Quantization and Dirac Cohomology
Under Construction

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1 Introduction

The subjects of quantum theory and that of Lie algebras and their representations are quite closely connected. For any Lie algebra $\mathfrak{g}$, the Lie bracket on $\mathfrak{g}$ determines a Poisson bracket on functions on $\mathfrak{g}^*$. One can think of this as a classical mechanical system, with quantization given by the universal enveloping algebra $U(\mathfrak{g})$. Another piece of structure is needed to actually have a quantum theory: a unitary representation of the Lie algebra, which realizes the algebra $U(\mathfrak{g})$ as an algebra of operators on a complex vector space, the quantum state space.

The problem of classifying and constructing all such representations is a difficult one. A general program for how to go about this is the "orbit method" or "orbit philosophy", according to which irreducible representations of a Lie group G with Lie algebra $\mathfrak{g}$ should correspond to orbits of the co-adjoint action of G on $\mathfrak{g}^*$. On such orbits the Poisson bracket is non-degenerate and the orbit is a symplectic manifold which can be taken to be the phase space of a Hamiltonian classical system. If there exists an appropriate method for quantizing this system, it should give a unitary irreducible representation of G.

Canonical quantization corresponds to the case of the $2d+1$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2d+1}$. The non-trivial co-adjoint orbits under the Heisenberg Lie group H_{2d+1} are, for each choice of a non-zero real constant, the usual $2d$ linear phase space, with the constant giving the normalization of the standard symplectic form. By the Stone-von Neumann theorem, quantization by any of various methods gives unitarily equivalent representations of the Heisenberg group on a Hilbert space, with the choice of constant corresponding to the choice of $\hbar$.

For reductive Lie algebras, the orbit method is just a "philosophy", providing neither a one-to-one map between orbits and irreducible representations nor a uniform method for quantizing orbits. For the simplest case of $\mathfrak{g}$ the Lie group of $SU(2)$, the orbits are spheres in $\mathfrak{su}(2)^* = \mathbf{R}^3$. Only spheres with areas satisfying an integrality condition will correspond to irreducible representations (the spin $\frac{n}{2}$ representations), and it is unclear whether to associate the trivial representation to the trivial orbit or to the smallest sphere. The orbit method (also known as "geometric quantization") constructs the representation using a holomorphic line bundle over the sphere, which requires a choice of invariant complex structure on the sphere. There are two such choices, in one of these the representation is on holomorphic sections (H^0), in the other it's on a higher cohomology group (H^1). To get a quantization method that does not depend on a choice of complex structure, one needs to introduce introduce spinor bundles on the orbits and find representations in the kernel of a Dirac operator.

We would like to argue here that both from the point of view of representation theory and from the point of view of physics, it is desirable to expand the above conception of quantization for a Lie algebra $\mathfrak{g}$ to one based on consideration of an extended version of $\mathfrak{g}$ we will denote $\widetilde{\mathfrak{g}}_d$. This is $\mathbf{Z}_2$-graded, adding an odd copy of $\mathfrak{g}$ to the usual even one, together with a differential d. One can then look for a quantization not of the dual of $\mathfrak{g}$, but of the (odd) dual of $\widetilde{\mathfrak{g}}_d$. When $\mathfrak{g}$ comes with a non-degenerate invariant bilinear form, the non-commutative algebra for such a quantization is given by the quantum Weil algebra

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$$

and quantum state spaces are given by tensoring representations of $\mathfrak{g}$ with modules for $\mathrm{Cliff}(\mathfrak{g})$. The new odd variables here can be quantized by the usual extension of canonical quantization to fermionic systems.

This expanded notion of the context for the relation of Lie algebra representations and quantization can be motivated in various ways. For instance:

- The quantization of the differential d is an algebraic version of the Dirac operator, so the Dirac operator and spinors play a fundamental role in the basic conception of quantization, explaining their appearance in the quantization of the orbits of $SU(2)$ mentioned above.

- Irreducible representations of $\mathfrak{g}$ can be characterized in terms of the action of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. After introduction of fermionic variables, the quadratic Casimir operator in $Z(\mathfrak{g})$ now has a square root, a Dirac operator. This is just a more general version of the original motivation for Dirac's discovery of the Dirac operator, which was for the special case of $\mathfrak{g}$ the Poincaré Lie algebra.

- The BRST method for handling gauge symmetries is based on exactly this sort of extension of the usual classical phase space to a pseudo-classical one with new fermionic variables.

- In recent years, there has been a significant amount of work in representation theory using $\mathcal{W}(\mathfrak{g})$ and the algebraic Dirac operator to define the Dirac cohomology of a representation. The original goal of this work was to understand the precise relationship between the Dirac cohomology story and the BRST cohomology story that physicists have been using.

2 Lie algebras and quantization

The canonical quantization method used by physicists can be understood in terms of a specific Lie algebra, the Heisenberg Lie algebra, together with the theory of its unitary representations. We'll first review this story, then discuss its generalization to arbitrary Lie algebras.

2.1 The Heisenberg Lie algebra and canonical quantization

The method of canonical quantization starts with a classical Hamiltonian system based on a linear phase space $\mathbf{R}^{2n}$ with position and momentum coordinates q_j, p_j $(j = 1, \cdots, n)$ and a Poisson bracket satisfying

$$\{q_j, p_k\} = c\delta_{jk}$$

The Poisson bracket provides a Lie algebra structure on the infinite-dimensional algebra of functions on the phase space $\mathbf{R}^{2n}$. The subspace of linear and constant functions is a Lie subalgebra, the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, with basis elements q_j, p_j, c. Here c is a constant with the same units as the action, usually normalized to 1.

Canonical quantization then proceeds by construction of a non-commutative algebra of operators. As a vector space this algebra is infinite-dimensional, with basis elements the ordered products of powers of the generators

$$C^i Q_1^{j_1} \cdots Q_n^{j_n} P_1^{k_1} \cdots P_n^{k_n}$$

and relations

$$[Q_j, P_k] = C\delta_{jk}$$

C is central, commuting with all other elements of the algebra. This algebra is also known as $U(\mathfrak{h}_{2n+1})$, the universal enveloping algebra of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. The operators are supposed to act irreducibly on a complex vector space with Hermitian inner product, the state space $\mathcal{H}$ of the quantum system. The central element C acts as the scalar $i\hbar$, chosen to be pure imaginary so that one has a unitary representation of the Lie algebra $\mathfrak{h}_{2n+1}$.

The algebra acting on $\mathcal{H}$ is thus the quotient

$$U(\mathfrak{h}_{2n+1})/(C - i\hbar\mathbf{1})$$

which is just the algebra generated by the Q_j, P_k, satisfying only the Heisenberg commutation relations

$$[Q_j, P_k] = i\hbar\delta_{jk}$$

2.2 Quantization and the universal enveloping algebra

For an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ of a Lie group G, one has much the same structures as in the special case of the Heisenberg Lie algebra above, providing a general notion of quantization, valid for any $\mathfrak{g}$. Since linear functions on $\mathfrak{g}^*$ are elements of $\mathfrak{g}$, they come with a Lie bracket operation. This can be extended to give a Poisson bracket on all functions on $\mathfrak{g}^*$ by defining

$$\{f_1, f_2\}(\xi) = \xi([df_1, df_2])$$

(here $\xi \in \mathfrak{g}^*$ and f_1, f_2 are functions on $\mathfrak{g}^*$). A co-adjoint orbit $\mathcal{O}_\xi$ of $\xi \in \mathfrak{g}^*$ is a symplectic manifold, with (Kirillov-Kostant-Souriau) symplectic two form given by

$$\omega_\xi(\widetilde{X}, \widetilde{Y}) = \pm\xi([X, Y])$$

(check sign) where $\widetilde{X}$ is the vector field on $\mathfrak{g}^*$ given by the co-adjoint action of an element $X \in \mathfrak{g}$. The Poisson bracket on functions on $\mathcal{O}_\xi$ is just the restriction of the one on $\mathfrak{g}^*$ given above. The orbits $\mathcal{O}_\xi$ with this Poisson bracket can be thought of as generalizations of the classical notion of phase space as a vector space of dimension $2n$, which is the special case $\mathfrak{g} = \mathfrak{h}_{2n+1}$.

Quantization of $\mathfrak{g}^*$ corresponds to considering the universal enveloping algebra $U(\mathfrak{g})$, which can be defined as the quotient

$$U(\mathfrak{g}) = T^*(\mathfrak{g})/(X \otimes Y - Y \otimes X - [X, Y]), \quad X, Y \in \mathfrak{g} = T^1(\mathfrak{g})$$

of the tensor algebra $T^*(\mathfrak{g})$ by an ideal of relations determined by the Lie bracket. If e_j are a basis of $\mathfrak{g}$ and dim $\mathfrak{g} = n$, then, as a vector space, $U(\mathfrak{g})$ has basis elements

$$e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n} \quad m_j = 0, 1, 2, \cdots$$

While $T^*(\mathfrak{g})$ is a graded algebra, $U(\mathfrak{g})$ is not, since the ideal that defines it is not homogeneous. $U(\mathfrak{g})$ does inherit a filtration from $T^*(\mathfrak{g})$, with $F_pU(\mathfrak{g})$

the subspace in $U(\mathfrak{g})$ spanned by products of up to p elements of $\mathfrak{g}$. This is an algebra filtration since $F_qU(\mathfrak{g}) \subset F_pU(\mathfrak{g})$ for $q < p$, and

$$F_pU(\mathfrak{g}) \cdot F_qU(\mathfrak{g}) \subset F_{p+q}U(\mathfrak{g})$$

As for any such filtered algebra, one can define an associated graded algebra as

$$Gr(U(\mathfrak{g})) = \bigoplus_{p=0}^{\infty} F_pU(\mathfrak{g})/F_{p-1}U(\mathfrak{g})$$

This graded algebra is commutative and, by the Poincaré-Birkhoff-Witt theorem, isomorphic to the symmetric algebra $S^*(\mathfrak{g})$, which in turn is isomorphic to the algebra of polynomials on $\mathfrak{g}^*$. An alternative form of the Poincaré-Birkhoff-Witt theorem is the statement that the symmetrization map

$$q : e_1 \cdots e_k \in S^k(\mathfrak{g}) \to \frac{1}{k!} \sum_{\sigma \in S_k} e_{\sigma(1)} \cdots e_{\sigma(k)} \in U(\mathfrak{g})$$

is an isomorphism of filtered vector spaces (but not of algebras). One can think of q as a "quantization" map

$$q : S^*(\mathfrak{g}) \to U(\mathfrak{g})$$

that takes elements of the commutative algebra of polynomial functions on $\mathfrak{g}^*$ to elements of the non-commutative algebra $U(\mathfrak{g})$, which will be realized as operators on a state space $\mathcal{H}$.

From the commutator on $U(\mathfrak{g})$ one can define a Poisson bracket on $S^*(\mathfrak{g})$ as follows. For

$$[\alpha] \in F_pU(\mathfrak{g})/F_{p-1}U(\mathfrak{g}),\ [\beta] \in F_qU(\mathfrak{g})/F_{q-1}U(\mathfrak{g})$$

take the Poisson bracket to be

$$\{[\alpha], [\beta]\} = [\alpha\beta - \beta\alpha] \in F_{p+q-1}U(\mathfrak{g})/F_{p+q-2}U(\mathfrak{g})$$

Such a Poisson bracket is anti-symmetric

$$\{[\alpha], [\beta]\} = -\{[\beta], [\alpha]\}$$

and satisfies the Jacobi identity.

Note that $X \in \mathfrak{g}$ acts on $U(\mathfrak{g})$ in three ways:

- The left regular representation

$$\alpha \in U(\mathfrak{g}) \to X\alpha$$

- The right regular representation

$$\alpha \in U(\mathfrak{g}) \to -\alpha X$$

- The adjoint representation

$$\alpha \in U(\mathfrak{g}) \rightarrow X\alpha - \alpha X = [X, \alpha]$$

The last of these is an example of the usual quantization of the action of a Lie group in Hamiltonian mechanics. In this case the infinitesimal generator of the action is the linear function X on the classical space $\mathfrak{g}^*$, and the group action is the co-adjoint action on $\mathfrak{g}^*$.

The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is the subalgebra $U(\mathfrak{g})^{\mathfrak{g}}$ of elements invariant under the adjoint $\mathfrak{g}$ action. This is a commuting subalgebra of elements that commute with everything in $U(\mathfrak{g})$. It can be identified with the adjoint-invariant subalgebra of $Gr(U(\mathfrak{g})) = S^*(\mathfrak{g})$, or equivalently the polynomial functions on $\mathfrak{g}^*$ invariant under the co-adjoint action on $\mathfrak{g}^*$. If $\mathfrak{g}$ has an invariant bilinear form, this gives a distinguished element of $Z(\mathfrak{g})$, the quadratic Casimir element.

2.3 Representations and the orbit philosophy

A quantization of $\mathfrak{g}^*$ requires not just the algebra $U(\mathfrak{g})$, but also a representation of the elements of this algebra as operators on a complex vector space $\mathcal{H}$, the state space of the quantum theory. This gives equivalently a module for the algebra $U(\mathfrak{g})$ or a representation of the Lie algebra $\mathfrak{g}$. While in general $\mathfrak{g}$ will be a real Lie algebra (i.e. a real vector space with Lie bracket), we will only consider complex representations, and in this case modules for $U(\mathfrak{g})$ are also modules for $U(\mathfrak{g}_{\mathbf{C}})$, where $\mathfrak{g}_{\mathbf{C}}$ is the complexification of $\mathfrak{g}$. Unitarity of the representation is a property (skew-adjointness) of the action of $\mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}$ on $\mathcal{H}$ (for a given inner product on $\mathcal{H}$).

If $\mathfrak{g}$ is the Lie algebra of the Lie group G, then elements of $\mathfrak{g}$ correspond to left-invariant vector fields on G, and are represented on functions on G as linear differential operators. Elements of $U(\mathfrak{g})$ then act on functions on G as left-invariant differential operators of all orders, with product the composition of differential operators. Such a representation is however a reducible representation, and what one really wants is to identify the possible irreducible representations. Each of these will give a quantization of $\mathfrak{g}^*$ that cannot be decomposed into simpler pieces.

Finding and constructing the irreducible representations for a given Lie algebra $\mathfrak{g}$ is in general a difficult problem. One way to try and characterize an irreducible representation is by using the fact that elements of the center $Z(\mathfrak{g})$ are invariant polynomials on $\mathfrak{g}^*$, and will act on the representation as scalars. As invariant polynomials, they take constant values on co-adjoint orbits, and one can try and identify these values with their eigenvalues on an irreducible representation.

The Kostant-Kirillov "orbit philosophy" (for two expository treatments see [9] and [21]) posits that one can associate co-adjoint orbits to irreducible representations. The difficult problem then is that of how to construct the representation corresponding to a given co-adjoint orbit. These orbits are symplectic manifolds with an action of G preserving the symplectic structure. Thinking of this as the phase space of a classical system, the problem of constructing the corresponding representation is exactly the physical problem of how to "quantize" a classical system. The method of "geometric quantization" in many cases provides a solution, but this does not always work (this is why the orbit philosophy is a "philosophy" and not a theorem). In the rest of this section we'll see some examples where it does work.

2.3.1 Canonical quantization

As discussed earlier in section 2.1, canonical quantization corresponds to the case where $\mathfrak{g} = \mathfrak{h}_{2n+1}$, the Heisenberg Lie algebra. In this case the center $Z(\mathfrak{h}_{2n+1})$ consists of polynomials in c, with co-adjoint orbits labeled by the constant value C that the element c takes on the orbit. For each non-zero value of C, the co-adjoint orbit is the usual phase space $\mathbf{R}^{2n}$ with the usual Poisson bracket (with normalization determined by C). The Stone-von Neumann theorem ensures that, for each choice of orbit, any two constructions of the corresponding irreducible representation will be unitarily equivalent (assuming that they integrate to representations of the Heisenberg group, not just the Lie algebra). This uniqueness property explains why usual discussions of canonical quantization just invoke the Heisenberg commutation relations

$$[Q_j, P_k] = i\hbar\mathbf{1}$$

thereby setting the value of C at $i\hbar$ (pure imaginary so that the representation is unitary), without making a choice of representation.

All constructions of a representation of Q_j, P_k on a state space $\mathcal{H}$ require a choice of an additional piece of structure, although by Stone-von Neumann different choices will give unitarily equivalent representations. Splitting the phase space coordinates into position and momentum coordinates allows the Schrödinger construction of $\mathcal{H}$ as complex-valued functions on position space $\mathbf{R}^n$ or (by Fourier transform) on momentum space $\mathbf{R}^n$.

