Notes on BRST

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1 Introduction

This is a collection of notes, mostly of an expository nature, giving background and explanation for the notion of "Dirac cohomology" and its relation to the BRST formalism for handling quantum gauge symmetry. It is currently being actively updated, check back for additional material in the near future.

2 Quantum Mechanics and Representation Theory

A quantum mechanical physical system is given by the following mathematical structure:

- A Hilbert space $\mathcal{H}$, the "space of states". A state of the physical system is determined by a vector $|\psi\rangle \in \mathcal{H}$, with unit norm (i.e. $||\psi||^2 = \langle\psi|\psi\rangle = 1$).
- An algebra $\mathcal{O}$ that acts on $\mathcal{H}$. To each physical observable corresponds a self-adjoint operator $O \in \mathcal{O}$. Eigenvectors in $\mathcal{H}$ of this operator correspond to states where the observable has a well-defined value, which is the eigenvalue.

If a physical system has a symmetry group G, there is a unitary representation $(\Pi, \mathcal{H})$ of G on $\mathcal{H}$. This means that for each $g \in G$ we get a unitary operator $\Pi(g)$ satisfying

$$\Pi(g_3) = \Pi(g_2)\Pi(g_1) \text{ if } g_3 = g_1 g_2$$

i.e. the map Π from group elements to unitary operators is a homomorphism. The $\Pi(g)$ act on $\mathcal{O}$ by taking an operator O to its conjugate $\Pi(g)O(\Pi(g))^{-1}$.

When G is a Lie group with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, differentiating Π gives a unitary representation $(\pi, \mathcal{H})$ of $\mathfrak{g}$ on $\mathcal{H}$. This means that for each $X \in \mathfrak{g}$ we get a skew-Hermitian operator $\pi(X)$ on $\mathcal{H}$, satisfying

$$\pi(X_3) = [\pi(X_1), \pi(X_2)] \text{ if } X_3 = [X_1, X_2]$$

i.e. the map π taking Lie algebra elements X (with the Lie bracket in $\mathfrak{g}$) to skew-Hermitian operators (with commutator of operators) is a homomorphism. On $\mathcal{O}$, $\mathfrak{g}$ acts by the differential of the conjugation action of G, this action is just that of taking the commutator with $\pi(X)$.

The Lie bracket is not associative, but to any Lie algebra $\mathfrak{g}$, one can construct an associative algebra $U(\mathfrak{g})$ called the universal enveloping algebra for $\mathfrak{g}$. If one identifies $X \in \mathfrak{g}$ with left-invariant vector fields on G, which are first-order differential operators on functions on G, then $U(\mathfrak{g})$ is the algebra of left-invariant differential operators on G of all orders, with product the composition of differential operators. A Lie algebra representation is precisely a module over $U(\mathfrak{g})$, i.e. a vector space with an action of $U(\mathfrak{g})$.

So, the state space $\mathcal{H}$ of a quantum system with symmetry group G carries not only a unitary representation of G, but also a unitary representation of $\mathfrak{g}$, or equivalently, an action of the algebra $U(\mathfrak{g})$. $X \in \mathfrak{g}$ acts by the operator $\pi(X)$. In this way a representation π gives a sub-algebra of the algebra $\mathcal{O}$ of observables. Most of the important observables that show up in practice come from a symmetry in this way. An interesting philosophical question is whether the quantum system that governs the real world is purely determined by symmetry, i.e. such that ALL its observables come from symmetries in this manner.

2.1 Some Examples

Much of the structure of common quantum mechanical systems is governed by the fact that they carry space-time symmetries. In our 3-space, 1-time dimensional world, these include:

- Translations in space: $G = \mathbf{R}^3, \mathfrak{g} = \mathbf{R}^3$, Lie Bracket is trivial. For each basis element $e_j \in \mathfrak{g}$ one gets a momentum operator $\pi(e_j) = iP_j$

- Translations in time: $G = \mathbf{R}, \mathfrak{g} = \mathbf{R}$. If e_0 is a basis of $\mathfrak{g}$, $i\pi(e_0) = H$, the Hamiltonian operator. The fact that this operator generates time-translations is just Schrödinger's equation.

