Quantum Field Theory for Mathematicians: Hamiltonian Mechanics and Symplectic Geometry

We'll begin with a quick review of classical mechanics, expressed in the language of modern geometry. There are two general formalisms used in classical mechanics to derive the classical equations of motion: the Hamiltonian and Lagrangian. Both formalisms lead to the same equations of motion in the cases where they both apply, but they provide rather different points of view on both classical mechanics and the subject we will turn to next, quantum mechanics.

For more detailed information about this subject, some good references are [1], [2], [3] and [4].

1 Hamiltonian Mechanics and Symplectic Geometry

The standard example of classical mechanics in its Hamiltonian form deals with a single particle moving in space ($\mathbf{R}^3$). The state of the system at a given time t is determined by six numbers, the coordinates of the position (q_1, q_2, q_3) and the momentum (p_1, p_2, p_3). The space $\mathbf{R}^6$ of positions and momenta is called "phase space." The time evolution of the system is determined by a single function of these six variables called the Hamiltonian and denoted H. For the case of a particle of mass m moving in a potential $V(q_1, q_2, q_3)$,

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$

The time evolution of the state of the system is given by the solution of the following equations, known as Hamilton's equations

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

There is an obvious generalization of this to the description of a particle in n dimensions, in the case phase space is $\mathbf{R}^{2n}$. Similarly, N particles can be described by the phase space given by taking the tensor product of N copies of $\mathbf{R}^{2n}$, which is just $\mathbf{R}^{2nN}$.

Besides the Hamiltonian function on it, phase space comes with an important algebraic structure: a bilinear, antisymmetric bracket on functions f and g on phase space called the Poisson bracket and given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial f}{\partial p_i}$$

Hamiltonian mechanics can be formulated in a geometric, coordinate invariant manner on a general class of manifolds of which $\mathbf{R}^{2nN}$ is just one kind of example. These are so-called symplectic manifolds, defined by

Definition 1 (Symplectic Manifold). *A symplectic manifold (M,ω) is a $2n$-dimensional manifold M with a two-form ω satisfying*

- *ω is non-degenerate, i.e. for each $m \in M$, the identification of T_m and T_m^* given by ω is an isomorphism*
- *ω is closed, i.e. $d\omega = 0$.*

The two main classes of examples of symplectic manifolds are

- Cotangent bundles: $M = T^*N$.

  In this case there is a canonical one-form θ defined at a point $(n, \alpha) \in T^*N$ $(n \in N,\ \alpha \in T_n^*(N))$ by

  $$\theta_{n,\alpha}(v) = \alpha(\pi_* v)$$

  where π is the projection from T^*N to N. The symplectic two-form on T^*N is

  $$\omega = d\theta$$

  Physically this case corresponds to a particle moving on an arbitrary manifold M. For the special case $N = \mathbf{R}^n$,

  $$\theta = \sum_{i=1}^{n} p_i \wedge dq_i$$

  and the symplectic form is

  $$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

- Kähler manifolds. Special cases here include the sphere $S^2 = \mathbf{C}P^1$ (with symplectic form proportional to the area 2-form), projective algebraic varieties, flag manifolds G/T where G is a compact Lie group and T is its maximal torus.

A map

$$f : M \to M$$

preserving the symplectic structure ($f^*(\omega) = \omega$) is called a symplectomorphism, and corresponds to the physicist's notion of a canonical transformation of phase space. The Darboux theorem states that locally any symplectic manifold is symplectomorphic to $\mathbf{R}^{2n}$, ω_0). Thus in symplectic geometry the only local geometric invariant is the dimension, unlike the case in Riemannian geometry where locally manifolds up to isometry do have geometric invariants: the

Riemannian curvature. Physicist's discussions of Hamiltonian mechanics often assume that one can globally choose "canonical coordinates" on phase space and identify it with $(\mathbf{R}^{2n}, \omega_0)$. This is not the case for a general symplectic manifold, with the simplest counter-example being S^2.

In many interesting cases of symplectic manifolds, there is another interesting geometrical structure that enters the story. Often the symplectic two-form ω is actually the curvature 2-form of a principal $U(1)$ bundle or of a complex line bundle.