Another possibility is the Bargmann-Fock representation, which depends on the choice of a complex structure J on $\mathbf{R}^{2n}$ (an operator J such that $J^2 = -1$). Complexified coordinates on $\mathbf{R}^{2n}$ then split up into n holomorphic coordinates z_j ($+i$ eigenvalues of J) and n anti-holomorphic coordinates $\overline{z}_j$ ($-i$ eigenvalues of J). Complex valued polynomials on $\mathbf{R}^{2n}$ are then a tensor product

$$S^*(\mathbf{C}^n) \otimes S^*(\overline{\mathbf{C}^n})$$

of the space of polynomials in the z_j and the space of polynomials in the $\overline{z}_j$. One can construct an irreducible representation of $U(\mathfrak{h}_{2n+1})$ on either one of these two spaces. The construction depends on a choice of J, but different J give unitarily equivalent representations. For a detailed discussion of this construction, see [22].

2.3.2 Representations of compact Lie groups

A simple example to keep in mind is the case of the Lie algebra $\mathfrak{su}(2)$, for which co-adjoint orbits are spheres in $\mathbf{R}^3$. Geometric quantization requires first choosing a complex line bundle L over the sphere, with curvature the area form on the sphere (this step is called "prequantization"). There is an integrality condition on the curvature and thus on the area form of the sphere. Only for spheres satisfying this integrality condition can one construct an appropriate line bundle L_k (for $k = 0, 1, \cdots$).

The sections of L_k are a representation of $SU(2)$, but this is an infinite-dimensional reducible representation. To get an irreducible representation, one needs to pick an invariant complex structure on the sphere, such that L_k is a holomorphic line bundle. The space of holomorphic sections ($H^0\mathcal{O}(L_k)$) of the line bundle will then give the spin $k/2$ representation of $SU(2)$. Two things to note are that

- One choice of invariant complex structure will give holomorphic line bundles L_k with sections, the other choice (equivalently, change of orientation of the sphere) will have no sections. For the choice with no sections, the sections appear in higher cohomology (H^1 rather than H^0).
- The general method of geometric quantization requires taking into account a "metaplectic correction", involving a square root of the canonical bundle. In this case the ambiguity in how one handles this is reflected in an ambiguity of whether one associates the trivial representation to the zero co-adjoint orbit or to the smallest non-zero one satisfying the quantization condition.

For a general simple compact Lie group G, there are different types of co-adjoint orbits, but generically the co-adjoint orbit will be the flag manifold G/T (T is the maximal torus). The integrality conditions will correspond to integrality of the weights λ of T. There will be a line bundle L_λ on G/T, as well as $|W|$ choices of invariant complex structure (here W is the Weyl group). The Borel-Weil theorem says that one gets the irreducible representations of G on the spaces $H^0\prime(L_\lambda)$ of holomorphic sections, for λ a "dominant" weight (a condition that depends on the complex structure). For non-dominant weights, the Borel-Weil-Bott theorem describes how the representations will appear in higher cohomology spaces. These conditions involve a "ρ-shift" by a special weight that is half the sum of the positive roots.

Instead of working with complex structures on G/T and Dolbeault cohomology groups of holomorphic line bundles, one can instead use the Dirac operator:

2.3.3 $SL(2, \mathbf{R})$ and discrete series representations

2.3.4 $SU(2,2)$ and minimal representations

3 The super Lie algebras $\widetilde{\mathfrak{g}}_d$ and $\widehat{\mathfrak{g}}_d$

A fundamental idea of modern mathematics is that it is often a good idea to consider not vector spaces, but complexes of vector spaces with a differential (see appendix B). One example is the de Rham complex, where one replaces consideration of the vector space of functions on a manifold M by the de Rham complex of differential forms on M. Given that in the orbit method we are interested in constructing representations on vector spaces built out of functions on orbits, it is natural to consider instead complexes of such representations, and one finds that larger structures than $\mathfrak{g}$ will act on such complexes.

3.1 The differential super Lie algebra $\widetilde{\mathfrak{g}}_d$

When a manifold M (such as an orbit) comes with an action of the Lie group G, for each $X \in \mathfrak{g}$ the derivative of the action will give a vector field on M (which we'll denote by X_M). The action of the vector field X_M on functions can be extended to an action on the space $\Omega^k(M)$ of k-forms by an operator called the Lie derivative, written L_X. The L_X satisfy the commutation relations

$$[L_X, L_Y] = L_{[X,Y]}$$

and provide a representation of $\mathfrak{g}$ on the $\Omega^k(M)$.

Also acting on the complex $\Omega^*(M)$ are operators

$$i_X : \Omega^k(M) \to \Omega^{k-1}$$

corresponding to contraction by the vector field X_M, which satisfy $i_X^2 = 0$. The relation between the de Rham differential d and the i_X, L_X is given by

$$L_X = di_X + i_X d = (d + i_X)^2$$

The structure that acts on $\Omega^*(M)$ extending the action of $\mathfrak{g}$ by the L_X can be thought of as a "differential super Lie algebra". By "super Lie algebra" we mean a vector space with $\mathbf{Z}_2$-graded (even and odd) components, and a super Lie bracket $[\cdot, \cdot]_\pm$ satisfying $\mathbf{Z}_2$-graded versions of the usual symmetry condition and Jacobi identity

$$[\alpha, \beta]_\pm = -(-1)^{|\alpha||\beta|}[\beta, \alpha]_\pm$$

and

$$[\alpha, [\beta, \gamma]_\pm]_\pm + (-1)^{|\alpha||\beta|}[\gamma, [\alpha, \beta]_\pm]_\pm + (-1)^{|\beta||\gamma|}[\beta, [\gamma, \alpha]_\pm]_\pm$$

where $|\alpha| = 0, 1$ depending on whether α is even or odd. An operator D is a super-derivation if it satisfies

$$D(\alpha\beta) = (D\alpha)\beta + (-1)^{|\alpha||D|}\alpha(D\beta)$$

We are interested in the super Lie algebra

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{R}[\epsilon] = \mathfrak{g} \oplus \epsilon\mathfrak{g}$$

where $\mathfrak{g}$ is the usual Lie algebra with even grading, and $\epsilon\mathfrak{g}$ is the same Lie algebra with odd grading. The super Lie bracket relations are given by

$$[X, Y]_{\widetilde{\mathfrak{g}}} = [X, Y]$$

$$[X, \epsilon X]_{\widetilde{\mathfrak{g}}} = \epsilon[X, Y]$$

$$[\epsilon X, \epsilon Y]_{\widetilde{\mathfrak{g}}} = 0$$

The differential d acts on generators by

$$d(\epsilon X) = X$$

$$dX = 0$$

and can be thought of as the differentiation operator $\frac{\partial}{\partial \epsilon}$. One can also think of d as an extra generator, and consider the Lie superalgebra

$$\widetilde{\mathfrak{g}}_d = \widetilde{\mathfrak{g}} \oplus \mathbf{R}d$$

with bracket relations

$$[d, d]_{\widetilde{\mathfrak{g}}} = 0$$

$$[d, X]_{\widetilde{\mathfrak{g}}} = 0$$

$$[d, \epsilon X]_{\widetilde{\mathfrak{g}}} = X$$

A representation of $\widetilde{\mathfrak{g}}$ will be a $\mathbf{Z}_2$-graded vector space $V = V^+ \oplus V^-$, with operators on V for each element of $\widetilde{\mathfrak{g}}$, satisfying the above commutation relations. When we have such a representation we will write the operators corresponding to $X, \epsilon X, d$ as L_X, i_X, d respectively, with the last of the relations above appearing as

$$L_X = di_X + i_X d \tag{1}$$

Elements of $V^{\widetilde{\mathfrak{g}}}$, i.e. elements of V invariant under the action of $\widetilde{\mathfrak{g}}$, are said to be "basic" elements of V. $V^{\widetilde{\mathfrak{g}}_d}$ will be the subspace of basic elements that are also closed (d acts as 0). Whereas (see appendix B) the Lie algebra cohomology of a representation of $\mathfrak{g}$ is the derived functor of the $\mathfrak{g}$-invariants functor, the derived functor of the $\widetilde{\mathfrak{g}}$-invariants functor will be the equivariant cohomology.

3.2 The central extension $\widehat{\mathfrak{g}}$

In the usual case of a phase space $\mathbf{R}^{2n}$, canonical quantization requires that we pick an antisymmetric non-degenerate bilinear form on $\mathbf{R}^{2n}$ and look for representations of a central extension of $\mathbf{R}^{2n}$, the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. Since we have introduced $\epsilon\mathfrak{g}$, a fermionic analog of phase space, quantization will now require picking a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ and replacing $\widetilde{\mathfrak{g}}$ with the central extension

$$\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbf{R}c$$

which has super Lie bracket relations

$$[X, Y]_{\widehat{\mathfrak{g}}} = [X, Y]$$

$$[X, \epsilon Y]_{\widehat{\mathfrak{g}}} = \epsilon[X, Y]$$

$$[\epsilon X, \epsilon Y]_{\widehat{\mathfrak{g}}} = (X, Y)c$$

This will be a super Lie algebra if (X, Y) is an invariant symmetric bilinear form on $\mathfrak{g}$ (check). The super Lie algebra $\widehat{\mathfrak{g}}$ can be extended (with d having the same bracket relations as for $\widetilde{\mathfrak{g}}$) to a super Lie algebra

$$\widehat{\mathfrak{g}}_d = \widehat{\mathfrak{g}} \oplus \mathbf{R}d$$

Note that the notation "$\widehat{\mathfrak{g}}$" is somewhat justified by the analogy between this situation and that of the affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[z, z^{-1}] \oplus \mathbf{R}c$$

which involves an analogous central extension and differential $(\frac{\partial}{\partial z})$.

3.3 The dual $\widetilde{\mathfrak{g}}_d^*$

When one has a $\mathbf{Z}_2$-graded vector spaces V, there are two possible notions of dual vector space:

- The even dual. This is just the usual space of linear maps from V to $\mathbf{R}$.
- The odd dual. This is the space of linear maps from V to $\mathbf{R}$, with $\mathbf{R}$ given the odd grading.

Just as $\mathfrak{g}$ acts by the co-adjoint representation on $\mathfrak{g}^*$, there is a co-adjoint action of $\widetilde{\mathfrak{g}}$ on $\widetilde{\mathfrak{g}}_d^*$ given by

$$L_X\mu = -ad_X^*\mu$$

$$L_X\epsilon\mu = -ad_X^*\epsilon\mu$$

$$i_X\mu = -ad_X^*\epsilon\mu$$

$$i_X\epsilon\mu = \mu(X)c$$

4 Clifford algebras and fermionic quantization

We would like to replace the usual quantization, which starts with a Lie algebra $\mathfrak{g}$ and gives the algebra $U(\mathfrak{g})$ realized as operators on some state space, by a quantization of the super Lie algebra $\widehat{\mathfrak{g}}$ and its differential d. $\widehat{\mathfrak{g}}$ contains as a subalgebra an extra piece, the super Lie algebra central extension

$$\epsilon\mathfrak{g} \oplus \mathbf{R}c$$

which uses an invariant symmetric bilinear form $(\cdot, \cdot)$ on the odd vector space $\epsilon\mathfrak{g}$. Quantization of this will give the Clifford algebra Cliff($\mathfrak{g}$).

In this section, we'll review the usual method of fermionic quantization. This is a precise analog of bosonic canonical quantization, where instead of starting with phase space (an even dimensional phase space with a non-degenerate antisymmetric bilinear form), one starts with a vector space V on any dimension, with a symmetric bilinear form. The table below gives some indication of the analogy, for much more detail see chapters 27-32 of [22].

Bosonic	**Fermionic**
$V = \mathbf{R}^{2d}$, even grading	$\epsilon V = \mathbf{R}^n$, odd grading
Non-degenerate antisymmetric bilinear form $\omega(\cdot,\cdot)$ on V	Non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on ϵV
Central extension $$V \oplus \mathbf{R}c$$ is a Lie algebra, $\mathfrak{h}_{2n+1}$	Central extension $$\epsilon V \oplus \mathbf{R}c$$ is a super Lie algebra
Weyl algebra $\text{Weyl}(V) = U(\mathfrak{h}_{2n+1})/(c-1)$	Clifford algebra $\text{Cliff}(V) = U(\epsilon V \oplus \mathbf{R}c)/(c-1)$
Classical observables $Gr(\text{Weyl}(V)) = S^*(V)$	Pseudo-classical observables $Gr(\text{Cliff}(V)) = \Lambda^*(V)$

4.1 Clifford algebras and exterior algebras

Given any vector space V of dimension n with a symmetric bilinear form $(\cdot,\cdot)$, one can define an associated Clifford algebra as a quotient of the tensor algebra

$$\text{Cliff}(V) = T^*(V)/(v \otimes w + w \otimes v - 2(v,w)1)$$

where $v, w \in V = T^1(V)$, $1 \in T^0(V)$. Identifying $V = T^1(V)$ we will denote by $\gamma(v)$ the element in $\text{Cliff}(V)$ corresponding to $v \in V$. Given a basis e_j of V, $\text{Cliff}(V)$ is the algebra generated by the γ_j, with relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2(\gamma_j,\gamma_k)1$$

This will be a finite dimensional algebra, of dimension 2^n.

If $(\cdot,\cdot)$ is the zero bilinear form, then the associated $\text{Cliff}(V)$ is just the exterior algebra $\Lambda^*(V)$ with the usual wedge product, which one can think of as the algebra of polynomials in anti-commuting coordinates on V^*. If $(\cdot,\cdot)$ is non-degenerate and one is working over $\mathbf{R}$, one can choose a basis such that

$$\gamma_j\gamma_k + \gamma_k\gamma_j = \pm 2\delta_{jk}$$

while over $\mathbf{C}$ there is always a basis such that

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}$$

Note that many authors use a different sign in the defining relation of the Clifford algebra, imposing the relation

$$v \otimes w + w \otimes v = -2(v,w)1$$

The sign and normalization used here follow Meinrenken[17] since we'll be using that reference extensively later on. Also note that the γ matrices used in discussions of the Dirac equation in relativistic quantum mechanics correspond to the case of the Lorentz metric on (according to convention) $V = \mathbf{R}^{3,1}$ or $\mathbf{R}^{1,3}$.

An alternate way to get the same definition of Cliff(V) is to use $(\cdot, \cdot)$ to define a super Lie algebra structure on the central extension

$$\epsilon V \oplus \mathbf{R}c$$

by defining the non-zero bracket to be

$$[\epsilon v, \epsilon w]_{\epsilon V \oplus \mathbf{R}c} = 2(v, w)c$$

The Clifford algebra is then given in terms of the universal enveloping algebra of this super Lie algebra as

$$\mathrm{Cliff}(V) = U(\epsilon V \oplus \mathbf{R}c)/(c - 1)$$

A basis for Cliff(V) is given by the 2^n elements

$$1$$

$$\gamma_{i_1}$$

$$\gamma_{i_1}\gamma_{i_2}$$

$$\cdots$$

$$\gamma_1\gamma_2\cdots\gamma_n$$

where $i_1 < i_2 < \cdots < i_k$. Cliff(V) is a filtered algebra, with $F_p\mathrm{Cliff}(V)$ the subspace of elements that are products of $\leq p$ generators. One has

$$F_p\mathrm{Cliff}(V)/F_{p-1}\mathrm{Cliff}(V) = \Lambda^p(V)$$

(actually, in this case

$$\Lambda^p(V) = F_p(\mathrm{Cliff}(V))/F_{p-2}(\mathrm{Cliff}(V))$$

since degrees change by two) so the associated graded algebra is

$$Gr(\mathrm{Cliff}(V)) = \Lambda^*(V)$$

with product the wedge product. While $\Lambda^*(V)$ is a $\mathbf{Z}$ graded algebra, $\mathrm{Cliff}(V) = \mathrm{Cliff}^{even}(V) \oplus \mathrm{Cliff}^{odd}(V)$ is only $\mathbf{Z}_2$-graded, since the Clifford product does not preserve degree but can change it by two when multiplying generators.

There is a quantization map

$$q : \Lambda^*(V) \to \mathrm{Cliff}(V)$$

which on an orthonormal basis e_j of V is given by

$$q(e_j) = \gamma(e_j) \equiv \gamma_j$$

On a wedge product of k vectors $v_i \in \Lambda^k(V)$ the quantization map is the anti-symmetrization map

$$q(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) q(v_1) q(v_2) \cdots q(v_k)$$

which on products of basis elements is just

$$q(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \gamma_{i_1} \gamma_{\imath_2} \cdots \gamma_{i_k}$$

The inverse $s = q^{-1}$: Cliff(V) $\rightarrow \Lambda^*(V)$ is sometimes called the "symbol map". This identification as vector spaces is known as the "Chevalley identification". Using it, one can think of the Clifford algebra as just an exterior algebra with a different product.