- Rotations in 3-space: $G = SO(3)$, or its double cover $G = Spin(3) = SU(2), \mathfrak{g} = \mathbf{R}^3$, with bracket given by the vector product. For each basis element $e_j \in \mathfrak{g}$ one gets an angular momentum operator $\pi(e_j) = iJ_j$. These operators do not commute, so cannot be simultaneously diagonalized.

- Another example is the symmetry of phase transformations of the state space $\mathcal{H}$. Here $G = U(1), \mathfrak{g} = \mathbf{R}$, and one gets an operator Q_e that can be normalized to have integral eigenvalues.

This last example also comes in a local version, where we make independent phase transformations at different points in space-time. This is an example of a "gauge symmetry", and the question of how it gets represented on the space of states is what will lead us into the BRST story.

3 Gauge Symmetry

My initial plan was to have this section of these notes be about gauge symmetry and the problems physicists have encountered in handling it, but as I started writing it quickly became apparent that explaining this in any detail would take me into various issues that are quite interesting, but far afield from what I want to get to. So, I hope to get back to this at some point, but for now will just assume that most of my readers know what gauge symmetry is, and that the rest just need to know that:

- The gauge group is an infinite dimensional Lie group. Locally (on space-time), it looks like a group of maps into a finite dimensional Lie group.
- The conventional assumption is that physics is invariant under the gauge group, so the gauge group and its Lie algebra should act trivially on physical states.

The actual situation is quite a bit more complicated than this, but for now we'll focus on the simplest version of the mathematical problem that comes up here, and see how the BRST formalism deals with it.

4 Lie Algebra Cohomology, Physicist's Version

This section will begin explaining one part of this story, starting with the simplest version of BRST cohomology, in a language familiar to physicists. Later sections will deal with Lie algebra cohomology in a more general mathematical context and work out some examples. For more about the material in this section, see, for instance, [1] Volume I, section 3.2.1, or [2].

Physicists always begin by choosing a basis, in this case a basis X_i of $\mathfrak{g}$ satisfying $[X_i, X_j] = f_{ij}^k X_k$, where f_{ij}^k are called the structure constants of $\mathfrak{g}$. A representation (π, V) is then a set of linear operators $K_i = \pi(X_i)$ on V satisfying $[K_i, K_j] = f_{ij}^k K_k$. Let α^i be a basis of the dual space $\mathfrak{g}^*$, dual to the basis X_i.

Now, extend V to $=V \otimes \Lambda^*(\mathfrak{g}^*)$, where $\Lambda^*(\mathfrak{g}^*)$ is the exterior algebra on $\mathfrak{g}^*$. On this space, define the "ghost" operator c^i to be wedge-product with α^i, and "anti-ghost" operator b_i to be contraction (interior product) with X_i. These operators satisfy "fermionic" anti-commutation relations

$$\{c^i, c^j\} = \{b_i, b_j\} = 0, \quad \{c^i, b_j\} = \delta_j^i$$

and one can get all vectors in $\mathcal{H}$ from linear combinations of decomposable elements of $\mathcal{H}$ (those given by repeated application of the c^i to the "vacuum vector" $V \otimes \mathbf{1}$).

The ghost number operator $N = c^i b_i$ on $\mathcal{H}$ has eigenvectors the decomposable elements, with integer eigenvalues from 0 to dim $\mathfrak{g}$, given by the number of ghost operators needed to produce the eigenvector from a vacuum vector.

The BRST operator is given by

$$Q = c^i K_i - \frac{1}{2} f_{ij}^k c^i c^j b_k$$

which increases the ghost number by one, and has the crucial property of $Q^2 = 0$ (this comes from the fact that the f_{ij}^k satisfy the Jacobi identity). The BRST cohomology is given by considering the space $ker\ Q$ of elements χ of $\mathcal{H}$ that are "BRST-closed", i.e. satisfy $Q\chi = 0$, and identifying two such elements if they are "BRST-exact", i.e. differ by $Q\lambda$ for some λ. So BRST cohomology is defined by

$$H_Q^*(V) = \frac{ker\ Q}{im\ Q}|_{V\otimes\Lambda^*(\mathfrak{g}^*)}$$

with $H_Q^j(V)$ the component of the BRST cohomology of ghost number j.

A vector $\chi = v \otimes \mathbf{1}$ of ghost number zero satisfies $Q\chi = 0$ iff and only if $K_i v = 0$ for all i, so we can identify $H_Q^0(V)$ with the space $V^{\mathfrak{g}}$ of $\mathfrak{g}$ - invariant vectors in V.