The symplectic structure can be used to write Hamilton's equations in a coordinate invariant manner. Note that these equations on $\mathbf{R}^{2n}$ are similar to the equations for a gradient flow in $2n$ dimensions

$$\frac{dp_i}{dt} = -\frac{\partial f}{\partial p_i}$$

$$\frac{dq_i}{dt} = -\frac{\partial f}{\partial q_i}$$

These equations correspond to flow along a vector field ∇_f which comes from choosing a function f, taking $-df$, then using an inner product on $\mathbf{R}^{2n}$ to dualize and get a vector field from this 1-form. In other words we use a symmetric non-degenerate 2-form (the inner product $< \cdot, \cdot >$) to produce a map from functions to vector fields as follows:

$$f \to \nabla_f : < \nabla_f, \cdot >= -df$$

Hamilton's equations correspond to a similar construction, with the symmetric 2-form coming from the inner product replaced by the antisymmetric symplectic 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

In this case, starting with a Hamiltonian function H, one produces a vector field X_H as follows

$$H \to X_H : \quad \omega(X_H, \cdot) = i_{X_H}\omega = -dH$$

Hamilton's equations are then the dynamical system for the vector field X_H. X_H is sometimes called the symplectic gradient of H. While the flow along a gradient vector field of f changes the magnitude of f as fast as possible, flow along X_H keeps the value of H constant since

$$dH = -\omega(X_H, \cdot)$$

$$dH(X_H) = -\omega(X_H, X_H) = 0$$

since ω is antisymmetric.

One can check that in the case of $\mathbf{R}^{2n}$, the equation

$$i_{X_H}\omega = -dH$$

implies Hamilton's equations for X_H since equating

$$-dH = -\sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i$$

and

$$i_{X_H} \sum_{i=1}^{n} dp_i \wedge dq_i$$

implies

$$X_H = -\sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}$$

The equation

$$i_{X_H}\omega = -dH$$

continues to make sense on any symplectic manifold, for any Hamiltonian function X_H, and H will be constant along the trajectories of X_H. Another important property of X_H is that

$$\mathcal{L}_{X_H}\omega = (di_{X_H} + i_{X_H}d)\omega = d(-dH) = 0$$

since $d\omega = 0$ (where $\mathcal{L}_{X_H}$ is the Lie derivative with respect to X_H). In general

Definition 2 (Hamiltonian Vector Field). *A vector field X that satisfies*

$$\mathcal{L}_X\omega = 0$$

is called a Hamiltonian vector field and the space of such vector fields on M, ω will be denoted $Vect(M, \omega)$.

Since ω is non-degenerate, the equation

$$i_{X_H}\omega = -dH$$

implies that if $X_H = 0$, then $dH = 0$ and $H = constant$. As a result, we have an exact sequence of maps

$$0 \to \mathbf{R} \to C^\infty(M) \to Vect(M, \omega)$$

One can also ask whether all Hamiltonian vector fields (elements of $Vect(M, \omega)$) actually come from a Hamiltonian function. The equation

$$\mathcal{L}_X\omega = (di_X + i_Xd)\omega = 0$$

implies

$$di_X\omega = 0$$

so $i_X\omega$ is a closed 1-form. When $H^1(M, \mathbf{R}) = 0$, closed 1-forms are all exact, so for any Hamiltonian vector field X one can find a Hamiltonian function H such that $X = X_H$. In general we have an exact sequence

$$0 \to \mathbf{R} \to C^\infty(M) \to Vect(M, \omega) \to H^1(M, \mathbf{R}) \to 0$$

Most of the examples we will be interested in will be simply connected and thus have $H^1 = 0$.

Just as we saw that $dH = 0$ along X_H, one can compute the derivative of an arbitrary function g along a Hamiltonian vector field X_f as

$$dg(\cdot) = -\omega(X_g, \cdot)$$

$$dg(X_f) = -\omega(X_g, X_f) = \omega(X_f, X_g)$$

which leads to the following definition

Definition 3 (Poisson Bracket). *The Poisson bracket of two functions on* M, ω *is*

$$\{f, g\} = \omega(X_f, X_g)$$

One can check that this definition of the Poisson bracket agrees with the standard one used by physicists for the case $\mathbf{R}^{2n}$,ω.

The Poisson bracket satisfies

$$\{f, g\} = -\{g, f\}$$

and

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$$

where the second of these equations can be proved by calculating

$$d\omega(X_{f_1}, X_{f_2}, X_{f_3}) = 0$$

These relations show that the Poisson bracket makes $C^\infty(M)$ into a Lie algebra. One can also show that

$$[X_f, X_g] = X_{\{f,g\}}$$

which is the condition that ensures that the map

$$f \in C^\infty(M) \to X_f \in Vect(M, \omega)$$

is a Lie algebra homomorphism, with the Lie bracket of vector fields the product in $Vect(M, \omega)$.