One can define a supercommutator on Cliff(V) by defining it on elements α, β that are products of $|\alpha|$ and $|\beta|$ generators respectively as

$$[\alpha, \beta]_{\pm} = \alpha\beta - (-1)^{|\alpha||\beta|} \alpha\beta$$

This induces a super-Poisson bracket on $\Lambda^* V$ by

$$\{[\alpha], [\beta]\}_{\pm} = [[\alpha, \beta]_{\pm}]$$

and gives a fermionic analog of the usual Hamiltonian formalism. V is now a pseudo-classical phase space with classical observables in $\Lambda^* V$, which after quantization become elements of Cliff(V).

4.2 Clifford algebra and the orthogonal Lie algebra

The Poisson bracket $\{\cdot, \cdot\}_{\pm}$ defined above is a Lie bracket on $\Lambda^2(V)$, making it into a Lie algebra isomorphic with the Lie algebra $\mathfrak{so}(V)$ of the group $SO(V)$ which preserves the bilinear form $(\cdot, \cdot)$. $\lambda \in \Lambda^2(V)$ acts on V by

$$v \rightarrow A_\lambda(v) = \{\lambda, v\}_{\pm}$$

The quantization map

$$q : \Lambda^2(V) \rightarrow \mathrm{Cliff}(V)$$

is a Lie algebra isomorphism between $\Lambda^2(V)$ and quadratic elements in Cliff(V), satisfying

$$[q(\lambda), q(\lambda')]_{\pm} = q(\{\lambda, \lambda'\}_{\pm})$$

Given two different orthonormal basis vectors e_j, e_k, the isomorphism above is

$$e_j \wedge e_k \leftrightarrow \frac{1}{2} \gamma_j \gamma_k$$

with this element of the Lie algebra the one that gives infinitesimal rotations in the jk plane. The expression of Lie algebra elements as quadratic elements of the Clifford algebra allows the construction (by exponentiation) of the group $Spin(n)$, which has the same Lie algebra of $SO(n)$, but is a double cover of this group.

4.3 Clifford modules and spinors

In the case of the universal enveloping algebra $U(\mathfrak{g})$, to construct a quantum system one needed to somehow construct an irreducible representation of $\mathfrak{g}$, which gives a state space $\mathcal{H}$ with elements of $U(\mathfrak{g})$ acting as operators on $\mathcal{H}$. For the fermionic case, one needs to similarly find an irreducible representation of the Clifford algebra, whose elements then become operators on a state space $\mathcal{H}$. The Clifford algebra case is much simpler than the $U(\mathfrak{g})$ case, especially if we complexify V. Over $\mathbf{C}$ there are isomorphisms

$$\text{Cliff}(V) = M(2^k, \mathbf{C})$$

(where $M(j, \mathbf{C})$ is the algebra of complex j by j matrices) for dim $V = n = 2k$ even, and

$$\text{Cliff}(V) = M(2^k, \mathbf{C}) \oplus M(2^k, \mathbf{C})$$

for dim $V = n = 2k + 1$ odd.

For simplicity, we will just discuss the $n = 2k$ even dimensional case here. There is then a single irreducible representation of the Clifford algebra, the spinor representation S on $\mathcal{H} = \mathbf{C}^{2^k}$. This representation breaks up into two irreducible representations S^+, S^- of the group $Spin(n)$, since this group is generated by quadratic elements in the Clifford algebra, so preserves the even-odd grading.

One would like to construct the irreducible representation S in a coordinate-invariant manner (i.e. not using a particular choice of identification of Clifford algebra generators γ_j with complex matrices). From the point of view of abstract algebra, a choice of spinor representation is a choice of a minimal left ideal in the Clifford algebra. The situation is closely analogous to that of the Heisenberg Lie algebra of section 2.1. In that case, given a phase space $\mathbf{R}^{2k}$, a choice of complex structure J on this space allowed one to decompose polynomials on phase space into the tensor product of polynomials in holomorphic and anti-holomorphic coordinates, given an irreducible representation on $S^*(\mathbf{C}^k)$. For the Clifford algebra case, for dim $V = n = 2k$, one can also choose a complex structure J, and construct the irreducible spinor representation, now on the antisymmetric tensors, on $\Lambda^*(\mathbf{C}^k)$. This gives the needed state space $\mathcal{H} = \mathbf{C}^{2^k}$

The spinor representation S is also a representation of the group $Spin(n)$, the non-trivial double cover of $SO(n)$. It is reducible, with irreducible components $S^+ = \Lambda^{even}(\mathbf{C}^k)$ and $S^- = \Lambda^{odd}(\mathbf{C}^k)$. Recall from section 4.2 that the Lie algebra of $Spin(n)$ can be identified with quadratic elements in the Clifford algebra. S^+ and S^- give representations of the Lie algebra of $Spin(n)$ using the action of these quadratic elements on S.

Note that this construction of S depends on the choice of J, although different choices of J will give unitarily equivalent representations. The way this works is rather subtle, for details see chapter 31 of [22]. One aspect of the story is that for a given choice of J one will get a subgroup

$$U(k) \subset SO(2k)$$

that preserves J, and a double cover $\widetilde{U(k)} \subset Spin(2k)$. As a representation of this $\widetilde{U(k)}$, one should think of S not as $\Lambda^*(\mathbf{C}^k)$, but as

$$\Lambda^*(\mathbf{C}^k) \otimes \Lambda^k(\mathbf{C}^k)^{\frac{1}{2}}$$

Projectively, the spin representation is just $\Lambda^*(\mathbf{C}^k)$, but the projective factor is a crucial part of the story, and the way it appears depends on the choice of J.

5 Quantum and classical Weil algebras

5.1 Introduction

Given an invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$, one can define as in section 3.2 the super Lie algebra $\widehat{\mathfrak{g}}$ and then construct a generalization of the quantization of $\mathfrak{g}$ to a quantization of $\widehat{\mathfrak{g}}$, with the role of $U(\mathfrak{g})$ now played by the quantum Weil algebra

$$\mathcal{W}(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/(c-1) = U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$$

Our argument will be that it is this algebra rather than $U(\mathfrak{g})$ which best captures fundamental phenomena in both quantum theory and representation theory. $\widehat{\mathfrak{g}}$ has a differential d, and as a result $\mathcal{W}(\mathfrak{g})$ will contain a remarkable element $\not{D}$ built out of both bosonic and fermionic generators, an algebraic version of the Dirac operator. It is a filtered algebra, with graded algebra the classical Weil algebra

$$W(\mathfrak{g}) = S^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g})$$

.

5.2 The quantum Weil algebra

When one has an invariant symmetric bilinear form on $\mathfrak{g}$, one can use it to define a Clifford algebra $\mathrm{Cliff}(\mathfrak{g})$ and an enlargement of the enveloping algebra

$$\mathcal{W}(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/(c-1) = U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$$

called the quantum Weil algebra. In terms of generators and relations, this is the filtered superalgebra generated by (for $X \in \mathfrak{g}$) even elements $X \otimes 1$ of degree two and odd elements $1 \otimes \epsilon X$ of degree one, satisfying the super-commutation relations

$$[X\otimes 1, Y\otimes 1]_{\mathcal{W}} = [X,Y]\otimes 1, \quad [X\otimes 1, 1\otimes Y]_{\mathcal{W}} = 0, \quad [1\otimes X, 1\otimes Y]_{\mathcal{W}} = 2(X,Y)$$

While $\mathcal{W}(\mathfrak{g})$ has a $\mathbf{Z}$-filtration, it is not $\mathbf{Z}$-graded, just $\mathbf{Z}_2$-graded (the $\mathbf{Z}_2$-grading is that of the Clifford algebra). An irreducible representation of this algebra will be of the form $\mathcal{H} = V \otimes S$, with V an irreducible representation of $\mathfrak{g}$ and S a spinor module for Cliff($\mathfrak{g}$). This will provide a state space for a quantum theory containing both fermionic and bosonic degrees of freedom, with elements of $\mathcal{W}(\mathfrak{g})$ acting as operators on $\mathcal{H}$.

Since the inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ is invariant, for $X \in \mathfrak{g}$ the adjoint action

$$ad_X(\cdot) = [X, \cdot]$$

gives a representation of the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{g})$ on $\mathfrak{g}$. We saw in section 4.2 that elements of this Lie algbra correspond to elements of $\Lambda^2(\mathfrak{g})$ (which we'll denote by λ_X) and to quadratic elements in Cliff($\mathfrak{g}$) (which we'll denote $q(\lambda_X)$). The Lie algebra $\mathfrak{g}$ acts on generators of $\mathcal{W}(\mathfrak{g})$ by

$$L_X(Y \otimes 1) = [X, Y] \otimes 1, \quad L_X(1 \otimes Y) = 1 \otimes [X, Y]$$

This is an inner action on $\mathcal{W}(\mathfrak{g})$, since it is given by

$$L_X(\cdot) = [X \otimes 1 + 1 \otimes q(\lambda_X), \cdot]_{\mathcal{W}}$$

This action of $\mathfrak{g}$ on $\mathcal{W}(\mathfrak{g})$ can be extended to an action of the Lie superalgebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \epsilon\mathfrak{g}$ by having odd elements act by

$$i_X(Y \otimes 1) = 0 \quad i_X(1 \otimes Y) = (X, Y)(1 \otimes 1)$$

This is an inner operator, since it can be written

$$i_X(\cdot) = [1 \otimes \frac{1}{2}X, \cdot]_{\mathcal{W}}$$

The elements of $\mathcal{W}(\mathfrak{g})$ invariant under the $\widetilde{\mathfrak{g}}$ action are those annihilated by the X and ϵX actions. These will be

$$\mathcal{W}(\mathfrak{g})^{\widetilde{\mathfrak{g}}} = U(\mathfrak{g})^{\mathfrak{g}} \otimes 1 = Z(\mathfrak{g}) \otimes 1$$

5.3 The Dirac operator $\not{D}_{\mathfrak{g}}$

So far we have been considering the quantum Weil algebra as a product of two independent algebras. The truly remarkable aspect of $\mathcal{W}(\mathfrak{g})$ is that it allows the construction of something quite new, an operator $\not{D}_{\mathfrak{g}}$ built out of generators of both $U(\mathfrak{g})$ and Cliff($\mathfrak{g}$), which is an algebraic version of the Dirac operator. This operator will also provide a differential d acting on $\mathcal{W}(\mathfrak{g})$.

Given an orthonormal basis e_j of $\mathfrak{g}$ (note that some e_j may satisfy $(e_j, e_j) = -1$) and corresponding generators e_i of the Clifford algebra, this operator can be defined as

$$\not{D}_{\mathfrak{g}} = \sum_j (e^j \otimes \gamma_j) + 1 \otimes q(\phi)$$

The first term corresponds to the usual construction of a Dirac operator. Here $e^j = \pm e_j$ is the dual vector to e_j, where we are identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ using $(\cdot,\cdot)$. Note that it is at this point, the construction of $\not{D}_{\mathfrak{g}}$, that we need $(\cdot,\cdot)$ to be non-degenerate.

One has

$$(\sum_j (e^j \otimes \gamma_j))^2 = \sum_j e^j e_j \otimes 1 = \Omega_{\mathfrak{g}} \otimes 1$$

where $\Omega_{\mathfrak{g}} \in Z(\mathfrak{g})$ is the quadratic Casimir element for $\mathfrak{g}$ corresponding to $(\cdot,\cdot)$. Note that this construction of a square root of a Casimir operator by introducing a Clifford algebra is very much in the spirit of Dirac's initial discovery of the Dirac operator.

The second term is a cubic element of the Clifford algebra, the quantization of

$$\phi = \frac{1}{3}\sum_i \lambda_{e_i} \wedge e_i \in \Lambda^3(\mathfrak{g})$$

$q(\phi)$ is the element of Cliff$(\mathfrak{g})$ such that

$$[\phi, q(X)]_{\pm} = 2q(\lambda_X)$$

One justification for the choice of coefficient of the $q(\phi)$ term in $\not{D}_{\mathfrak{g}}$ is that it gives the following simple formula for the square of $\not{D}_{\mathfrak{g}}$

$$\not{D}_{\mathfrak{g}}^2 = \Omega_{\mathfrak{g}} \otimes 1 + \frac{1}{24} tr_{\mathfrak{g}}(\Omega_{\mathfrak{g}})(1 \otimes 1)$$

where $\Omega_{\mathfrak{g}}$ is the quadratic Casimir element in $U(\mathfrak{g})$.

We now can define a d acting on $\mathcal{W}(\mathfrak{g})$ and an inner action of the full $\widetilde{\mathfrak{g}}_d$, given on generators by

$$d(1 \otimes X) = [\not{D}_{\mathfrak{g}}, 1 \otimes X]_{\mathcal{W}} = 2(X \otimes 1 + 1 \otimes q(\lambda_X))$$

$$d(X \otimes 1) = [\not{D}_{\mathfrak{g}}, X \otimes 1]_{\mathcal{W}} =$$

This satisfies the relations of $\widetilde{\mathfrak{g}}_d$ since d, L_X, i_X acting on $\mathcal{W}(\mathfrak{g})$ satisfy the relation 1. The relation $d^2 = 0$ holds since

$$d^2(\cdot) = [\not{D}_{\mathfrak{g}}, [\not{D}_{\mathfrak{g}}, \cdot]_{\mathcal{W}}]_{\mathcal{W}} = [\not{D}_{\mathfrak{g}}^2, \cdot]_{\mathcal{W}}$$

and $\not{D}_{\mathfrak{g}}^2$ is in the center $Z(\mathfrak{g})$.

The cohomology of $\mathcal{W}(\mathfrak{g})$ with respect to this differential $d^{\mathcal{W}}$ is just the constants.

For more details about the above, see [17] and [13].

5.4 $\mathcal{W}(\mathfrak{g})$ as an enveloping algebra

To be expanded

$$\mathcal{W}(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/(c-1)$$

It is sometimes convenient to work with

$$\overline{X} = 2(X \otimes 1 + 1 \otimes q(\lambda_X))$$

instead of $X \otimes 1$, and then the action of $\widetilde{\mathfrak{g}}$ is given by

$$X \cdot (\cdot) = [\frac{1}{2}\overline{X}, \cdot]$$

and one has

$$d(1 \otimes X) = \overline{X}, \quad d\overline{X} = 0$$

and

$$\mathcal{W}(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/(c-1)$$

5.5 The classical Weil algebra

The quantum Weil algebra $\mathcal{W}(\mathfrak{g})$ is a filtered algebra, with generators of $U(\mathfrak{g})$ in degree two and generators of Cliff($\mathfrak{g}$) in degree one. The associated graded algebra is the classical Weil algebra:

$$\mathrm{Gr}(\mathcal{W}(\mathfrak{g})) = W_c(g) = S^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g})$$

This is a graded algebra with generators of $S^*(\mathfrak{g})$ carrying degree two, generators of $\Lambda^*(\mathfrak{g})$ degree one. It comes with a super-Poisson bracket, induced from the super-commutator on $\mathcal{W}(\mathfrak{g})$. The relations satisfied by generators and the action of $\widetilde{\mathfrak{g}}_d$ are essentially the same as for $\mathcal{W}(\mathfrak{g})$, replacing super-commutators by super-Poisson brackets.

The super-Poisson bracket on generators in $S^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g})$ is given by

$$\{X \otimes 1, Y \otimes 1\}_\pm = [X,Y]_\mathfrak{g} \otimes 1, \ \ \{X \otimes 1, 1 \otimes Y\}_\pm = 0, \ \ \{1 \otimes X, 1 \otimes Y\}_\pm = 2(X,Y)$$

We may also want to use as generator instead of $X \otimes 1$

$$\overline{X} = 2(X \otimes 1 + 1 \otimes \lambda_X)$$

The classical version of the Dirac operator is

$$D = \sum_j (e^j \otimes e_j) + 1 \otimes \phi$$

where e_j are an orthonormal basis in $\mathfrak{g} = \Lambda^1(\mathfrak{g})$ and $\phi \in \Lambda^3(\mathfrak{g})$ satisfies

$$\{\phi, X\}_\pm = 2\lambda_X$$

(note, likely typo on page 164 of Meinrenken).