The essence of the BRST method is to replace the problem of finding the invariant subspace $V^{\mathfrak{g}}$ of a representation V by the problem of finding the degree zero BRST cohomology $H_Q^0(V)$.

There are two different ways of putting an inner product on $\Lambda^*(\mathfrak{g}*)$ and thus getting an inner product on $\mathcal{H}$ ((π, V) is assumed to be unitary, so preserves a given inner product on V).

- Given $\omega_1, \omega_2 \in \Lambda^*(\mathfrak{g}*)$, one can define

  $$<\omega_1, \omega_2> = \int \omega_1\omega_2 \equiv coeff.\ of\ \alpha_1 \wedge \cdots \wedge \alpha_{dim\ \mathfrak{g}}\ in\ \omega_1 \wedge \omega_2$$

  (this uses the "fermionic" or "Berezin" integral $\int$, although I have not properly dealt with signs here). This inner product is indefinite, but it makes the BRST operator Q and ghost-operator c^i self-adjoint.

- Use an inner product on $\mathfrak{g}$, e.g. the Killing form for a semi-simple Lie algebra, to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. This gives a Hodge operator $*_{Hodge}$ on $\Lambda^*(\mathfrak{g}*)$ that takes $\Lambda^i(\mathfrak{g}*)$ to $\Lambda^{dim\ \mathfrak{g}-i}(\mathfrak{g}*)$, and one can define

  $$<\omega_1, \omega_2> = \int_G \omega_1 \wedge *_{Hodge}\omega_2$$

  (Note, here the integral sign is not Berezin integration, but the usual integration of differential forms over a compact manifold, in this case G)

  With this inner product Q and c^i are not self-adjoint on $\mathcal{H}$. To get something self-adjoint, one can consider the operator $Q + Q^\dagger$ where $Q^\dagger$ is the adjoint of Q, but this operator does not have a definite ghost-number.

5 Lie Algebra Cohomology

The last section discussed one of the simplest incarnations of BRST cohomology, in a formalism familiar to physicists. This fits into a much more abstract mathematical context, and that's what we'll turn to now.

5.1 The Invariants Functor

Given a Lie algebra $\mathfrak{g}$, we'll consider Lie algebra representations as modules over $U(\mathfrak{g})$. Such modules form a category $\mathcal{C}_{\mathfrak{g}}$: what is interesting is not just the objects of the category (the equivalence classes of modules), but also the morphisms between the objects. For two representations V_1 and V_2 the set of morphisms between them is a linear space denoted $Hom_{U(\mathfrak{g})}(V_1, V_2)$. This is just the set of linear maps from V_1 to V_2 that commute with the action of $\mathfrak{g}$:

$$Hom_{U(\mathfrak{g})}(V_1, V_2) = \{\phi \in Hom_{\mathbf{C}}(V_1, V_2) : \pi(X)\phi = \phi\pi(X)\ \forall X \in \mathfrak{g}\}$$

Another conventional name for this is the space of intertwining operators between the two representations.

For any representation V, its $\mathfrak{g}$-invariant subspace $V^{\mathfrak{g}}$ can be identified with the space $Hom_{U(\mathfrak{g})}(\mathbf{C}, V)$, where here $\mathbf{C}$ is the trivial one-dimensional representation. Having a way to pick out the invariant piece of a representation also allows one to solve the more general problem of picking out the subspace that transforms like a specific irreducible W: just find the invariant subspace of $V \otimes W^*$.

The map $V \rightarrow V^{\mathfrak{g}}$ that takes a representation to its $\mathfrak{g}$-invariant subspace is a functor: it takes the category $\mathcal{C}_{\mathfrak{g}}$ to $\mathcal{C}_{\mathbf{C}}$, the category of vector spaces and linear maps ($\mathbf{C}$ - modules and $\mathbf{C}$ - homomorphisms). If, instead of taking

$$V \rightarrow V^{\mathfrak{g}}$$

one takes

$$V \rightarrow V^{\mathfrak{h}}$$

where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, one again gets a functor. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (so that $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra), then this functor takes $\mathcal{C}_{\mathfrak{g}}$ to $\mathcal{C}_{\mathfrak{g}/\mathfrak{h}}$. This is a simple version of the situation of interest in the case of gauge theory: if V is a state space with $\mathfrak{h}$ acting as a gauge symmetry, then $V^{\mathfrak{h}}$ will be the physical subspace, carrying an action of the algebra of operators $U(\mathfrak{g}/\mathfrak{h})$.