So, for a symplectic manifold (M, ω) with $H^1(M) = 0$, we have an exact sequence of Lie algebra homomorphisms

$$0 \to \mathbf{R} \to C^\infty(M) \to Vect(M, \omega) \to 0$$

Note that the Lie algebras involved are infinite dimensional.

2 Examples

The standard example in mechanics is that of a particle moving in $\mathbf{R}^3$, subject to a potential $V(\mathbf{q})$. The Hamiltonian function is

$$H(\mathbf{p},\mathbf{q}) = \frac{||\mathbf{p}||^2}{2m} + V(\mathbf{q})$$

where m is the particle mass and we are using the standard metric on $\mathbf{R}^3$. Hamilton's equations become

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}$$

and

$$\frac{d\mathbf{p}}{dt} = -\nabla V$$

The first equation just says that the momentum is the mass times the velocity, combining this with the second equation is just the familiar

$$F = -\nabla V = m\mathbf{a}$$

A special case of this occurs when V is quadratic $\mathbf{q}$, this is the case of the harmonic oscillator, which can be solved exactly. The harmonic oscillator problem is of great importance in physics, since a standard way of approaching physical problems involves building approximate solutions to problems starting from the harmonic oscillator. If one takes the quadratic approximation to V near one of its critical points, a first approximation to motion of a particle near that critical point will be given by the harmonic oscillator solution, and one can try and find better approximations as small perturbations of the harmonic oscillator. This method is fundamental both in quantum mechanics and in quantum field theory.

The simplest version of the harmonic oscillator is the Hamiltonian system $M = \mathbf{R}^2$ with Hamiltonian

$$H(p,q) = \frac{1}{2}(\frac{p^2}{m} + kq^2)$$

By rescaling variables, we just need to consider

$$H(p,q) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

for some real variable ω. Hamilton's equations are

$$\frac{dq}{dt} = p$$

and

$$\frac{dp}{dt} = -\omega^2 q$$

so

$$\frac{d^2q}{dt^2} = -\omega^2 q$$

with solutions

$$q(t) = A\sin\omega t + B\cos\omega t$$

One can choose a complex structure and identify $\mathbf{R}^2 = \mathbf{C}$ by

$$z = p + i\omega q$$

in which case

$$H = \frac{1}{2}|z|^2$$

and solutions to the Hamiltonian system are given by

$$z(t) = Ce^{i\omega t}$$

This has an obvious generalization to more variables, although in that case there are a lot more ways to choose the complex structure on phase space. This will be discussed in detail when we come to the quantum harmonic oscillator.

Another important example corresponds physically to the behavior of a particle with magnetic moment proportional to its "spin", moving in a magnetic field. If one considers the case of a very massive particle and ignores its motion in $\mathbf{R}^3$, just considering the motion of its spin vector $\mathbf{s}$, the appropriate phase space is S^2, with a point in phase space corresponding to a choice of direction of the spin vector. Coordinates on S^2 can be the polar angle ϕ and the azimuthal angle θ or equivalently θ and s_z, the z-component of $\mathbf{s}$.

The symplectic form ω is just proportional to the area form dA

$$\omega = ds_z \wedge d\theta$$

Ignoring the physical constants involved in the problem, and taking the magnetic field $\mathbf{B}$ to be in the z direction with magnitude $B = |\mathbf{B}|$, the Hamiltonian system is (S^2, ω) with Hamiltonian function

$$H(\theta, s_z) = s_z B$$

Hamilton's equations are that the vector field X_H satisfies

$$i_{X_H}(ds_z \wedge d\theta) = -d(s_z B)$$

with solution

$$X_H = B\frac{\partial}{\partial\theta}$$

so the motion of the spin is given by precession about the magnetic field vector, with precession velocity proportional to the strength of the magnetic field. One can see that this must be the trajectory just by noting H must be constant along trajectories and thus the inner product of the spin vector and the magnetic field vector will be constant, which is just true on the circle given by rotating the spin about the magnetic field.

References

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[3] Woodhouse, N., *Geometric Quantization,* Oxford University Press, 1980.

[4] Segal, G., *Notes on Symplectic Manifolds and Quantization,* `http://online.itp.ucsb.edu/online/geom99/segal2_n`