There is an action of $\widetilde{\mathfrak{g}}_d$ on $W(\mathfrak{g})$ is given by

$$L_X(\cdot) = \{\frac{1}{2}\overline{X}, \cdot\}_\pm, \quad i_X(\cdot) = \{1 \otimes \frac{1}{2}X, \cdot\}_\pm, \quad d(\cdot) = \{D, \cdot\}_\pm$$

D satisfies

$$\{D, D\}_\pm = 2\sum_j e^j e_j \otimes 1$$

which Poisson commutes with everything in $W(\mathfrak{g})$, so $d^2 = 0$. The cohomology of $W(\mathfrak{g})$ with this d will be trivial, i.e. only in degree zero, with

$$H^0(W^*(\mathfrak{g})) = \mathbf{R}$$

On the separate factors $S^*(\mathfrak{g})$ and $\Lambda^*(\mathfrak{g})$ the quantization map q is just the symmetrization maps to $U(\mathfrak{g})$ and $\mathrm{Cliff}(\mathfrak{g})$ discussed earlier. Alekseev and Meinrenken[1] show however that the full quantization map

$$q : W^*(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*) \to \mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$$

is not just the tensor product map but...

Restricted to the basic subalgebras it becomes the Duflo map

$$q : S^*(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}} = Z(\mathfrak{g})$$

which is an isomorphism.

5.6 The Weil algebra, Chern-Weil theory and equivariant cohomology

The conventional Weil algebra is defined by

$$W(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*)$$

Since it involves $S^*(\mathfrak{g}^*)$ instead of $S^*(\mathfrak{g}) = Gr(U(\mathfrak{g}))$, unlike $W_c(\mathfrak{g})$ it does not naturally come with a notion of quantization or a super-Poisson bracket. However, when $\mathfrak{g}$ has a non-degenerate inner product, this provides an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, which can be used to identify $W(\mathfrak{g})$ and $W_c(\mathfrak{g})$.

$W^*(\mathfrak{g})$ does come with an action of $\widetilde{\mathfrak{g}}_d$, given on generators by

$$L_X(\mu \otimes 1) =, \quad L_X(1 \otimes \mu) =$$

$$i_X(\mu \otimes 1) =, \quad i_X(1 \otimes \mu) =$$

$$d(\mu \otimes 1) =, \quad d(1 \otimes \mu) =$$

Same story about change of variables as in the non-commutative case. Define the dual of $\widetilde{g}$, and dual of $\widehat{g}$, then

$$W(\mathfrak{g}) = S^*(\widehat{g}^*)/(c-1)$$

Before doing this, need to rewrite section of the dual of the super Lie algebra.

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, and P is a principal G-bundle over a manifold M, then the free G action on P gives an action of $\widetilde{\mathfrak{g}}_d$ on the differential forms $\Omega^*(P)$. A connection on P corresponds to an equivariant map

$$A : \mathfrak{g}^* \to \Omega^*(P)$$

This can be extended to a $\widetilde{\mathfrak{g}}_d$-equivariant algebra homomorphism

$$\theta : W^*(\mathfrak{g}) \to \Omega^*(P)$$

which takes the generators of $\Lambda^*(\mathfrak{g}^*)$ to connection one-forms on P, the generators of $S^*(\mathfrak{g}^*)$ to curvature two-forms.

In each case one can define a basic sub-complex of the algebra, the sub-complex annihilated by these operators. One has $(\Omega^*(P))^{basic} = \Omega^*(M)$, and $(W^*(\mathfrak{g}))^{basic} = S^*(\mathfrak{g}^*)^{\mathfrak{g}}$, the invariant polynomials on $\mathfrak{g}$.

Restricting to basic sub-complexes, a connection gives a homomorphism

$$\theta : S^*(\mathfrak{g}^*)^{\mathfrak{g}} \to \Omega^*(M)$$

This is the Chern-Weil homomorphism which takes invariant polynomials on $\mathfrak{g}$ to differential forms on M constructed out of curvature two-forms. Taking cohomology, for compact G, this homomorphism is independent of the connection and gives invariants of the bundle in $H^*(M)$.

Discuss Chern-Simons A remarkable property of the Dirac operator $\not{D}_{\mathfrak{g}}$ is that it can be defined by

$$\not{D}_{\mathfrak{g}} = q(CS)$$

i.e., as the quantization of the Chern-Simons element $CS \in W(\mathfrak{g})$. Given a connection $\theta : W(g) \to \Omega^*(P)$, CS is the element that maps to the Chern-Simons form of the connection. It satisfies $d(CS) = C_2$, where C_2 is the quadratic element of $S^*(\mathfrak{g}^*)$ constructed from the Killing form.

Another remarkable property of $\not{D}_{\mathfrak{g}}$ is the formula for its square. Using

$$\not{D}_{\mathfrak{g}}^2 = \frac{1}{2}[\not{D}_{\mathfrak{g}}, \not{D}_{\mathfrak{g}}]_{\pm} = \frac{1}{2}d(\not{D}_{\mathfrak{g}}) = \frac{1}{2}dq(CS) = \frac{1}{2}q(dCS) = \frac{1}{2}q(C_2)$$

Some comments on equivariant cohomology:

In this case one can think of the complex $W^*(\mathfrak{g})$ as a subcomplex of the de Rham complex $\Omega^*(EG)$ of a homotopically trivial space EG with free G-action, one that carries its cohomology, just as the Chevalley-Eilenberg complex is a subcomplex of $\Omega^*(G)$ carrying the cohomology of the manifold G. For the classifying space $BG = EG/G$, its cohomology ring can be computed as the cohomology of $W^*(\mathfrak{g})^{basic}$, which, since the differential is trivial, is just the ring of invariant polynomials $S^*(\mathfrak{g}^*)^{\mathfrak{g}}$.

For more details on the Weil algebra, a good reference is the book of Guillemin and Sternberg[6].

6 Subalgebras and Dirac cohomology

Since the quantum Weil algebra $\mathcal{W}(\mathfrak{g})$ has a differential d satisfying $d^2 = 0$, we can define a "Dirac cohomology" algebra as

$$H_{\not{D}}(\mathfrak{g}) = \frac{\mathrm{Ker(d)}}{\mathrm{Im(d)}}$$

but this turns out to just be the constants. To get something more interesting, we need to chose a subalgebra $\mathfrak{r} \subset \mathfrak{g}$ and look at the $\mathfrak{r}$-basic sub-algebra

$$\mathcal{W}(\mathfrak{g})^{\mathfrak{r}}$$

This will also come with a differential d satisfying $d^2 = 0$, with a more interesting cohomology given by

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}) = U(\mathfrak{r})^{\mathfrak{r}}$$

say something about the relation to equivariant cohomology?

On representations V of $\mathfrak{g}$ we can define

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}, V)$$

which will give an interesting characterization of the representation, and come with an action of $H_{\not{D}}(\mathfrak{g}, \mathfrak{r})$.

Given a subgroup $R \subset G$ with Lie algebra $\mathfrak{r}$, the space EG has a free action of R, and $EG/R = BR$. The Weil algebra $W(\mathfrak{g})$ provides a tractable algebraic model for the de Rham cohomology of EG, and the subalgebra $W(\mathfrak{g})^{\mathfrak{r}-basic}$ of basic elements for the $\mathfrak{r}$ action (annihilated by $\mathcal{L}_X,\ i_X$, for $X \in \mathfrak{r}$) provides a model for the deRham cohomology of BR. One has

$$H^*(W(\mathfrak{g})^{\mathfrak{r}-basic}) = H^*(BR) = (S^*(\mathfrak{r}^*))^{\mathfrak{r}}$$

In the quantum Weil algebra case, one has

$$H^*(\mathcal{W}(\mathfrak{g})^{\mathfrak{g}-basic}) = Z(\mathfrak{g})$$

and expects that

$$H^*(\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}) = Z(\mathfrak{r})$$

In a more explicit form, this sort of statement was first conjectured by Vogan, then proved by Huang-Pandzic[7], and more generally, by Kostant[12]. For a more abstract proof, see Alekseev-Meinrenken[2] and Kumar[16].

6.1 Quadratic subalgebras and the Dirac operator for a pair $\mathfrak{r} \subset \mathfrak{g}$

In this section, start by defining quadratic subalgebras

In such a situation one can define

- $\mathfrak{s}$ is the orthocomplement to $\mathfrak{r}$ with respect to B. One has $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ as vector spaces and $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{s}$. In general $\mathfrak{s}$ is just a vector subspace, it is not a Lie sub-algebra of $\mathfrak{g}$.

- $\mathrm{Cliff}(\mathfrak{s})$ is the Clifford algebra associated to $\mathfrak{s}$ and the inner product $B(\cdot,\cdot)|_{\mathfrak{s}}$.

- S is a spinor module for $\mathrm{Cliff}(\mathfrak{s})$. For $\mathfrak{s}$ even (complex) dimensional, this is unique up to isomorphism.

To make this isomorphism more explicit, note first that

$$\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic} = (U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$$

The Lie algebra also acts by the inner action the $\mathfrak{r}$-invariant sub-complex of $U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s})$. $\mathfrak{r}$ acts "diagonally" here, not just on the $U(\mathfrak{g})$ factor. In other words, we are using the homomorphism

$$\zeta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s})$$

defined by

$$\zeta(X) = X \otimes 1 + 1 \otimes \nu(X)$$

where

$$\nu : \mathfrak{r} \to Lie(SO(\mathfrak{s})) \subset \mathrm{Cliff}(\mathfrak{s})$$

is the representation of $\mathfrak{r}$ on the spinor module S as quadratic elements in $\mathrm{Cliff}(\mathfrak{s})$ coming from the fact that the adjoint action of $\mathfrak{r}$ on $\mathfrak{s}$ is an orthogonal action.

The Kostant Dirac operator is defined as the difference of the algebraic Dirac operators for $\mathfrak{g}$ and $\mathfrak{r}$

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = \not{D}_{\mathfrak{g}} - \not{D}_{\mathfrak{r}}$$

It is an element of $\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}$ and the restriction of the differential d on $\mathcal{W}(\mathfrak{g})$ to $\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}$ is given by

$$d(\cdot) = [\not{D}_{\mathfrak{g},\mathfrak{r}}, \cdot]$$

The map

$$Z(\mathfrak{r}) = U(\mathfrak{r})^{\mathfrak{r}} \to \mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic} = (U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$$

takes values on cocycles for d and is an isomorphism on cohomology.

Even more explicitly, one can write

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = \sum_{i=1}^{n} Z_i \otimes Z_i + 1 \otimes v$$

where

$$v = \frac{1}{2} \sum_{1 \le i,j,k \le n} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k$$

and the Z_i are an orthonormal basis of $\mathfrak{s}$.

The square of the Kostant Dirac operator is given by

$$\not{D}^2_{\mathfrak{g},\mathfrak{r}} = -\Omega_{\mathfrak{g}} \otimes 1 + \zeta(\Omega_{\mathfrak{r}}) + (|\rho_{\mathfrak{r}}|^2 - |\rho_{\mathfrak{g}}|^2) 1 \otimes 1$$

where $\rho_{\mathfrak{g}}$ is half the sum of the positive roots for $\mathfrak{g}$, $\rho_{\mathfrak{r}}$ the same for $\mathfrak{r}$.

$\not{D}^2_{\mathfrak{g},\mathfrak{r}}$ commutes with all elements of $U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s})$ so the differential d satisfies $d^2 = 0$. The differential d is an equivariant map for the $U(\mathfrak{r})$ action given by ζ, so it is also a differential on $(U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$. The operator Dirac cohomology is defined as

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}) = Ker\ d / Im\ d$$

on $(U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$. The Vogan conjecture says that it is isomorphic to $Z(\zeta(\mathfrak{r}))$. Note that in general this will be not $\mathbf{Z}$-graded, but just $\mathbf{Z}_2$-graded, using the $\mathbf{Z}_2$ grading of $\text{Cliff}(\mathfrak{s})$).

So, for any representation V of $\mathfrak{g}$, we have an algebra of operators $(U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$ acting on $V \otimes S$, with differential $d = ad(\not{D}_{\mathfrak{g},\mathfrak{r}})$ and cohomology isomorphic to the center $Z(\mathfrak{r})$.

6.2 Dirac cohomology: states

$\not{D}_{\mathfrak{g},\mathfrak{r}}$ acts on $V \otimes S$, and one can define the state Dirac cohomology as

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} / (Im\ \not{D}_{\mathfrak{g},\mathfrak{r}} \cap Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}})$$

Since $\not{D}^2_{\mathfrak{g},\mathfrak{r}} \neq 0$, this is not a standard sort of homological differential. In particular, one has no assurance in general that

$$Im\ \not{D}_{\mathfrak{g},\mathfrak{r}} \subset Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}}$$

However, if V is either finite-dimensional or a unitary representation, then an inner product on $V \otimes S$ can be chosen so that $\not{D}_{\mathfrak{g},\mathfrak{r}}$ will be skew self-adjoint. In that case $Ker(\not{D}_{\mathfrak{g},\mathfrak{r}}) = Ker(\not{D}^2_{\mathfrak{g},\mathfrak{r}})$ and one has

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}}$$

7 Examples: reductive Lie algebras

Note that the same Dirac operator appears earlier in the physics literature as the supersymmetry generator for the superparticle on the group G (see, e.g. [4]).

Dirac cohomology gives a version of the standard highest-weight theory for representations of semi-simple Lie algebras, as well as other applications in representation theory. For an exposition of some of these applications, see the recent book by Huang and Pandzic[8].

For an irreducible representation V of $\mathfrak{g}$, a well-known invariant is the infinitesimal character $\chi(V)$. Such infinitesimal characters can be identified with orbits in $\mathfrak{h}^*$ ($\mathfrak{h}$ is a Cartan sub-algebra) under the Weyl group $W_{\mathfrak{g}}$, with a representation of highest weight $\lambda \in \mathfrak{h}^*$ corresponding to the orbit of $\lambda + \delta_{\mathfrak{g}}$. The

Dirac cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V)$ of a representation V also provides an invariant of the representation V. For a finite-dimensional V, it will consist of a collection of $|W_{\mathfrak{g}}|/|W_{\mathfrak{r}}|$ $\mathfrak{r}$ irreducibles. These will all have the same infinitesimal character as V, when one includes $\mathfrak{t}^* \subset \mathfrak{h}^*$ ($\mathfrak{t}$ is the Cartan sub-algebra of $\mathfrak{r}$) by extending element of $\mathfrak{t}^*$ as zero on $\mathfrak{h}/\mathfrak{t}$. This phenomenon was first noticed by Pengpan and Ramond for the case $G = F_4$ and $R = Spin(9)$ and was explained in general in [5], which led to Kostant's discovery[15] of his version of the Dirac operator.

For specific choices of $\mathfrak{g}$ and $\mathfrak{r}$, one can compute the Dirac cohomology of representations. In each case, determination of the Dirac cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V)$ depends upon the formula

$$\not{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \zeta(\Omega_{\mathfrak{r}}) + (||\rho_{\mathfrak{r}}||^2 - ||\rho_{\mathfrak{g}}||^2)1 \otimes 1$$

and the fact that $\Omega_{\mathfrak{g}}$ acts by the scalar

$$||\lambda + \rho||^2 - ||\rho||^2$$

on an irreducible representation of highest-weight λ.

7.1 The cases $\mathfrak{r} = 0$ and $\mathfrak{r} = \mathfrak{g}$

- $\mathfrak{r} = 0$

  This is the case of no un-gauged symmetry, corresponding to the quantum Weil algebra itself. Here

  $$H_{\not{D}}(\mathfrak{g}, 0) = \mathbf{C}$$

  and

  $$H_{\not{D}}(\mathfrak{g}, 0; V) = 0$$

  for any V. On an irreducible representation of highest weight λ,

  $$\not{D}_{\mathfrak{g}}^2 = -|\lambda + \rho_{\mathfrak{g}}|^2 Id$$

  where $\rho_{\mathfrak{g}}$ is half the sum of the positive roots. This is negative-definite, so $ker\ \not{D}_{\mathfrak{g}} = 0$ for the action of $D_{\mathfrak{g}}$ on $V \otimes S$, for any representation V.

- $\mathfrak{r} = \mathfrak{g}$

  This is the case of no gauged symmetry. Here one has

  $$H_{\not{D}}(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g})$$

  and

  $$H_{\not{D}}(\mathfrak{g}, \mathfrak{g}; V) = V$$

7.2 Generalized highest-weight theory

The case $\mathfrak{r} = \mathfrak{h}$, the Cartan subalgebra of a complex semi-simple Lie algebra reproduces the Cartan-Weyl highest-weight theory, of finite dimensional representations, as generalized by Bott[3] and Kostant[14]. In this case

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$$

and one can identify Dirac cohomology and Lie algebra cohomology in the manner discussed above. One has

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{h}) = Z(\mathfrak{h}) = S^*(\mathfrak{h}^*)$$

and

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{h}; V_\lambda) = \sum_{w \in W} \mathbf{C}_{w(\lambda+\delta_{\mathfrak{g}})}$$

which has dimension $|W|$.