5.2 Some Homological Algebra

It turns out that when one has a category of modules like $\mathcal{C}_{\mathfrak{g}}$, these can usefully be studied by considering complexes of modules, and this is the subject of homological algebra. A complex of modules is a sequence of modules and homomorphisms

$$\cdots \xrightarrow{\partial} U \xrightarrow{\partial} V \xrightarrow{\partial} W \xrightarrow{\partial} \cdots$$

such that $\partial \circ \partial = 0$. If the complex satisfies $im\ \partial = ker\ \partial$ at each module, the complex is said to be an "exact complex".

To motivate the notion of exact complex, note that

$$0 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

is exact iff V_0 is isomorphic to V, and an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

represents the module V as the quotient V_0/V_1. Using longer complexes, one gets the notion of a *resolution* of a module V by a sequence of n modules V_i. This is an exact complex

$$0 \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow V \longrightarrow 0$$

The deviation of a sequence from being exact is measured by its homology, $H_* = \frac{ker\ \partial}{im\ \partial}$. Note that if one deletes V from its resolution, the sequence

$$0 \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

is exact except at V_0. Indexing the homology in the obvious way, one has $H_i = 0$ for $i > 0$, and $H^0 = V$. A sequence like this whose only homology is V at H_0 is another manifestation of a resolution of V.

The reason this construction is useful is that, for many purposes, it allows us to replace a module whose structure we may not understand by a sequence of modules whose structure we do understand. In particular, we can replace a $U(\mathfrak{g})$ module V by a sequence of free modules, i.e. modules that are just sums of copies of $U(\mathfrak{g})$ itself. This is called a free resolution, and more generally one can work with projective modules (direct summands of free modules).

A functor that takes exact complexes to exact complexes is called an exact functor. Homological invariants of modules come about in cases where one has a functor on a category of modules that is not exact. Applying such a functor to a free or projective resolution gives the homological invariants.

5.3 The Koszul Resolution and Lie Algebra Cohomology

There are many possible choices of a free resolution of a module. For the case of $U(\mathfrak{g})$ modules, one convenient choice is known as the Koszul (or Chevalley-Eilenberg) resolution. To construct a resolution of the trivial module $\mathbf{C}$, one uses the exterior algebra on $\mathfrak{g}$ to make free modules

$$Y_k = U(\mathfrak{g}) \otimes_{\mathbf{C}} \Lambda^k(\mathfrak{g})$$

and get a resolution of $\mathbf{C}$

$$0 \longrightarrow Y_{dim\ \mathfrak{g}} \xrightarrow{\partial_{dim\ \mathfrak{g}-1}} \cdots \xrightarrow{\partial_1} Y_1 \xrightarrow{\partial_0} Y_0 \xrightarrow{\epsilon} \mathbf{C} \longrightarrow 0$$

The maps are given by

$$\epsilon : u \in Y_0 = U(\mathfrak{g}) \rightarrow \epsilon(u) = const.\ term\ of\ u$$

and

$$\begin{aligned}\partial_{k-1}(u \otimes X_1 \wedge \cdots \wedge X_k) &= \sum_{i=1}^{k}(-1)^{i+1}(uX_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k) \\ &+ \sum_{i<j}(-1)^{i+j}(u \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k)\end{aligned}$$

To get Lie algebra cohomology, we apply the invariants functor

$$V \longrightarrow V^{\mathfrak{g}} = Hom_{U(\mathfrak{g})}(\mathbf{C}, V)$$

replacing the trivial representation by its Koszul resolution. This gives us a complex with terms

$$\begin{aligned} C^k(\mathfrak{g}, V) = Hom_{U(\mathfrak{g})}(Y_k, V) &= Hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes \Lambda^k(\mathfrak{g}), V) \\ &= Hom_{U(\mathfrak{g})}(U(\mathfrak{g}), Hom_{\mathbf{C}}(\Lambda^k(\mathfrak{g}), V)) \\ &= Hom_{\mathbf{C}}(\Lambda^k(\mathfrak{g}), V) = V \otimes \Lambda^k(\mathfrak{g}^*) \end{aligned}$$

and induced maps d_i

$$0 \longrightarrow C^0(\mathfrak{g}, V) \xrightarrow{d_0} C^1(\mathfrak{g}, V) \cdots \xrightarrow{d_{dim\ \mathfrak{g}-1}} C^{dim\ \mathfrak{g}}(\mathfrak{g}, V) \longrightarrow 0$$

