful adaptation of powerful concepts of (Tomita-Takesaki) modular theory within the setting of operator algebras to the problem of classifying and constructing nonperturbative QFT. These developments took place largely unnoticed in the shadow of the great globalized string theory caravan. Although these new ideas are still in their beginnings, they are already forcing us to rethink the conceptual basis of QFT.

We start by reminding the reader of some of the conceptual and mathematical limitations the functional approach faces even in the best of its mathematical settings in QM. In a subsequent section the perturbative Lagrangian setting is compared with that done in the spirit of AQFT. Whereas the algebraic perturbative approach includes the results of the Lagrangian method it goes in two important issues beyond it. The main power of the algebraic method shows up in a new programmatic intrinsic setting for QFT whose first success has been to provide a new conceptual platform for the bootstrap-formfactor construction of factorizing models.

2 Functional Lagrangian method versus representation theory in QM

Quantum mechanics was discovered in terms of its operator algebraic structure in the form of Heisenberg (or the bounded Weyl algebra) commutation relations. The functional (Feynman-Kac) representation for the (imaginary time) propagation kernel and ground state correlations can be derived from the operator form (for those systems where it applies); its rigorous formulation requires a measure-theoretic setting which is closely related to the Wiener measure.

The functional approach is a valuable addition to the algebraic operator approach which shows its computational power especially for quasiclassical approximations; but even though these two settings are mathematically equivalent for standard systems (at least for systems for which the free part of the Hamiltonian has the standard nonrelativistic kinetic energy form), the former is not quite a conceptual replacement for the operator formulation.

Just try to contemplate a hypothetical situation in which the transition from the old Bohr-Sommerfeld quantum theory to the new quantum mechanics would have been made with the help of functional integrals (the idea of inverting Heisenberg-Feynman is not so absurd in view of the fact that those integral representations are much closer to the quasiclassical concepts of the old theory). The transition from the quasiclassical B-S setting would have been deceptively smooth but instead of a rapid and profound insight into the new mechanics there may have been years of agonizing struggle with a good conceptual understanding of what quantum mechanics really is about.

When we look at geometrically more complex dynamics as the quantum theory of the rigid body which in the Feynman setting would be the movement of

a particle on a group manifold, one would have run against a serious paradox which I have described before. Whereas the group-representation based operator approach leads to the result in a straightforward way, the functional setting produces an unsolved paradox. Namely the action of the spinning top has an infinite number of saddle points and if one evaluates the functional integral around them up to the quadratic fluctuations and adds up all the contributions one obtains the exact result known from the operator approach. The paradoxical situation consists in the fact that although the higher fluctuation terms (higher perturbations) are nonvanishing, *one must ignore them in order to arrive at the rigorous result*. In finite dimensional integration one observes a similar phenomenon if the manifold to be integrated over is a flag-manifold. Duistermaat and Heckmann unraveled the cohomological reason behind it. Formally the path-manifold of the rigid top is an infinite dimensional flag manifold, but since the D-H theory is finite dimensional one must approximate the infinite dimensional functional manifold by a sequence of finite-dimensional ones which coincides with the necessity to regularize the path integral in order to convert it into a mathematically well-defined object. But a regularization which maintains the cohomological D-H property has never been found and it is not clear whether it exists at all.

This difficulty is typical for the functional integral setting as soon as one goes beyond the standard setting of quantum mechanics. Since e.g. sigma model field theories can be considered as field-theoretic higher dimensional generalizations of the quantum mechanical rigid top, the functional approach becomes even more ill-defined and unmanageable. But even though its mathematical status is dubious (see next section), it is of significant “artistic” (non-mathematical) value because it carries potentially the locality and spectral principles (hidden in the reflection positivity) in a rather compact manner (for more see next section). Here “potentially” means that you do not follow any mathematical rules but artistically implement your intuitive ideas you have about what QFT is about, including a distinction between Lagrangian- and physical- parameters as a result of vacuum fluctuations. With other words you “vivisect” the functional integral into its gruesome details and apply renormalization theory to the infinitely many different pieces, not giving a damn whether the resulting renormalized physical series can be rewritten again into a functional integral representation of the kind you started with (for strictly renormalizable models it cannot!).

Since many quantum physicists subconsciously share Einstein’s nostalgia for a return to a classical geometric setting, the popularity of the functional approach will be also guaranteed in the future notwithstanding its mathematical shortcomings.

3 The deceiving beauty of functional integration in QFT

For reasons mentioned before the mathematical status of functional integral representations worsens considerably in QFT. At the same time the absence of a powerful operator approach and the ease of injecting geometrical structures into actions generate an irresistible temptation to use it as a short-hand starting point and to repair its ill-defined intermediate steps by renormalization. So the functional approach gains its popularity by its deceiving formal beauty (never mind the gruesome vivisection you have to go into afterwards) and the feeling that there is no alternative.

It is important to clarify one common point of confusion before going on. *Constructing covariant local free fields for a given (m,s) one-particle Wigner representation is not the same as Lagrangian quantization.* Whereas the Wigner approach is totally intrinsic (no reference to any classical parallelism) and unique, the Lagrangian quantization approach requires as an input a classical Lagrangian (quadratic in free case) and only after its quantization (replacing the classical canonical equal time Poisson (or covariant Peierls bracket) structure with the commutation relation) one knows whether it is consistent with the Hilbert space structure and what is its particle content.