The Weyl character formula for the character $ch(V_\lambda)$ function on the Cartan sub-algebra can be derived from this, by taking a supertrace, i.e., the difference between the part of the Dirac cohomology lying in the half spinor S^+ and that lying in the half spinor S^-. One has

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W} (-1)^{l(w)} \mathbf{C}_{w(\lambda+\delta_{\mathfrak{g}})}$$

as $\mathfrak{h}$ represention. So the character satisfies

$$ch(V_\lambda) = \frac{ch(V_\lambda \otimes S^+ - V_\lambda \otimes S^-)}{ch(S^+ - S^-)}$$

which is

$$\frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\delta_{\mathfrak{g}})}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\delta_{\mathfrak{g}})}}$$

More generally, if $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic sub-algebra of $\mathfrak{g}$, with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{u}$, then again Dirac cohomology can be identified with the Lie algebra cohomology for the subalgebra $\mathfrak{u}$. One finds

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{l}) = Z(\mathfrak{l})$$

and

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{l}; V_\lambda) = \sum_{w \in W/W_{\mathfrak{l}}} V_{w(\lambda+\delta_{\mathfrak{g}})-\delta_{\mathfrak{l}}}$$

which is a sum of $|W/W_{\mathfrak{l}}|$ $\mathfrak{l}$-modules.

7.3 $(\mathfrak{g}, K)$ modules

For real semi-simple Lie algebras $\mathfrak{g}_0$, corresponding to real Lie groups G with maximal compact subgroup K (with Lie algebra $\mathfrak{k}_0$), the interesting unitary representations are infinite-dimensional. The simplest example here is $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbf{R})$, $K = SO(2)$ which has important applications in the theory of automorphic forms. These representations can be studied in terms of the corresponding Harish-Chandra $(\mathfrak{g}, K)$ modules, using relative Lie algebra cohomology $H^*(\mathfrak{g}, K; V)$ to produce invariants of the representations (see [10] and [11]). Here $\mathfrak{g}, \mathfrak{k}$ are the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$. Dirac cohomology can also be used in this context, and one finds (see [8], chapter 8)

$$H^*(\mathfrak{g}, K; V \otimes F^*) = Hom_{\mathfrak{k}}(H_{\not{D}}(\mathfrak{g}, \mathfrak{k}; F), H_{\not{D}}(\mathfrak{g}, \mathfrak{k}; V))$$

when V is an irreducible unitary $(\mathfrak{g}, K)$ module with the same infinitesimal character as a finite dimensional $(\mathfrak{g}, K)$ module F.

7.4 Examples not related to Lie algebra cohomology

A remarkable property of Dirac cohomology is that it exists and allows the definition of a physical space of states for a system with a symmetry group G, where a specified subgroup R remains ungauged, even in cases where $\mathfrak{g}/\mathfrak{r}$ is not a Lie algebra and is not even of the form $\mathfrak{u} \oplus \overline{\mathfrak{u}}$ for $\mathfrak{u}$ a Lie algebra. In other words, there is no Lie algebra to apply Lie algebra cohomology and the BRST method to.

A simple example is the case $\mathfrak{g} = \mathfrak{spin}(2n+1), \mathfrak{r} = \mathfrak{spin}(2n)$. Here for $n > 1$ $\mathfrak{spin}(2n+1)/\mathfrak{spin}(2n)$ cannot be decomposed as $\mathfrak{u} \oplus \overline{\mathfrak{u}}$, which corresponds to the fact that even dimensional spheres S^{2n} cannot be given an invariant complex structure for $n > 1$. In this case

$$H_{\not{D}}(\mathfrak{spin}(2n+1), \mathfrak{spin}(2n)) = Z(\mathfrak{spin}(2n))$$

and

$$H_{\not{D}}(\mathfrak{spin}(2n+1), \mathfrak{spin}(2n); V_\lambda)$$

is a sum of two $\mathfrak{spin}(2n)$ representations. For the trivial representation, these are just the two half-spinor representations of $\mathfrak{spin}(2n)$.

8 Relating BRST/Lie algebra cohomology to Dirac cohomology

The BRST method uses an operator Q satisfying $Q^2 = 0$, and describes states in terms of Lie algebra cohomology of the gauged subgroup, whereas the Dirac cohomology construction described above appears to be rather different. It depends on a choice of un-gauged subgroup R and defines states as the kernel of an operator that does not square to zero. It turns out though that these two methods give essentially the same thing in the case that

$$\mathfrak{g}/\mathfrak{r} = \mathfrak{u} \oplus \overline{\mathfrak{u}}$$

In this case, $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are isotropic subspaces with respect to the symmetric bilinear form B, and one can identify $\mathfrak{u}^* = \overline{\mathfrak{u}}$. The spinor module S for Cliff($\mathfrak{s}$) can be realized explicitly on either $\Lambda^*(\mathfrak{u})$ or on $\Lambda^*(\overline{\mathfrak{u}})$. However, when one does this, the adjoint $\mathfrak{r}$ action on $\Lambda^*(\mathfrak{u})$ differs from the $\mathfrak{spin}(\mathfrak{s})$ action on $S = \Lambda^*(\overline{\mathfrak{u}})$ by a scalar factor $\mathbf{C}_{\rho(\mathfrak{u})}$. Here $\rho(\mathfrak{u})$ is half the sum of the weights in $\mathfrak{u}$.

The Dirac operator in this situation can be written (for details of this, see [?]) as the sum

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = C^+ + C^-$$

where (using dual bases u_i for $\mathfrak{u}$ and u_i^* for $\mathfrak{u}^*$)

$$C^+ = \sum_i u_i^* \otimes u_i + 1 \otimes \frac{1}{4} \sum_{i,j} u_i u_j [u_i^*, u_j^*]$$

$$C^- = \sum_i u_i \otimes u_i^* + 1 \otimes \frac{1}{4} \sum_{i,j} u_i^* u_j^* [u_i, u_j]$$

The operators C^+ and C^- are differentials satisfying $(C^-)^2 = (C^+)^2 = 0$, and negative adjoints of each other. This is very much like the standard Hodge theory set-up, and one has

$$V \otimes S = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^+ \oplus Im\ C^-$$

$$Ker\ C^+ = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^+$$

$$Ker\ C^- = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^-$$

If one identifies $S = \Lambda^*(\mathfrak{u})$, then $V \otimes S$ with differential C^+ is the complex with cohomology $H^*(\overline{\mathfrak{u}}, V)$, and if one identifies $S = \Lambda^*(\overline{\mathfrak{u}})$, then $V \otimes S$ with differential C^- is the complex with homology $H_*(\mathfrak{u}, V)$. Both of these can be identified with the cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\ D_{\mathfrak{g},\mathfrak{r}}$. Note that one gets not the usual $\mathfrak{r}$ action on $H^*(\overline{\mathfrak{u}}, V)$ or $H_*(\mathfrak{u}, V)$, but the action twisted by the one-dimensional representation $\mathbf{C}_{\rho(\mathfrak{u})}$.

For the details of this, see Chapter 9 of [8].

9 Constructing representations

Using Dirac cohomology of the regular representation to get irreducible representations.

9.1 Dirac operators on G/R

:Work out Dirac cohomology in geometric case on G/R

The Dirac operator appearing here is an algebraic version of the Dirac operator. For the case of a group manifold, one can get geometric Dirac operators by taking the representation to be functions on the group.

Relation to the supersymmetric quantum mechanics proof of the index theorem, which has sometimes been claimed to come from a BRST-fixing of infinitesimal translations on the manifold.

A Appendix: Semi-simple Lie algebras

A.1 Structure of semi-simple Lie algebras

A semi-simple Lie algebra is a direct sum of non-abelian simple Lie algebras. Over the complex numbers, every such Lie algebra is the complexification $\mathfrak{g}_{\mathbf{C}}$ of some real Lie algebra $\mathfrak{g}$ of a compact, connected Lie group. The Lie algebra $\mathfrak{g}$ of a compact Lie group G is, as a vector space, the direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t}$$

where $\mathfrak{t}$ is a commutative sub-algebra (the Cartan sub-algebra), the Lie algebra of T, a maximal torus subgroup of G.

Note that $\mathfrak{t}$ is not an ideal in $\mathfrak{g}$, so $\mathfrak{g}/\mathfrak{t}$ is not a subalgebra. $\mathfrak{g}$ is itself a representation of $\mathfrak{g}$ (the adjoint representation: $\pi(X)Y = [X, Y]$), and thus a representation of the subalgebra $\mathfrak{t}$. On any complex representation V of $\mathfrak{g}$, the action of $\mathfrak{t}$ can be diagonalized, with eigenspaces V^λ labeled by the corresponding eigenvalues, given by the weights λ. These weights $\lambda \in \mathfrak{t}^*_{\mathbf{C}}$ are defined by (for $v \in V^\lambda,\ H \in \mathfrak{t}$):

$$\pi(H)v = \lambda(H)v$$

Complexifying the adjoint representation, the non-zero weights of this representation are called roots, and we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus ((\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C})$$

The second term on the right is the sum of the root spaces V^α for the roots α. If α is a root, so is $-\alpha$, and one can choose decompositions of the set of roots into “positive roots” and “negative roots” such that:

$$\mathfrak{n}^+ = \bigoplus_{+\text{ roots } \alpha} (\mathfrak{g}_{\mathbf{C}})^\alpha,\ \mathfrak{n}^- = \bigoplus_{-\text{ roots } \alpha} (\mathfrak{g}_{\mathbf{C}})^\alpha$$

where $\mathfrak{n}^+$ (the ”nilpotent radical”) and $\mathfrak{n}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$. So, while $\mathfrak{g}/\mathfrak{t}$ is not a subalgebra of $\mathfrak{g}$, after complexifying we have decompositions

$$(\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

The choice of such a decomposition is not unique, with the Weyl group W (for a compact group G, W is the finite group $N(T)/T$, $N(T)$ the normalizer of T in G) permuting the possible choices.

Recall that a complex structure on a real vector space V is given by a decomposition

$$V \otimes \mathbf{C} = W \oplus \overline{W}$$

so the above construction gives $|W|$ different invariant choices of complex structure on $\mathfrak{g}/\mathfrak{t}$, which in turn give $|W|$ invariant ways of making G/T into a complex manifold.

The simplest example to keep in mind is $G = SU(2),\ T = U(1),\ W = \mathbf{Z}_2$, where $\mathfrak{g} = \mathfrak{su}(2)$, and $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(2, \mathbf{C})$. One can choose T to be the diagonal matrices, with a basis of $\mathfrak{t}$ given by

$$\frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and bases of $\mathfrak{n}^+,\ \mathfrak{n}^-$ given by

$$\frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(here the σ_i are the Pauli matrices). The Weyl group in this case just interchanges $\mathfrak{n}^+ \leftrightarrow \mathfrak{n}^-$.

A.2 Highest weight theory

Irreducible representations V of a compact Lie group G are finite dimensional and correspond to finite dimensional representations of $\mathfrak{g}_{\mathbf{C}}$. For a given choice of $\mathfrak{n}^+$, such representations can be characterized by their subspace $V^{\mathfrak{n}^+}$, the subspace of vectors annihilated by $\mathfrak{n}^+$. Since $\mathfrak{n}^+$ acts as "raising operators", taking subspaces of a given weight to ones with weights that are more positive, this is called the "highest weight" space since it consists of vectors whose weight cannot be raised by the action of $\mathfrak{g}_{\mathbf{C}}$. For an irreducible representation, this space is one dimensional, and we can label irreducible representations by the weight of $V^{\mathfrak{n}^+}$. The irreducible representation with highest weight λ is denoted V_λ. Note that this labeling depends on the choice of $\mathfrak{n}^+$.

A.3 Casimir operators

For the case of $G = SU(2)$, it is well-known from the discussion of angular momentum in any quantum mechanics textbook that irreducible representations can be labeled either by j, the highest weight (here, highest eigenvalue of J_3), or by $j(j+1)$, the eigenvalue of $\mathbf{J} \cdot \mathbf{J}$. The first of these requires making a choice (the z-axis) and looking at a specific vector in the representation, the second doesn't. It was a physicist (Hendrik Casimir), who first recognized the existence of an analog of $\mathbf{J} \cdot \mathbf{J}$ for general semi-simple Lie algebras, and the important role that this plays in representation theory.

Recall that for a semi-simple Lie algebra $\mathfrak{g}$ one has a non-degenerate, invariant, symmetric bi-linear form $(\cdot,\cdot)$, the Killing form, given by

$$(X,Y) = tr(ad(X)ad(Y))$$

If one starts with $\mathfrak{g}$ the Lie algebra of a compact group, this bilinear form is defined on $\mathfrak{g}_{\mathbf{C}}$, and negative-definite on $\mathfrak{g}$. For a simple Lie algebra, taking the trace in a different representation gives the same bilinear form up to a constant. As an example, for the case $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n,\mathbf{C})$, one can show that

$$(X,Y) = 2n\ tr(XY)$$

here taking the trace in the fundamental representation as n by n complex matrices.

One can use the Killing form to define a distinguished quadratic element Ω of $U(\mathfrak{g})$, the Casimir element

$$\Omega = \sum_i X_i X^i$$

where X_i is an orthonormal basis with respect to the Killing form and X^i is the dual basis. On any representation V, this gives a Casimir operator

$$\Omega_V = \sum_i \pi(X_i)\pi(X^i)$$

Note that, taking the representation V to be the space of functions $C^\infty(G)$ on the compact Lie group G, Ω_V is an invariant second-order differential operator, (minus) the Laplacian.

Ω is independent of the choice of basis, and belongs to $U(\mathfrak{g})^{\mathfrak{g}}$, the subalgebra of $U(\mathfrak{g})$ invariant under the adjoint action. It turns out that $U(\mathfrak{g})^{\mathfrak{g}} = Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. By Schur's lemma, anything in the center $Z(\mathfrak{g})$ must act on an irreducible representation by a scalar. One can compute the scalar for an irreducible representation (π, V) as follows:

Choose a basis $(H_i, X_\alpha, X_{-\alpha})$ of $\mathfrak{g}_{\mathbf{C}}$ with H_i an orthonormal basis of the Cartan subalgebra $\mathfrak{t}_{\mathbf{C}}$, and $X_{\pm\alpha}$ elements of $\mathfrak{n}^\pm$ in the $\pm\alpha$ root-spaces of $\mathfrak{g}_{\mathbf{C}}$, orthonormal in the sense of satisfying

$$(X_\alpha, X_{-\alpha}) = 1$$

Then one has the following expression for Ω:

$$\Omega = \sum_i H_i^2 + \sum_{+\ roots} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)$$

To compute the scalar eigenvalue of this on an irreducible representation (π, V_λ) of highest weight λ, one can just act on a highest weight vector $v \in V^\lambda = V^{\mathfrak{n}^+}$. On this vector the raising operators $\pi(X_\alpha)$ act trivially, and using the commutation relations

$$[X_\alpha, X_{-\alpha}] = H_\alpha$$

(H_α is the element of $\mathfrak{t}_{\mathbf{C}}$ satisfying $(H, H_\alpha) = \alpha(H)$) one finds

$$\Omega = \sum_i H_i^2 + \sum_{+roots} H_\alpha = \sum_i H_i^2 + 2H_\rho$$

where ρ is half the sum of the positive roots, a quantity which keeps appearing in this story. Acting on $v \in V^\lambda$ one finds

$$\Omega_{V_\lambda} v = (\sum_i \lambda(H_i)^2 + 2\lambda(H_\rho))v$$

Using the inner-product $\langle \cdot, \cdot \rangle$ induced on $\mathfrak{t}^*$ by the Killing form, this eigenvalue can be written as:

$$\langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle = ||\lambda + \rho||^2 - ||\rho||^2$$

In the special case $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{s}l(2, \mathbf{C})$, there is just one positive root, and one can take

$$H_1 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X_\alpha = e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ X_{-\alpha} = f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Computing the Killing form, one finds

$$(h, h) = 8, \ (e, f) = 4$$

and

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}(ef + fe) = \frac{1}{8}h^2 + \frac{1}{4}(h + 2fe)$$

On a highest weight vector Ω acts as

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}h = \frac{1}{8}h(h+2) = \frac{1}{2}(\frac{h}{2}(\frac{h}{2} + 1))$$

This is 1/2 times the physicist's operator $\mathbf{J} \cdot \mathbf{J}$, and in the irreducible representation V_n of spin $j = n/2$, it acts with eigenvalue $\frac{1}{2}j(j+1)$.