The Lie algebra cohomology $H^*(\mathfrak{g}, V)$ is just the cohomology of this complex, i.e.

$$H^i(\mathfrak{g}, V) = \frac{ker\ d_i}{im\ d_{i-1}}|_{C^i(\mathfrak{g}, V)}$$

This is exactly the same definition as that of the BRST cohomology defined in physicist's formalism in the last section with $\mathcal{H} = C^*(\mathfrak{g}, V)$.

One has $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$ and so gets the $\mathfrak{g}$-invariants as expected, but in general the cohomology will be non-zero also in other degrees.

This is all rather abstract, so in the next section some examples will be worked out, as well as the relationship of all this to the de Rham cohomology of the group. Anthony Knapp's book *Lie Groups, Lie Algebras, and Cohomology* [3] is an excellent reference for details on Lie algebra cohomology.

6 Lie Algebra Cohomology for Semi-simple Lie Algebras

In this section I'll work out some examples of Lie algebra cohomology, still for finite dimensional Lie algebras and representations.

If G is a compact, connected Lie group, it can be thought of as a compact manifold, and as such one can define its de Rham cohomology $H^*_{deRham}(G)$ as the cohomology of the complex

$$0 \longrightarrow \Omega^0(G) \xrightarrow{d} \Omega^1(G) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{dim\ G}(G) \longrightarrow 0$$

where $\Omega^i(G)$ are the differential i-forms on G (note, we'll use complex-valued forms), and d is the deRham differential.

For a compact group, one has a bi-invariant Haar measure $\int_G$, and can use this to "average" over an action of the group on a space. For a representation (π, V), we get a projection operator $\int_g \Pi(g)$ onto the invariant subspace V^G. This projection operator gives explicitly the invariants functor on $\mathcal{C}_{\mathfrak{g}}$. It is an exact functor, taking exact sequences to exact sequences.

The differential forms $\Omega^*(G)$ give a representation of G in two ways, taking the induced action on forms by pullback, using either left or right translation on the group. If $(\Pi(g), \Omega^*(G))$ is the representation by left translations, we can use this to apply our "averaging over G" projection operator to the de Rham complex. This action commutes with the de Rham differential, so we get a sub-complex of left-invariant forms

$$0 \longrightarrow \Omega^0(G)^G \xrightarrow{d} \Omega^1(G)^G \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{dim\ G}(G)^G \longrightarrow 0$$

Since elements of the Lie algebra $\mathfrak{g}$ are precisely left-invariant 1-forms, it turns out that this complex is nothing but the Chevalley-Eilenberg complex considered last time to represent Lie algebra cohomology, for the case of the trivial representation. This means we have $C^*(\mathfrak{g}, \mathbf{R}) = \Lambda^*(\mathfrak{g}^*) = \Omega^*(G)^G$, and the differentials coincide. So, what we have shown is that

$$H^*(\mathfrak{g}, \mathbf{C}) = H^*_{deRham}(G)$$

If one knows the cohomology of G, the Lie algebra cohomology is thus known, but this identity is normally used in the other direction, to find the cohomology of G from that of the Lie algebra. To compute the Lie-algebra cohomology, we can exploit the right-action of G on the group, averaging over the induced action on the left-invariant forms $\Lambda^*(\mathfrak{g})$, which again commutes with the differential. We end up with a complex

$$0 \longrightarrow (\Lambda^0(\mathfrak{g}^*))^G \longrightarrow (\Lambda^1(\mathfrak{g}^*))^G \longrightarrow \cdots \longrightarrow (\Lambda^{\dim\ \mathfrak{g}}(\mathfrak{g}^*))^G \longrightarrow 0$$

where all the differentials are zero, so the cohomology is given by

$$H^*(\mathfrak{g}, \mathbf{C}) = (\Lambda^*(\mathfrak{g}^*))^G = (\Lambda^*(\mathfrak{g}^*))^{\mathfrak{g}}$$

the adjoint-invariant pieces of the exterior algebra on $\mathfrak{g}^*$. Finding the cohomology has now been turned into a purely algebraic problem in invariant theory.