It is very easy to write down quadratic Lagrangians which contain in addition to the desired (m,s) elementary objects additional "stuff", the difficulty (especially in case of higher spin) is to find Lagrangians whose Euler-Lagrange equations possess a totality of solutions which describe precisely the desired (m,s) object. Most of the countably infinitely many covariant fields for a given Wigner representation (m,s) with finite spin (helicity) have no Lagrangian e.i. such pointlike fields do not obey Euler-Lagrange equations in the aforementioned sense. One particular choice of intertwiner from the Wigner- to the covariant-setting leads to the well-known Bargmann-Wigner spinorial (dotted-undotted spinor indices) fields which solve a kind of generalized Dirac equation, only for Wigner spin ≤ 2 these equations have the Euler-Lagrange property (or can be brought into a E-L form). Of course at that point you may want to argue that for formulating the Standard model you don't need to know this. This seems to be Weinberg's point of view who starts his book with a very detailed presentation of Wigner's representation approach only in order to dismiss one of the three big representation classes (the "infinite spin" representation) and to work with low spin Lagrangians.

The important point here is that for the representation in terms of a functional integral representation one needs the Lagrangian setting, whereas for causal perturbation theory (which follows the intrinsic spirit of AQFT) any of the infinitely many covariantizations can be used to define interaction densities by coupling the spinorial free fields into invariant interaction polynomials (having specified the interaction in terms of one field-coordinatization the interaction polynomial of course has to be transformed appropriately). The infinite set of (m,s) covariant free fields is only the "linear part" of an infinite set of all

relatively local covariant fields in the same Fock space, the so-called Borchers class; the linear part consists precisely of those fields whose one-time application to the vacuum is a genuine one-particle state without any vacuum-polarization admixture. In view of formal equivalence theorems securing that the different members of this huge class have the same S-matrix, it is very suggestive to expect that there exist a formulation of perturbation theory which is fully independent of the field-coordinatization from this Borchers class which one uses for the perturbation; indeed there does, it is AQFT.

Besides the greater generality there is a second advantage in the AQFT supported causal perturbation theory. Whereas Lagrangian formalism in particular in its canonical quantization form (inherited by the functional approach) still contains a lot of quantum mechanical relics (By this I mean the idea of occupying energy levels which tends to ignore the pivotal role of vacuum polarization through spacetime localization see a paper by Hollands and Wald gr-qc/0405082) which prevent a clear QFT perspective on its own merits, causal perturbation theory is better suited to keep track of the necessarily singular nature of pointlike fields (operator-valued distribution). The renormalization theory in this setting shows its true nature as a distributional extension problem (subject to a minimal scaling requirement) without the necessity of any cutoff or regularization in intermediate steps. In this way one obtains a perturbative series which is completely rigorous in the mathematical sense of a formal power series (i.e. in the same sense as mathematicians treat the vertex algebra construction of Kac-Moody algebras). Physically one wants it to be asymptotic to a genuine solution. In the case of superrenormalizable interactions (which are only available in low spacetime dimensions e.g. the book by Glim-Jaffe) where the vacuum fluctuations are very mild (so that functional-analytic theorems of quantum mechanics are still applicable) one has been able to proof the existence of a nonperturbative solution as well as the property that the perturbative series is asymptotic to the exact solution.

Perhaps the greatest achievement of this causal perturbation theory is that it is capable to be adjusted to any curved spacetime metric with local quantum matter. In order to implement this one has to separate the algebraic perturbative structure from the structure of states. The Feynman approach and its more concise systematic extensions aim directly at vacuum expectation values i.e. products of operators (the algebraic part) evaluated in a reference state (in the standard Minkowskian case the vacuum). However permitting oneself to be guided by the concepts of AQFT it is possible to construct field-generators of the net of local algebras and place them into selected states in a second step. In QFT in curved spacetime there is a priori no preferred state but only a distinguished folium of states (those which satisfy the micro-causal spectrum condition) and therefore the dichotomy between the algebraic structure (the perturbation is done on the algebraic level) and states (which the Lagrangian approach cannot implement) is extremely important. Whereas the causality-preserving isometric diffeomorphisms have a simple functorial algebraic image, the action on such transformations on states is more involved, in general states are changed in such a way that only the folium is preserved. This observation

explains why in (localizable) the search for diffeomorphism invariant states led to negative results. Theories which aim at diffeomorphism invariant states as loop gravity seems to do have necessarily a very different algebraic structure than that of laboratory supported QFT for which localization is the central structure in the formulation and physical interpretation of the theory.

The algebraic perturbative approach may lack the compactness of a single formal for the functional integral representation, but we have seen that this elegance is deceiving because you have to vivisect in gruesome detail and at the end the physical renormalized result does not fulfil the formula from which you started. Getting perturbative AQFT going may be less compact, but at least it is an honest enterprise in that your result fulfills exactly those requirements which you set out to demand and in achieving this you did not have to wade through ultraviolet infinities. All the claims in this section can be backed up by a body of papers of which most of them are mentioned in the recent Fredenhagen-Rehren-Seiler review.

In some previous communication I wondered why such a powerful approach is often dismissed as offering no computational access to particle physics. I speculated that the reason for this is that Nobel prize winners cannot only direct public interests towards aphysical fashions but also (wittingly or unwittingly) keep newcomers away from real promising new physical ideas. It is not necessary for them to say something negative; given the trend towards monocultures it is simply enough to ignore such contributions altogether.

To say that Lagrangian QFT is computationally more powerful than the algebraic approach is absolute rubbish. The only treatment of the Hawking radiation (in the way as Hawking intended) as a nonstationary collapse of a star was given by Fredenhagen and Haag using ideas of AQFT. One only has to look at Wald's book on QFT in CST and Black-Hole Thermodynamics in order to realize how deeply this contribution was appreciated. The list of examples can be continued. Of course articles in AQFT deliver precisely what they promise; may be the modern reader prefers more the imaginative style of story-telling and speculations as produced in string theory.

If there is enough interest I can write a separate section about recent inroads into nonperturbative QFT under a title: *How Local Quantum Physics implements the principles underlying QFT*