In the next section we'll discuss the Harish-Chandra homomorphism, and the question of how the Casimir acts not just on $V^{\mathfrak{n}^+} = H^0(\mathfrak{n}^+, V)$, but on all of the cohomology $H^*(\mathfrak{n}^+, V)$. After that, taking note that the Casimir is in some sense a Laplacian, we'll follow Dirac and introduce Clifford algebras and spinors in order to take its square root.

A.4 The Harish-Chandra homomorphism

The Casimir element discussed in the last section is a distinguished quadratic element of the center $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ (note, here $\mathfrak{g}$ is a complex semi-simple Lie algebra), but there are others, all of which will act as scalars on irreducible representations. The information about an irreducible representation V contained in these scalars can be packaged as the so-called *infinitesimal character* of V, a homomorphism

$$\chi_V : Z(\mathfrak{g}) \to \mathbf{C}$$

defined by $zv = \chi_V(z)v$ for any $z \in Z(\mathfrak{g})$, $v \in V$. Just as was done for the Casimir, this can be computed by studying the action of $Z(\mathfrak{g})$ on a highest-weight vector.

Note: this is not the same thing as the usual (or global) character of a representation, which is a conjugation-invariant function on the group G with Lie algebra $\mathfrak{g}$, given by taking the trace of a matrix representation. For infinite dimensional representations V, the character is not a function on G, but a distribution Θ_V. The link between the global and infinitesimal characters is given by

$$\Theta_V(zf) = \chi_V(z)\Theta_V(f)$$

i.e. Θ_V is a conjugation-invariant eigendistribution on G, with eigenvalues for the action of $Z(\mathfrak{g})$ given by the infinitesimal character. Knowing the infinitesimal character gives differential equations for the global character.

The Poincare-Birkhoff-Witt theorem implies that for a simple complex Lie algebra $\mathfrak{g}$ one can use the decomposition (here the Cartan subalgebra is $\mathfrak{h} = \mathfrak{t}_{\mathbf{C}}$)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

to decompose $U(\mathfrak{g})$ as

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}))$$

and show that If $z \in Z(\mathfrak{g})$, then the projection of z onto the second factor is in $U(\mathfrak{g})\mathfrak{n}^+ \cap \mathfrak{n}^- U(\mathfrak{g})$. This will give zero acting on a highest-weight vector. Defining $\gamma' : Z(\mathfrak{g}) \to Z(\mathfrak{h})$ to be the projection onto the first factor, the infinitesimal character can be computed by seeing how $\gamma'(z)$ acts on a highest-weight vector.

Remarkably, it turns out that one gets something much simpler if one composes γ' with a translation operator

$$t_\rho : U(\mathfrak{h}) \to U(\mathfrak{h})$$

corresponding to the mysterious $\rho \in \mathfrak{h}^*$, half the sum of the positive roots. To define this, note that since $\mathfrak{h}$ is commutative, $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$, the symmetric algebra on $\mathfrak{h}$, which is isomorphic to the polynomial algebra on $\mathfrak{h}^*$. Then one can define

$$t_\rho(\phi(\lambda)) = \phi(\lambda - \rho)$$

where $\phi \in \mathbf{C}[\mathfrak{h}^*]$ is a polynomial on $\mathfrak{h}^*$, and $\lambda \in \mathfrak{h}^*$.

The composition map

$$\gamma = t_\rho \circ \gamma' : Z(\mathfrak{g}) \to U(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$$

is also homomorphism, known as the Harish-Chandra homomorphism. One can show that the image is invariant under the action of the Weyl group, and the map is actually an isomorphism

$$\gamma : Z(\mathfrak{g}) \to \mathbf{C}[\mathfrak{h}^*]^W$$

It turns out that the ring $\mathbf{C}[\mathfrak{h}^*]^W$ is generated by $dim\ \mathfrak{h}$ independent homogeneous polynomials. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ these are of degree $2, 3, \cdots, n$ (where the first is the Casimir).

To see how things work in the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, where there is one generator, the Casimir Ω, recall that

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}(ef + fe) = \frac{1}{8}h^2 + \frac{1}{4}(h + 2fe)$$

so one has

$$\gamma'(\Omega) = \frac{1}{4}(h + \frac{1}{2}h^2)$$

Here $t_\rho(h) = h - 1$, so

$$\gamma(\Omega) = \frac{1}{4}((h-1) + \frac{1}{2}(h-1)^2) = \frac{1}{8}(h^2 - 1)$$

which is invariant under the Weyl group action $h \to -h$. Once one has the Harish-Chandra homomorphism γ, for each$\lambda \in \mathfrak{h}^*$ one has a homomorphism

$$\chi_\lambda : z \in Z(\mathfrak{g}) \to \chi_\lambda(z) = \gamma(z)(\lambda) \in \mathbf{C}$$

and the infinitesimal character of an irreducible representation of highest weight λ is $\chi_{\lambda+\rho}$.

B Lie algebra and equivariant cohomology

B.1 Lie algebra cohomology

In the case of $\mathfrak{g}^*$ and its quantization, the Lie algebra $\mathfrak{g}$ acts on the algebras of classical and quantum observables $S^*(\mathfrak{g})$ and $U(\mathfrak{g})$ in the obvious way induced from the adjoint action. In the Fermionic case, again there is an action of $\mathfrak{g}$ on $\Lambda^*(\mathfrak{g})$ and Cliff($\mathfrak{g}$) induced from the adjoint action, but there is also a larger structure which is acting, involving a differential and a super-Lie algebra. In this section we'll discuss the differential, in the next the super-Lie algebra.

To understand this larger structure, it is perhaps best to think about the geometry of a Lie group G with Lie algebra $\mathfrak{g}$, and the differential form $\Omega^*(G)$ on G. Elements of $\mathfrak{g}$ can be identified with left-invariant vector fields, and then elements of $\Lambda^*(\mathfrak{g})$ are the left-invariant differential forms, which we'll denote $\Omega^*(G)^G$. The action of $\mathfrak{g}$ induced from the adjoint action is given by the Lie derivative operator L_X, for X the left-invariant vector field corresponding to $X \in \mathfrak{g}$. These operators preserve the degree of a differential form, and satisfy

$$[L_X, L_Y] = L_{[X,Y]}$$

There de Rham differential d acts by

$$d : \Omega^k(G) \to \Omega^{k+1}(G)$$

It can be written in terms of its action on a basis α_j of left-invariant one-forms dual to a basis X_j of $\mathfrak{g}$ as

$$d\alpha_i =$$

where the structure constants are

$$[X_j, X_k] = c_{jkl} X_l$$

The operator d satisfies $d^2 = 0$ and thus gives a complex

$$0 \longrightarrow \Omega^0(G)^G \xrightarrow{d} \Omega^1(G)^G \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{dim\ G}(G)^G \longrightarrow 0$$

The cohomology of this complex is the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbf{C})$.

If G is a compact, connected Lie group, its de Rham cohomology $H^*_{deRham}(G)$ is the cohomology of the complex

$$0 \longrightarrow \Omega^0(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{dim\ G}(G) \longrightarrow 0$$

In this case, the operation of averaging over differential forms by the left action of G to get the left-invariant ones commutes with d and does not change the cohomology, so

$$H^*_{deRham}(G) = H^*(\mathfrak{g}, \mathbf{C})$$

The right-action of G on the group induces an action on the left-invariant forms $\Lambda^*(\mathfrak{g})$, which again commutes with the differential (infinitesimally, this is given by the L_X). Again, averaging gives a complex

$$0 \longrightarrow (\Lambda^0(\mathfrak{g}^*))^G \longrightarrow (\Lambda^1(\mathfrak{g}^*))^G \longrightarrow \cdots \longrightarrow (\Lambda^{\dim\ \mathfrak{g}}(\mathfrak{g}^*))^G \longrightarrow 0$$

where now all the differentials are zero, so the cohomology is given by

$$H^*(\mathfrak{g}, \mathbf{C}) = (\Lambda^*(\mathfrak{g}^*))^G = (\Lambda^*(\mathfrak{g}^*))^{\mathfrak{g}}$$

the adjoint-invariant pieces of the exterior algebra on $\mathfrak{g}^*$. Finding the cohomology has now been turned into a purely algebraic problem in invariant theory.

Note that complexifying the Lie algebra and working with $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ commutes with taking cohomology, so we get

$$H^*(\mathfrak{g}_{\mathbf{C}}, \mathbf{C}) = H^*(\mathfrak{g}, \mathbf{C}) \otimes \mathbf{C}$$

Complexifying the Lie algebra of a compact semi-simple Lie group gives a complex semi-simple Lie algebra, and we have now computed the cohomology of these as

$$H^*(\mathfrak{g}_{\mathbf{C}}, \mathbf{C}) = (\Lambda^*(\mathfrak{g}_{\mathbf{C}}))^{\mathfrak{g}_{\mathbf{C}}}$$

Besides H^0, one always gets a non-trivial H^3, since one can use the Killing form $< \cdot, \cdot >$ to produce an adjoint-invariant 3-form

$$\omega_3(X_1, X_2, X_3) = < x_1, [X_2, X_3] >$$

For $G = SU(n)$, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n, \mathbf{C})$, and one gets non-trivial cohomology classes ω_{2i+1} for $i = 1, 2, \cdots n$, such that

$$H^*(\mathfrak{sl}(n, \mathbf{C})) = \Lambda^*(\omega_3, \omega_5, \cdots, \omega_{2n+1})$$

the exterior algebra generated by the ω_{2i+1}.

To compute Lie algebra cohomology $H^*(\mathfrak{g}, V)$ with coefficients in a representation V, we can go through the same procedure as above, starting with $\Omega^*(G, V)$, the differential forms on G taking values in V, or we can just use exactness of the averaging functor that takes V to V^G. Either way, we end up with the result

$$H^*(\mathfrak{g}, V) = H^*(\mathfrak{g}, \mathbf{C}) \otimes V^{\mathfrak{g}}$$

The H^0 piece of this is just the $V^{\mathfrak{g}}$ that we want when we are doing BRST, but we also get quite a bit else: *dim* $V^{\mathfrak{g}}$ copies of the higher degree pieces of the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbf{C})$. The Lie algebra cohomology here is quite non-trivial, but doesn't interact in a non-trivial way with the process of identifying the invariants $V^{\mathfrak{g}}$ in V.

Rest of what needs to happen here: need to reformulate everything in terms of the super Poisson Lie bracket, show that L_X is given by a Poisson bracket, and that d is given by a Poisson bracket.

Remarkably (this was first noticed by Kostant-Sternberg[13]) one can use the invariant inner product on $\mathfrak{g}$ to get the final operator needed to make $\Lambda^*(\mathfrak{g}^*)$ a $\mathfrak{g}$-differential algebra, the differential d, by taking the Poisson bracket with a degree 3 element $\Omega \in \Lambda^3(\mathfrak{g}^*)$, where

$$\Omega(X, Y, Z) = -(X, [Y, Z])$$

To show this, we just need to show that Poisson bracket with Ω gives the action of d on $\Lambda^1(\mathfrak{g}^*)$. Recall the following formula for d acting on $\theta = \Lambda^1(\mathfrak{g}^*)$ which one gets when one computes the de Rham differential on left-invariant 1-forms:

$$d\theta(Y, Z) = -\theta([Y, Z])$$

Taking the one-form $\theta_X = (X, \cdot)$, one has

$$d\theta_X(Y, Z) = -\theta_X([Y, Z]) = -(X, [Y, Z]) = \{X, \Omega\} = \{\Omega, X\}$$

so

$$d = \{\Omega, \cdot\}$$

B.1.1 Lie algebra cohomology as a derived functor

The last section discussed one of the simplest incarnations of BRST cohomology, in a formalism familiar to physicists. This fits into a much more abstract mathematical context, and that's what we'll turn to now.

Given a Lie algebra $\mathfrak{g}$, we'll consider Lie algebra representations as modules over $U(\mathfrak{g})$. Such modules form a category $\mathcal{C}_{\mathfrak{g}}$: what is interesting is not just the objects of the category (the equivalence classes of modules), but also the morphisms between the objects. For two representations V_1 and V_2 the set of morphisms between them is a linear space denoted $Hom_{U(\mathfrak{g})}(V_1, V_2)$. This is just the set of linear maps from V_1 to V_2 that commute with the action of $\mathfrak{g}$:

$$Hom_{U(\mathfrak{g})}(V_1, V_2) = \{\phi \in Hom_{\mathbf{C}}(V_1, V_2) : \pi(X)\phi = \phi\pi(X)\ \forall X \in \mathfrak{g}\}$$

Another conventional name for this is the space of intertwining operators between the two representations.

For any representation V, its $\mathfrak{g}$-invariant subspace $V^{\mathfrak{g}}$ can be identified with the space $Hom_{U(\mathfrak{g})}(\mathbf{C}, V)$, where here $\mathbf{C}$ is the trivial one-dimensional representation. Having a way to pick out the invariant piece of a representation also allows one to solve the more general problem of picking out the subspace that transforms like a specific irreducible W: just find the invariant subspace of $V \otimes W^*$.

The map $V \to V^{\mathfrak{g}}$ that takes a representation to its $\mathfrak{g}$-invariant subspace is a functor: it takes the category $\mathcal{C}_{\mathfrak{g}}$ to $\mathcal{C}_{\mathbf{C}}$, the category of vector spaces and linear maps ($\mathbf{C}$ - modules and $\mathbf{C}$ - homomorphisms).

It turns out that when one has a category of modules like $\mathcal{C}_{\mathfrak{g}}$, these can usefully be studied by considering complexes of modules, and this is the subject of homological algebra. A complex of modules is a sequence of modules and homomorphisms

$$\cdots \xrightarrow{\partial} U \xrightarrow{\partial} V \xrightarrow{\partial} W \xrightarrow{\partial} \cdots$$

such that $\partial \circ \partial = 0$. If the complex satisfies $im\ \partial = ker\ \partial$ at each module, the complex is said to be an "exact complex".

To motivate the notion of exact complex, note that

$$0 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

is exact iff V_0 is isomorphic to V, and an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

represents the module V as the quotient V_0/V_1. Using longer complexes, one gets the notion of a *resolution* of a module V by a sequence of n modules V_i. This is an exact complex

$$0 \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

The deviation of a sequence from being exact is measured by its homology, $H_* = \frac{ker\ \partial}{im\ \partial}$. Note that if one deletes V from its resolution, the sequence

$$0 \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

is exact except at V_0. Indexing the homology in the obvious way, one has $H_i = 0$ for $i > 0$, and $H^0 = V$. A sequence like this whose only homology is V at H_0 is another manifestation of a resolution of V.

The reason this construction is useful is that, for many purposes, it allows us to replace a module whose structure we may not understand by a sequence of modules whose structure we do understand. In particular, we can replace a $U(\mathfrak{g})$ module V by a sequence of free modules, i.e. modules that are just sums of copies of $U(\mathfrak{g})$ itself. This is called a free resolution, and more generally one can work with projective modules (direct summands of free modules).

A functor that takes exact complexes to exact complexes is called an exact functor. Homological invariants of modules come about in cases where one has a functor on a category of modules that is not exact. Applying such a functor to a free or projective resolution gives the homological invariants.

There are many possible choices of a free resolution of a module. For the case of $U(\mathfrak{g})$ modules, one convenient choice is known as the Koszul (or Chevalley-Eilenberg) resolution. To construct a resolution of the trivial module $\mathbf{C}$, one uses the exterior algebra on $\mathfrak{g}$ to make free modules

$$Y_k = U(\mathfrak{g}) \otimes_{\mathbf{C}} \Lambda^k(\mathfrak{g})$$

and get a resolution of $\mathbf{C}$

$$0 \longrightarrow Y_{dim\ \mathfrak{g}} \xrightarrow{\partial_{dim\ \mathfrak{g}-1}} \cdots \xrightarrow{\partial_1} Y_1 \xrightarrow{\partial_0} Y_0 \xrightarrow{\epsilon} \mathbf{C} \longrightarrow 0$$

The maps are given by

$$\epsilon : u \in Y_0 = U(\mathfrak{g}) \to \epsilon(u) = const.\ term\ of\ u$$

and

$$\begin{aligned}
\partial_{k-1}(u \otimes X_1 \wedge \cdots \wedge X_k) &= \sum_{i=1}^{k} (-1)^{i+1}(uX_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k) \\
&+ \sum_{i<j} (-1)^{i+j}(u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k)
\end{aligned}$$

To get Lie algebra cohomology, we apply the invariants functor

$$V \longrightarrow V^{\mathfrak{g}} = Hom_{U(\mathfrak{g})}(\mathbf{C}, V)$$

replacing the trivial representation by its Koszul resolution. This gives us a complex with terms

$$\begin{aligned}
C^k(\mathfrak{g}, V) = Hom_{U(\mathfrak{g})}(Y_k, V) &= Hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes \Lambda^k(\mathfrak{g}), V) \\
&= Hom_{U(\mathfrak{g})}(U(\mathfrak{g}), Hom_{\mathbf{C}}(\Lambda^k(\mathfrak{g}), V)) \\
&= Hom_{\mathbf{C}}(\Lambda^k(\mathfrak{g}), V) = V \otimes \Lambda^k(\mathfrak{g}^*)
\end{aligned}$$

and induced maps d_i

$$0 \longrightarrow C^0(\mathfrak{g}, V) \xrightarrow{d_0} C^1(\mathfrak{g}, V) \cdots \xrightarrow{d_{dim\ \mathfrak{g}-1}} C^{dim\ \mathfrak{g}}(\mathfrak{g}, V) \longrightarrow 0$$

The Lie algebra cohomology $H^*(\mathfrak{g}, V)$ is just the cohomology of this complex, i.e.

$$H^i(\mathfrak{g}, V) = \frac{ker\ d_i}{im\ d_{i-1}}|_{C^i(\mathfrak{g},V)}$$

This is exactly the same definition as that of the BRST cohomology defined in physicist's formalism in the last section with $\mathcal{H} = C^*(\mathfrak{g}, V)$.