For $G = U(1)$, $\mathfrak{g} = \mathbf{R}$, and we have shown that $H^*(\mathbf{R}, \mathbf{C}) = \Lambda^*(\mathbf{C})$, this is $\mathbf{C}$ in degrees 0, and 1, as expected for the de Rham cohomology of the circle $U(1) = S^1$. For $G = U(1)^n$, we get

$$H^*(\mathbf{R}^n, \mathbf{C}) = \Lambda^*(\mathbf{C}^n)$$

Note that complexifying the Lie algebra and working with $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ commutes with taking cohomology, so we get

$$H^*(\mathfrak{g}_{\mathbf{C}}, \mathbf{C}) = H^*(\mathfrak{g}, \mathbf{C}) \otimes \mathbf{C}$$

Complexifying the Lie algebra of a compact semi-simple Lie group gives a complex semi-simple Lie algebra, and we have now computed the cohomology of these as

$$H^*(\mathfrak{g}_{\mathbf{C}}, \mathbf{C}) = (\Lambda^*(\mathfrak{g}_{\mathbf{C}}))^{\mathfrak{g}_{\mathbf{C}}}$$

Besides H^0, one always gets a non-trivial H^3, since one can use the Killing form $< \cdot, \cdot >$ to produce an adjoint-invariant 3-form

$$\omega_3(X_1, X_2, X_3) =< x_1, [X_2, X_3] >$$

For $G = SU(n)$, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n, \mathbf{C})$, and one gets non-trivial cohomology classes ω_{2i+1} for $i = 1, 2, \cdots n$, such that

$$H^*(\mathfrak{sl}(n, \mathbf{C})) = \Lambda^*(\omega_3, \omega_5, \cdots, \omega_{2n+1})$$

the exterior algebra generated by the ω_{2i+1}.

To compute Lie algebra cohomology $H^*(\mathfrak{g}, V)$ with coefficients in a representation V, we can go through the same procedure as above, starting with differential forms on G taking values in V, or we can just use exactness of the averaging functor that takes V to V^G. Either way, we end up with the result

$$H^*(\mathfrak{g}, V) = H^*(\mathfrak{g}, \mathbf{C}) \otimes V^{\mathfrak{g}}$$

The H^0 piece of this is just the $V^{\mathfrak{g}}$ that we want when we are doing BRST, but we also get quite a bit else: *dim* $V^{\mathfrak{g}}$ copies of the higher degree pieces of the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbf{C})$. The Lie algebra cohomology here is quite non-trivial, but doesn't interact in a non-trivial way with the process of identifying the invariants $V^{\mathfrak{g}}$ in V.

7 Highest Weight Theory

In the last section we discussed the Lie algebra cohomology $H^*(\mathfrak{g}, V)$ for $\mathfrak{g}$ a semi-simple Lie algebra. Because the invariants functor is exact here, this tells us nothing about the structure of irreducible representations in this case. In this section we'll consider a different sort of example of Lie algebra cohomology, one that is intimately involved with the structure of irreducible $\mathfrak{g}$-representations.

7.1 Structure of semi-simple Lie algebras

A semi-simple Lie algebra is a direct sum of non-abelian simple Lie algebras. Over the complex numbers, every such Lie algebra is the complexification $\mathfrak{g}_{\mathbf{C}}$ of some real Lie algebra $\mathfrak{g}$ of a compact, connected Lie group. The Lie algebra $\mathfrak{g}$ of a compact Lie group G is, as a vector space, the direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t}$$

where $\mathfrak{t}$ is a commutative sub-algebra (the Cartan sub-algebra), the Lie algebra of T, a maximal torus subgroup of G.

