Groups and Representations II: Problem Set 4

Due Monday, April 23

Problem 1:

Show that the real Clifford algebra for $\mathbf{R}^3$ is isomorphic to the sum of two copies of the quaternion algebra, i.e.

$$C(3) = \mathbf{H} \oplus \mathbf{H}$$

Problem 2: Using the results of Problem 1, explicitly construct the group $Spin(3)$ and its Lie algebra in terms of the Clifford algebra. Show that $Spin(3) = Sp(1)$, the group of unit quaternions.

Problem 3: Let $Spin(3,1)$ be the double cover of $SO(3,1)$, the orthogonal group that preserves a real quadratic form which, when diagonalized had 3 $+1$'s and one -1. Show that $Spin(3,1) = SL(2,\mathbf{C})$. This is the spin version of the Lorentz group of special relativity.

Problem 4: Using the definition of Fock space given in class, show that the operators a_j and $a_j^\dagger$ are adjoint operators on this space.

Problem 5: Using Fock space, explicitly construct the metaplectic representation of $Mp(4,\mathbf{R})$. Identify the maximal torus of this group and compute the character of the representation.