One has $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$ and so gets the $\mathfrak{g}$-invariants as expected, but in general the cohomology will be non-zero also in other degrees.

For a much more detailed exposition of Lie algebra cohomology, see Anthony Knapp's book *Lie Groups, Lie Algebras, and Cohomology* [10].

B.2 Lie algebra cohomology of the nilpotent Lie subalgebra

In the last section we discussed the Lie algebra cohomology $H^*(\mathfrak{g}, V)$ for $\mathfrak{g}$ a semi-simple Lie algebra. Because the invariants functor is exact here, this tells us nothing about the structure of irreducible representations in this case. In this section we'll consider a different sort of example of Lie algebra cohomology, one that is intimately involved with the structure of irreducible $\mathfrak{g}$-representations.

Getting back to Lie algebra cohomology, while $H^*(\mathfrak{g}, V) = 0$ for an irreducible representation V, the Lie algebra cohomology for $\mathfrak{n}^+$ is more interesting, with $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+}$, the highest weight space. $\mathfrak{t}$ acts not just on V, but on the entire complex $C(\mathfrak{n}^+, V)$, in such a way that the cohomology spaces $H^i(\mathfrak{n}^+, V)$ are representations of $\mathfrak{t}$, so can be characterized by their weights.

For an irreducible representation V_λ, one would like to know which higher cohomology spaces are non-zero and what their weights are. The answer to this question involves a surprising "ρ - shift", a shift in the weights by a weight ρ, where

$$\rho = \frac{1}{2}\sum_{+roots} \alpha$$

half the sum of the positive roots. This is a first indication that it might be better to work with spinors rather than with the exterior algebra that is used in the Koszul resolution used to define Lie algebra cohomology. Much more about this in a later section.

One finds that $dim\ H^*(\mathfrak{n}^+, V_\lambda) = |W|$, and the weights occuring in $H^i(\mathfrak{n}^+, V_\lambda)$ are all weights of the form $w(\lambda + \rho) - \rho$, where $w \in W$ is an element of length i. The Weyl group can be realized as a reflection group action on $\mathfrak{t}^*$, generated by one reflection for each "simple" root. The length of a Weyl group element is the minimal number of reflections necessary to realize it. So, in dimension 0, one gets $H^0(\mathfrak{n}^+, V_\lambda) = V^{\mathfrak{n}^+}$ with weight λ, but there is also higher cohomology. Changing one's choice of $\mathfrak{n}^+$ by acting with the Weyl group permutes the different weight spaces making up $H^*(\mathfrak{n}^+, V)$. For an irreducible representation, to characterize it in a manner that is invariant under change in choice of $\mathfrak{n}^+$, one should take the entire Weyl group orbit of the ρ - shifted highest weight λ, i.e. the set of weights

$$\{w(\lambda+\rho),\ w \in W\}$$

In our $G = SU(2)$ example, highest weights can be labeled by non-negative half integral values (the "spin" s of the representation)

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

with $\rho = \frac{1}{2}$. The irreducible representation V_s is of dimension $2s+1$, and one finds that $H^0(\mathfrak{n}^+, V_s)$ is one-dimensional of weight s, while $H^1(\mathfrak{n}^+, V_s)$ is one-dimensional of weight $-s-1$.

The character of a representation is given by a positive integral combination of the weights

$$char(V) = \sum_{weights\ \omega} (dim\ V^\omega)\omega$$

(here V^ω is the ω weight space). The Weyl character formula expresses this as a quotient of expressions involving weights taken with both positive and negative integral coefficients. The numerator and denominator have an interpretation in terms of Lie algebra cohomology:

$$char(V) = \frac{\chi(H^*(\mathfrak{n}^+, V))}{\chi(H^*(\mathfrak{n}^+, \mathbf{C}))}$$

Here χ is the Euler characteristic: the difference between even-dimensional cohomology (a sum of weights taken with a + sign), and odd-dimensional cohomology (a sum of weights taken with a - sign). Note that these Euler characteristics are independent of the choice of $\mathfrak{n}^+$.

The material in this last section goes back to Bott's 1957 paper *Homogeneous Vector Bundles*[3], with more of the Lie algebra story worked out by Kostant in his 1961 *Lie Algebra Cohomology and the Generalized Borel-Weil Theorem*[14]. For an expository treatment with details, showing how one actually computes the Lie algebra cohomology in this case, for U(n) see chapter VI.3 of Knapp's *Lie Groups, Lie Algebras and Cohomology*[10], or for the general case see chapter IV.9 of Knapp and Vogan's *Cohomological Induction and Unitary Representations*[11].

We have computed the infinitesimal character of a representation of highest weight λ by looking at how $Z(\mathfrak{g})$ acts on $V^{\mathfrak{n}^+} = H^0(\mathfrak{n}^+, V)$. On $V^{\mathfrak{n}^+}, z \in Z(\mathfrak{g})$ acts by

$$z \cdot v = \chi_V(z)v$$

This space has weight λ, so $U(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$ acts by evaluation at λ

$$\phi \cdot v = \phi(\lambda)v$$

These two actions are related by the map $\gamma' : Z(\mathfrak{g}) \to U(\mathfrak{h})$ and we have

$$\chi_V(z) = (\gamma'(z))(\lambda) = (\gamma(z))(\lambda+\rho)$$

It turns out that one can consider the same question, but for the higher cohomology groups $H^k(\mathfrak{n}^+, V)$. Here one again has an action of $Z(\mathfrak{g})$ and an action of $U(\mathfrak{h})$. $Z(\mathfrak{g})$ acts on k-cochains $C^k(\mathfrak{n}^+, V) = Hom_{\mathbf{C}}(\Lambda^k\mathfrak{n}^+, V)$ just by acting on V, and this action commutes with d so is an action on cohomology. $U(\mathfrak{h})$ acts simultaneously on $\mathfrak{n}^+$ and on V, again in a way that descends to cohomology. The content of the Casselman-Osborne lemma is that these two actions are again related in the same way by the Harish-Chandra homomorphism. If μ is a weight for the $\mathfrak{h}$ action on $H^k(\mathfrak{n}^+, V)$, then

$$\chi_V(z) = (\gamma'(z))(\mu) = (\gamma(z))(\mu + \rho)$$

Since $\chi_V(z) = (\gamma(z))(\lambda + \rho)$, one can use this equality to show that the weights occurring in $H^k(\mathfrak{n}^+, V)$ must satisfy

$$(\mu + \rho) = w(\lambda + \rho)$$

and thus

$$\mu = w(\lambda + \rho) - \rho$$

for some element $w \in W$. Non zero elements of $H^k(\mathfrak{n}^+, V)$ can be constructed with these weights, and the Casselman-Osborne lemma used to show that these are the only possible weights. This gives the computation of $H^k(\mathfrak{n}^+, V)$ as an $\mathfrak{h}$ - module referred to earlier in these notes, which is known as Kostant's theorem (the algebraic proof was due to Kostant[14], an earlier one using geometry and sheaf cohomology was due to Bott[3]).

For more details about this and a proof of the Casselman-Osborne lemma, see Knapp's *Lie Groups, Lie Algebras and Cohomology*[10], where things are worked out for the case of $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{C})$ in chapter VI.

So far we have been considering the case of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and its orthogonal complement with a choice of splitting into two conjugate subalgebras, $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. Equivalently, we have a choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. At the group level, this corresponds to a choice of Borel subgroup $B \subset G$, with the space G/B a complex projective variety known as a flag manifold. More generally, much of the same structure appears if we choose larger subgroups $P \subset G$ containing B such that G/P is a complex projective variety of lower dimension. In these cases $Lie\ P = \mathfrak{l} \oplus \mathfrak{u}^+$, with $\mathfrak{l}$ (the Levi subalgebra) a reductive algebra playing the role of the Cartan subalgebra, and $\mathfrak{u}^+$ playing the role of $\mathfrak{n}^+$.

In this more general setting, there is a generalization of the Harish-Chandra homomorphism, now taking $Z(\mathfrak{g})$ to $Z(\mathfrak{l})$. This acts on the cohomology groups $H^k(\mathfrak{u}^+, V)$, with a generalization of the Casselman-Osborne lemma determining what representations of $\mathfrak{l}$ occur in this cohomology. The Dirac cohomology formalism to be discussed later generalizes this even more, to cases of a reductive subalgebra $\mathfrak{r}$ with orthogonal complement that cannot be given a complex structure and split into conjugate subalgebras. It also provides a compelling explanation for the continual appearance of ρ, as the highest weight of the spin representation.

B.2.1 Kostant's Theorem

The computation of the Lie algebra cohomology of the nilpotent radical was done by Kostant in 1961, with the result

Theorem 1 (Kostant's Theorem)**.** *For a finite dimensional highest-weight representation V^λ of a complex semi-simple Lie algebra $\mathfrak{g}$*

$$H^i(\mathfrak{n}^+, V^\lambda) = \bigoplus_{w\in W: l(w)=i} \mathbf{C}_{w(\lambda+\rho)-\rho}$$

There are at least four possible approaches to proving this:

- One can use the BGG resolution and the fact that for Verma modules $H^i(\mathfrak{g}, V(\mu))$ is $\mathbf{C}_\mu$ for $i = 0$, 0 for $i > 0$. This requires knowing the BGG resolution, which is a stronger result since it carries information about homomorphisms between Verma modules.
- One can prove Borel-Weil-Bott by other (e.g. topological) methods, then use this to prove Kostant's theorem. For an example of such a proof of Borel-Weil-Bott, see Jacob Lurie's notes[**?**].
- One can find explicit elements in $H^*(\mathfrak{n}^+, V^\lambda)$ that represent the cohomology classes in Kostant's theorem. One way to do this is to look for elements in
$$C^i(\mathfrak{n}^+, V^\lambda) = \Lambda^i(\mathfrak{n}^+)^* \otimes V^\lambda$$
that represent these cohomology classes. Note that the weights of $(\mathfrak{n}^+)^*$ are multiples of $-\alpha$ where $\alpha \in R^+$, the positive roots. A choice that gives the right element in degree i for each Weyl group element w such that $l(w) = i$ is:
$$\omega_{-\beta_1} \wedge \omega_{-\beta_2} \wedge \cdots \wedge \omega_{-\beta_i} \otimes V^\lambda(w\lambda)$$
where
$$\omega_{-\beta_j} \in (\mathfrak{n}^+)^*_{-\beta_j}$$
for β_j a positive root such that $w\beta_j$ is a negative root. $V^\lambda(w\lambda)$ is the transform by w of the highest weight space. The more difficult part of this sort of proof is showing that only these elements can occur. One way to do this is to construct an analog of the Laplacian, and show that it acts like the Casimir on cohomology (this was Kostant's original method). A generalization of this uses the full center of the enveloping algebra, and the Casselman-Osborne lemma, which says that the center much act on the higher cohomology in just the way that the Harish-Chandra ismorphism says it acts in degree zero cohomology (the highest weight space). For more details on this argument see Goodman-Wallach[**?**].
- One can replace the use of the exterior algebra and a Laplacian by closely related spinors, and a "square-root" of the Laplacian known as the Dirac operator. We'll try and come back to this argument after developing the technology of spinors and Clifford algebras in the next couple weeks.

B.3 Equivariant cohomology

B.4 Verma modules and resolutions

C Appendix: Borel-Weil and Borel-Weil-Bott

C.1 The Borel-Weil theorem

We'll now turn to a geometric construction of irreducible finite-dimensional representations, using induction on group representations, rather than at the Lie algebra level. In this section G will be a compact Lie group, T a maximal torus of G.

Recall that according to the Peter-Weyl theorem

$$L^2(G) = \widehat{\bigoplus}_i V_i \otimes V_i^*$$

with the left regular representation of G on $L^2(G)$ corresponding to the action on the factor V_i and the right regular representation corresponding to the dual action on the factor V_i^*. Under the left regular representation $L^2(G)$ decomposes into irreducibles as a sum over all irreducibles, with each one occuring with multiplicity dim V_i (which is the dimension of V_i^*). To make things simpler later, we'll interchange our labeling of representations and use Peter-Weyl in the form

$$L^2(G) = \widehat{\bigoplus}_i V_i^* \otimes V_i$$

i.e. the left regular representation will be on V_i^*, the right regular on V_i.

Recall also that we can induce from representations $\mathbf{C}_\lambda$ of T to representations of G, with the induced representation interpretable as a space of sections of a line bundle L_λ over G/T.

$$Ind_T^G(\mathbf{C}_\lambda) = \Gamma(L_\lambda)$$

Here

$$\Gamma(L_\lambda) \subset L^2(G)$$

is the subspace of the left-regular representation picked out by the condition

$$f(gt) = \rho_\lambda(t^{-1})f(g)$$

Here the representation $(\rho_\lambda, \mathbf{C})$of T is one dimensional and if $t = e^H$

$$\rho_\lambda(t^{-1}) = e^{-\lambda(H)}$$

where λ is an integral weight in $\mathbf{t}^*$.

This condition says that under the action of the subgroup $T_R \subset G_R$, $\Gamma(L_\lambda)$ is the subspace of $L^2(G)$ that has weight $-\lambda$. In other words

$$\Gamma(L_\lambda) = \widehat{\bigoplus}_i V_i^* \otimes (V_i)_{-\lambda}$$

or

$$\Gamma(L_{-\lambda}) = \widehat{\bigoplus}_i V_i^* \otimes (V_i)_\lambda$$

where $(V_i)_\lambda$ is the λ weight space of V_i.

Our induced representation $\Gamma(L_{-\lambda})$ thus breaks up into irreducibles as a sum over all irreducibles V_i^*, with multiplicity given by the dimension of the weight λ in V_i. We can get a single irreducible if we impose the condition that λ be a highest weight since by the highest-weight theorem λ will be the highest weight for just one irreducible representation.

So if we impose the condition on $\Gamma(L_{-\lambda})$ that infinitesimal right translation by an element of a positive root space give zero, we will finally have a construction of a single irreducible: it will be the dual of the irreducible with highest weight λ. It turns out (see discussion of complex structures in next section, more detail in [20]) that imposing the condition of infinitesimal right invariance under the action of $\mathfrak{n}^+$ on $\Gamma(L_{-\lambda})$ is exactly the holomorphicity condition corresponding to using the complex structure on G/T to give L_λ the structure of a holomorphic line bundle. So, on the subspace

$$\Gamma_{hol}(L_{-\lambda}) \subset \Gamma(L_{-\lambda}) \subset L^2(G)$$

we have

$$\begin{aligned}
\Gamma_{\mathrm{hol}}(L_{-\lambda}) &= \widehat{\bigoplus}_i V_i^* \otimes \{v \in V_i : \begin{cases} \mathfrak{n}^+ v = 0 \\ v \in (V_i)_\lambda \end{cases} \} \\
&= \bigoplus_{V_i \text{ has highest-weight } \lambda} V_i^* \otimes (V_i)_\lambda \\
&= (V^\lambda)^* \otimes \mathbf{C}
\end{aligned}$$

where V^λ is the irreducible representation of highest weight λ.