Note that $\mathfrak{t}$ is not an ideal in $\mathfrak{g}$, so $\mathfrak{g}/\mathfrak{t}$ is not a subalgebra. $\mathfrak{g}$ is itself a representation of $\mathfrak{g}$ (the adjoint representation: $\pi(X)Y = [X, Y]$), and thus a representation of the subalgebra $\mathfrak{t}$. On any complex representation V of $\mathfrak{g}$, the action of $\mathfrak{t}$ can be diagonalized, with eigenspaces V^λ labeled by the corresponding eigenvalues, given by the weights λ. These weights $\lambda \in \mathfrak{t}^*_{\mathbf{C}}$ are defined by (for $v \in V^\lambda,\ H \in \mathfrak{t}$):

$$\pi(H)v = \lambda(H)v$$

Complexifying the adjoint representation, the non-zero weights of this representation are called roots, and we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus ((\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C})$$

The second term on the right is the sum of the root spaces V^α for the roots α. If α is a root, so is $-\alpha$, and one can choose decompositions of the set of roots into "positive roots" and "negative roots" such that:

$$\mathfrak{n}^+ = \bigoplus_{+roots\ \alpha} (\mathfrak{g}_{\mathbf{C}})^\alpha,\ \mathfrak{n}^- = \bigoplus_{-roots\ \alpha} (\mathfrak{g}_{\mathbf{C}})^\alpha$$

where $\mathfrak{n}^+$ (the "nilpotent radical") and $\mathfrak{n}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$. So, while $\mathfrak{g}/\mathfrak{t}$ is not a subalgebra of $\mathfrak{g}$, after complexifying we have decompositions

$$(\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

The choice of such a decomposition is not unique, with the Weyl group W (for a compact group G, W is the finite group $N(T)/T$, $N(T)$ the normalizer of T in G) permuting the possible choices.

Recall that a complex structure on a real vector space V is given by a decomposition

$$V \otimes \mathbf{C} = W \oplus \overline{W}$$

so the above construction gives $|W|$ different invariant choices of complex structure on $\mathfrak{g}/\mathfrak{t}$, which in turn give $|W|$ invariant ways of making G/T into a complex manifold.

The simplest example to keep in mind is $G = SU(2),\ T = U(1),\ W = \mathbf{Z}_2$, where $\mathfrak{g} = \mathfrak{su}(2)$, and $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(2, \mathbf{C})$. One can choose T to be the diagonal matrices, with a basis of $\mathfrak{t}$ given by

$$\frac{i}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and bases of $\mathfrak{n}^+,\ \mathfrak{n}^-$ given by

$$\frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(here the σ_i are the Pauli matrices). The Weyl group in this case just interchanges $\mathfrak{n}^+ \leftrightarrow \mathfrak{n}^-$.

7.2 Highest weight theory

Irreducible representations V of a compact Lie group G are finite dimensional and correspond to finite dimensional representations of $\mathfrak{g}_{\mathbf{C}}$. For a given choice of $\mathfrak{n}^+$, such representations can be characterized by their subspace $V^{\mathfrak{n}^+}$, the subspace of vectors annihilated by $\mathfrak{n}^+$. Since $\mathfrak{n}^+$ acts as "raising operators", taking subspaces of a given weight to ones with weights that are more positive, this is called the "highest weight" space since it consists of vectors whose weight cannot be raised by the action of $\mathfrak{g}_{\mathbf{C}}$. For an irreducible representation, this space is one dimensional, and we can label irreducible representations by the weight of $V^{\mathfrak{n}^+}$. The irreducible representation with highest weight λ is denoted V_λ. Note that this labeling depends on the choice of $\mathfrak{n}^+$.

Getting back to Lie algebra cohomology, while $H^*(\mathfrak{g}, V) = 0$ for an irreducible representation V, the Lie algebra cohomology for $\mathfrak{n}^+$ is more interesting, with $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+}$, the highest weight space. $\mathfrak{t}$ acts not just on V, but on the entire complex $C(\mathfrak{n}^+, V)$, in such a way that the cohomology spaces $H^i(\mathfrak{n}^+, V)$ are representations of $\mathfrak{t}$, so can be characterized by their weights.

For an irreducible representation V_λ, one would like to know which higher cohomology spaces are non-zero and what their weights are. The answer to this question involves a surprising "ρ - shift", a shift in the weights by a weight ρ, where

$$\rho = \frac{1}{2}\sum_{+roots}\alpha$$

half the sum of the positive roots. This is a first indication that it might be better to work with spinors rather than with the exterior algebra that is used in the Koszul resolution used to define Lie algebra cohomology. Much more about this in a later section.