The Borel-Weil version of the highest-weight theorem is thus:

Theorem 2 (Borel-Weil)**.** *As a representation of G, for a dominant weight λ, $\Gamma_{hol}(L_{-\lambda})$ is the dual of a non-zero, irreducible representation of highest weight λ. All irreducible representations of G can be constructed in this way.*

For an outline of the proof from the point of view of complex analysis, see [19] chapter 14. Note however, that we have shown how the Borel-Weil theorem is related to the highest-weight classification of irreducible representations discussed using Lie algebras and Verma modules, with a different explicit construction of the representation. The Lie algebra argument made clear that having a dominant, integral highest weight is a necessary condition for a finite dimensional irreducible. The Verma module construction was such that it was not so easy to see that these conditions were sufficient for finite dimensionality. Finite dimensionality can be proved for the Borel-Weil construction using either

- General theorems of algebraic geometry to show that

$$\Gamma_{hol}(L_\lambda) = H^0(G/T, \mathcal{O}(L_\lambda))$$

is finite dimensional (sheaf cohomology of a holomorphic bundle over a compact projective variety is finite-dimensional), or

- Hodge theory. Picking a metric the Cauchy-Riemann operator has an adjoint, and the corresponding Laplacian is elliptic. An elliptic operator on a compact manifold has finite-dimensional kernel.

C.2 Flag manifolds and complex structures.

We need to show that the holomorphicity condition on the space of sections $\Gamma(L_\lambda)$ corresponds to imposing the condition that Lie algebra elements in the positive root spaces act trivially, as vector fields corresponding to the infinitesimal right-action of the group. For a detailed argument, see [20], section 7. 4.3. The complex geometry involved goes as follows.

We'll need the general notion of what a complex structure on a manifold is. To begin, on real vector spaces:

Definition 1 (Complex structure on a vector space)**.** *Given a real vector space V of dimension n, a complex structure is a non-degenerate operator J such that $J^2 = -\mathbf{1}$. On the complexification $V \otimes \mathbf{C}$ it has eigenvalues i and $-i$, and an eigenspace decomposition*

$$V \otimes \mathbf{C} = V^{1,0} \oplus V^{0,1}$$

$V^{1,0}$ is a complex vector space with complex dimension n, with multiplication by i given by the action of J, $V^{0,1}$ its complex conjugate.

For manifolds

Definition 2 (Almost complex manifold)**.** *A manifold with a smooth choice of a complex structure J_x on each tangent space $T_x(M)$ is called an almost complex manifold.*

and

Definition 3 (Complex manifold)**.** *A complex manifold is an almost complex manifold with an integrable complex structure, i.e. the Lie bracket of vector fields satisfies*

$$[T^{1,0}, T^{1,0}] \subset T^{1,0}$$

By the Newlander-Nirenberg theorem, having an integrable complex structure implies that one can choose complex coordinate charts, with holomorphic transition functions, and thus have a notion of which functions are holomorphic.

Going back to Lie groups and case of the manifold G/T, to even define the positive root space, we need to begin by complexifying the Lie algebra

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and then making a choice of positive roots $R^+ \subset R$

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

Note that the complexified tangent space to G/T at the identity coset is

$$T_{eT}(G/T) \otimes \mathbf{C} = \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and a choice of positive roots gives a choice of complex structure on $T_{eT}(G/T)$, with decomposition

$$T_{eT}(G/T) \otimes \mathbf{C} = T^{1,0}_{eT}(G/T) \oplus T^{0,1}_{eT}(G/T) = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

While $\mathfrak{g}/\mathfrak{t}$ is not a Lie algebra,

$$\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \text{ and } \mathfrak{n}^- = \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

are each Lie algebras, subalgebras of $\mathfrak{g}_{\mathbf{C}}$ since

$$[\mathfrak{n}^+, \mathfrak{n}^+] \subset \mathfrak{n}^+$$

(and similarly for $\mathfrak{n}^-$). This follows from

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The fact that these are subalgebras also implies that the almost complex structure they define on G/T is actually integrable, so G/T is a complex manifold.

Note that the choice of positive roots is not unique or canonical. There are $|W|$ inequivalent choices that will work. The Weyl group acts on the inequivalent complex structures. In particular, it permutes the Weyl chambers, and it is the choice of positive roots that determines which Weyl chamber is the dominant one. Changing choice of positive roots will correspond to changing choice of complex structure. We'll see later on that a representation appearing as a holomorphic section (in H^0) with respect to one complex structure, appears in higher cohomology when one changes the complex structure.

The choice of a decomposition into positive and negative roots takes the original $\mathfrak{g}/\mathfrak{t}$, which is not a Lie subalgebra of $\mathfrak{g}$, and, upon complexification, gives two Lie subalgebras instead:

$$\mathfrak{g}/\mathfrak{t} \otimes \mathbf{C} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

Recall that in the case $\mathfrak{u}(n)$, this corresponds to the fact that, upon complexification to $\mathfrak{gl}(n, \mathbf{C})$, the non-diagonal entries split into two nilpotent subalgebras: the upper and lower triangular matrices.

Another important Lie subalgebra is

$$\mathfrak{b} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{n}^+$$

This is the Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$. One also has parabolic subalgebras, those satisfying

$$\mathfrak{b} \subseteq \mathfrak{p} \subset \mathfrak{g}_{\mathbf{C}}$$

The Borel subalgebra is the minimal parabolic subalgebra. Other parabolic subalgebras can be constructed by adding to the positive roots some of the negative roots, with the possible choices corresponding to the nodes of the Dynkin diagram.

Corresponding to the Lie sub-algebras $\mathfrak{n}^+, \mathfrak{b}, \mathfrak{p}$ one has Lie subgroups N^+, B, P of $G_{\mathbf{C}}$. One can identify

$$G/T = G_{\mathbf{C}}/B$$

and another approach to Borel-Weil theory would be to do "holomorphic induction", inducing from a one-dimensional representation of B on $\mathbf{C}$ to a representation of $G_{\mathbf{C}}$ using holomorphic functions on $G_{\mathbf{C}}$.

One can see that the complex manifolds $G/T = G_{\mathbf{C}}/B$ and $G_{\mathbf{C}}/P$ are actually projective varieties as follows (for more details, see [18]):

Pick a highest-weight vector $v_\lambda \in V^\lambda$ and look at the map

$$g \in G_{\mathbf{C}} \rightarrow [gv_\lambda] \in P(V^\lambda)$$

i.e. the orbit in projective space of the line defined by the highest-weight vector. For a generic dominant weight, the Borel subgroup B will act trivially on this line, for weights on the boundary of the dominant Weyl chamber one gets larger stabilizer groups, the parabolic groups P. The orbit can be identified with $G_{\mathbf{C}}/B$ or $G_{\mathbf{C}}/P$, and this gives a projective embedding.

In the special case that the orbit is the full projective space, one can understand the Borel-Weil theorem in the following way:

Given a projective space $P(V)$, one can construct a "tautological" line bundle above it by taking the fiber above a line to be the line itself. In the complex case, this give a holomorphic bundle L, one that has no holomorphic sections. But for each element of V^*, one can restrict this to the line L, getting a section of the dual bundle $\Gamma(L^*)$. It turns out this is an isomorphism

$$V^* = \Gamma_{hol}(L^*)$$

and more generally one has

$$\text{homogeneous polys on V of degree n} = \Gamma_{hol}((L^*)^{\otimes n})$$

i.e. the sections of the n'th power of the dual of the tautological bundle are the homogeneous polynomials on V of degree n. These give special cases of the Borel-Weil theorem, and we'll see explicitly how this works for $V = \mathbf{C}^2$ in the next section.

C.3 Borel-Weil-Bott

Recall that in our discussion of the Borel-Weil theorem we were using a complex line bundle L_λ over the flag manifold G/T (G is a compact simple Lie group, T a maximal torus). The integral weight λ labels a T representation ρ_λ on $\mathbf{C}$. Sections of this line bundle are explicitly

$$\begin{aligned}\Gamma(L_\lambda) &= \{f : G \to \mathbf{C},\ f(gt) = \rho_\lambda(t^{-1})f(g)\} \\ &= (C^\infty(G) \otimes \mathbf{C}_\lambda)^T \\ &= (C^\infty(G))_{-\lambda}\end{aligned}$$

and holomorphic sections are the subspace of this invariant under the right action of $\mathfrak{n}^+$.

We are interested now in using the structure of G/T as a complex manifold (which depends on the choice of positive roots) to define a holomorphic version of cohomology. The usual topological cohomology computes the derived functor of the functor of taking global sections of the sheaf of locally constant functions. For a complex manifold, we instead use the sheaf of local holomorphic functions, or more generally the sheaf of local holomorphic sections of a holomorphic line bundle such as L_λ. Just as the de Rham theorem allows computation of topological cohomology using differential forms, the Dolbeault theorem says we can compute holomorphic cohomology using the the bi-graded complex

$$(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial})$$

of differential forms with coefficients in line bundle L_λ, of degree i in local variables $d\overline{z}$ (and degree 0 in the dz). In degree 0 we just get

$$H^0(G/T, L_\lambda) = \Gamma_{hol}(L_\lambda)$$

the holomorphic sections, but we can also get higher cohomology, in degrees up to the complex dimension of G/T.

If one works out explicitly what the Dolbeault complex is in this case, generalizing the case of holomorphic sections, one finds

$$(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial}) = ((Hom(\Lambda^i(\mathfrak{n}^+), C^\infty(G) \otimes \mathbf{C}_\lambda))^T, d)$$

where T acts on $\mathfrak{n}^+$ by the adjoint representation, and the d is the d of Lie algebra cohomology for $\mathfrak{n}^+$, with $\mathfrak{n}^+$ acting on $C^\infty(G)$ by infinitesimal right translation.

Note that one has a commuting action of G on this complex, coming from the left G action on functions on G, so we will get G representations on the cohomology spaces

$$H^i(G/T, L_\lambda)$$

Recall that the way Borel-Weil works is that one uses Peter-Weyl to see that

$$\begin{aligned}\Gamma(L_\lambda) &= (C^\infty(G)\otimes \mathbf{C}_\lambda)^T\\ &= (C^\infty(G))_{-\lambda}\\ &= \bigoplus_{\mu\ dominant} (V^\mu)^*\otimes V^\mu_{-\lambda}\end{aligned}$$

and thus that

$$\Gamma_{hol}(L_{-\lambda}) = (V^\lambda)^*$$

For higher cohomology, one has

$$\begin{aligned}H(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial}) &= H((Hom(\Lambda^i(\mathfrak{n}^+), C^\infty(G)\otimes \mathbf{C}_\lambda))^T, d)\\ &= H(\bigoplus_{\mu\ dominant} (V^\mu)^*\otimes (Hom(\Lambda^i(\mathfrak{n}^+), V^\mu\otimes \mathbf{C}_\lambda))^T, d)\\ &= \bigoplus_{\mu\ dominant} (V^\mu)^*\otimes (H^i(\mathfrak{n}^+, V^\mu\otimes \mathbf{C}_\lambda))^T\\ &= \bigoplus_{\mu\ dominant} (V^\mu)^*\otimes H^i(\mathfrak{n}^+, V^\mu)_{-\lambda}\end{aligned}$$

so

$$H(\Omega^{0,i}(G/T, L_{-\lambda}), \overline{\partial}) = \bigoplus_{\mu\ dominant} (V^\mu)^*\otimes H^i(\mathfrak{n}^+, V^\mu)_\lambda$$

This show that in this case computing holomorphic cohomology comes down to computing $\mathfrak{n}^+$ Lie algebra cohomology.

C.4 Borel-Weil-Bott and the Weyl Character Formula

Kostant's theorem gives the Borel-Weil-Bott theorem very directly. Recall that

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = \bigoplus_\mu (V^\mu)^*\otimes H^i(\mathfrak{n}^+, V^\mu)_\lambda$$

where the sum is over dominant integral weights μ. By Kostant's theorem we have

$$H^i(\mathfrak{n}^+, V^\mu)_\lambda = (\bigoplus_{w\in W: l(w)=i} \mathbf{C}_{w(\mu+\rho)-\rho})_\lambda$$

and this has a one-dimensional contribution iff

$$w(\mu+\rho)-\rho = \lambda$$

or equivalently

$$w(\mu+\rho) = \lambda+\rho$$

Note that the set of weights of the form $\mu + \rho$ for μ dominant integral are in the interior of the dominant Weyl chamber, and acting on these by Weyl group elements gives us sets of weights in the interiors of the other Weyl chambers. Weights λ such that $\lambda + \rho$ is on the boundary of a Weyl chamber will not occur. In summary, we have

Theorem 3 (Borel-Weil-Bott)**.** *If $\lambda + \rho$ is a singular weight then for all i we have*

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = 0$$

If $\lambda+\rho$ is a non-singular weight, there will be an i such that $w(\lambda+\rho) = \mu+\rho$ is in the interior of the dominant Weyl chamber for a $w : l(w) = i$ and

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = (V^\mu)^*$$

As usual, the simplest example is $G = SU(2),\ G/T = \mathbf{C}P^1$, and the Borel-Weil-Bott theorem can be proved via Serre duality, which says that for line bundles L on a curve C one has

$$H^1(C, L) = H^0(C, L^* \otimes \omega_C)$$

where ω_C is the canonical bundle on C. In our case $C = \mathbf{C}P^1$, and line bundles L_n are labeled by an integer n with ρ corresponding to $n = 1$. The canonical bundle is L_2.

For $n \geq 0$ we have, as in the Borel-Weil theorem

$$H^0(\mathbf{C}P^1, L_{-n}) = (V^n)^*$$

where V^n is the irreducible $SU(2)$ representation of dimension $n+1$. By Serre duality

$$H^1(\mathbf{C}P^1, L_{-n}) = H^0(\mathbf{C}P^1, L_{n+2})$$

which is consistent with Borel-Weil-Bott which tells us that

$$H^1(\mathbf{C}P^1, L_{-n}) = (V^{-n-2})^*$$

when $n < -1$ and, in the singular $n = -1$ case

$$H^1(\mathbf{C}P^1, L_1) = H^0(\mathbf{C}P^1, L_1) = 0$$

So, for $n > 0$ one gets all irreducibles as holomorphic sections, whereas for $n < -1$ one gets all irreducibles again, but in higher cohomology (H^1).

Another quick corollary of Kostant's theorem is the Weyl character formula. Recall that this says that the character $ch(V^\lambda)$ of a finite-dim irreducible of highest weight λ is given by

$$ch(V^\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho}}$$

This follows from an application of the Euler-Poincaré principle, which says that in the case of an Abelian invariant like the character, its value on the alternating sum of the cohomology groups (the Euler characteristic) is the same as its value on the alternating sum of whatever co-chains ones uses to define cohomology, i.e. here we have

$$\sum_i (-1)^i ch(H^i(\mathfrak{n}^+, V)) = \sum_i (-1)^i ch(C^i(\mathfrak{n}^+, V))$$

This follows from two facts: the first is that

$$ch(C^i(\mathfrak{n}^+, V)) = ch(Z^i(\mathfrak{n}^+, V)) + ch(B^{i+1}(\mathfrak{n}^+, V))$$

since we have an exact sequence

$$0 \to Z^i(\mathfrak{n}^+, V) \longrightarrow C^i(\mathfrak{n}^+, V) \xrightarrow{d} B^{i+1}(\mathfrak{n}^+, V) \to 0$$

(here $Z^i(\mathfrak{n}^+, V)$ are the co-cycles on which $d = 0$, $B^{i+1}(\mathfrak{n}^+, V)$ are the co-boundaries which are in the image of d. Since

$$H^i(\mathfrak{n}^+, V) = Z^i(\mathfrak{n}^+, V)/B^i(\mathfrak{n}^+, V)$$

we also have a second fact

$$ch(H^i(\mathfrak{n}^+, V)) = ch(Z^i(\mathfrak{n}^+, V)) - ch(B^i(\mathfrak{n}^+, V))$$

and this together with our first fact gives the Euler-Poincaré principle.

Recall that

$$C^i(\mathfrak{n}^+, V) = Hom(\Lambda^i(\mathfrak{n}^+), V) = \Lambda^i(\mathfrak{n}^+)^* \otimes V$$

so we have

$$\sum_i (-1)^i ch(C^i(\mathfrak{n}^+, V^\lambda)) = \sum_i (-1)^i ch(\Lambda^i(\mathfrak{n}^+)^*) ch(V^\lambda)$$

whereas Kostant's theorem tells us that the Euler characteristic is

$$\sum_i (-1)^i ch(H^i(\mathfrak{n}^+, V^\lambda)) = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}$$

Applying the Euler-Poincaré principle in the case $\lambda = 0$ gives

$$\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho} = \sum_i (-1)^i ch(\Lambda^i(\mathfrak{n}^+)^*)$$

and thus in the general case the Weyl character formula as

$$\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho} = (\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho}) ch(V^\lambda)$$

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