One finds that $dim\, H^*(\mathfrak{n}^+, V_\lambda) = |W|$, and the weights occuring in $H^i(\mathfrak{n}^+, V_\lambda)$ are all weights of the form $w(\lambda+\rho)-\rho$, where $w \in W$ is an element of length i. The Weyl group can be realized as a reflection group action on $\mathfrak{t}^*$, generated by one reflection for each "simple" root. The length of a Weyl group element is the minimal number of reflections necessary to realize it. So, in dimension 0, one gets $H^0(\mathfrak{n}^+, V_\lambda) = V^{\mathfrak{n}^+}$ with weight λ, but there is also higher cohomology. Changing one's choice of $\mathfrak{n}^+$ by acting with the Weyl group permutes the different weight spaces making up $H^*(\mathfrak{n}^+, V)$. For an irreducible representation, to characterize it in a manner that is invariant under change in choice of $\mathfrak{n}^+$, one should take the entire Weyl group orbit of the ρ - shifted highest weight λ, i.e. the set of weights

$$\{w(\lambda+\rho),\ w \in W\}$$

In our $G = SU(2)$ example, highest weights can be labeled by non-negative half integral values (the "spin" s of the representation)

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

with $\rho = \frac{1}{2}$. The irreducible representation V_s is of dimension $2s+1$, and one finds that $H^0(\mathfrak{n}^+, V_s)$ is one-dimensional of weight s, while $H^1(\mathfrak{n}^+, V_s)$ is one-dimensional of weight $-s-1$.

The character of a representation is given by a positive integral combination of the weights

$$char(V) = \sum_{weights\ \omega} (dim\ V^\omega)\omega$$

(here V^ω is the ω weight space). The Weyl character formula expresses this as a quotient of expressions involving weights taken with both positive and negative integral coefficients. The numerator and denominator have an interpretation in terms of Lie algebra cohomology:

$$char(V) = \frac{\chi(H^*(\mathfrak{n}^+, V))}{\chi(H^*(\mathfrak{n}^+, \mathbf{C}))}$$

Here χ is the Euler characteristic: the difference between even-dimensional cohomology (a sum of weights taken with a + sign), and odd-dimensional cohomology (a sum of weights taken with a - sign). Note that these Euler characteristics are independent of the choice of $\mathfrak{n}^+$.

The material in this last section goes back to Bott's 1957 paper *Homogeneous Vector Bundles*[4], with more of the Lie algebra story worked out by Kostant in his 1961 *Lie Algebra Cohomology and the Generalized Borel-Weil Theorem*[5]. For an expository treatment with details, showing how one actually computes the Lie algebra cohomology in this case, for U(n) see chapter VI.3 of Knapp's *Lie Groups, Lie Algebras and Cohomology*[3], or for the general case see chapter IV.9 of Knapp and Vogan's *Cohomological Induction and Unitary Representations*[6].

8 Casimir Operators

For the case of $G = SU(2)$, it is well-known from the discussion of angular momentum in any quantum mechanics textbook that irreducible representations can be labeled either by j, the highest weight (here, highest eigenvalue of J_3), or by $j(j+1)$, the eigenvalue of $\mathbf{J}\cdot\mathbf{J}$. The first of these requires making a choice (the z-axis) and looking at a specific vector in the representation, the second doesn't. It was a physicist (Hendrik Casimir), who first recognized the existence of an analog of $\mathbf{J}\cdot\mathbf{J}$ for general semi-simple Lie algebras, and the important role that this plays in representation theory.

Recall that for a semi-simple Lie algebra $\mathfrak{g}$ one has a non-degenerate, invariant, symmetric bi-linear form $(\cdot,\cdot)$, the Killing form, given by

$$(X, Y) = tr(ad(X)ad(Y))$$

If one starts with $\mathfrak{g}$ the Lie algebra of a compact group, this bilinear form is defined on $\mathfrak{g}_{\mathbf{C}}$, and negative-definite on $\mathfrak{g}$. For a simple Lie algebra, taking the trace in a different representation gives the same bilinear form up to a constant.

As an example, for the case $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n, \mathbf{C})$, one can show that

$$(X,Y) = 2n\ tr(XY)$$

here taking the trace in the fundamental representation as n by n complex matrices.

One can use the Killing form to define a distinguished quadratic element Ω of $U(\mathfrak{g})$, the Casimir element

$$\Omega = \sum_i X_i X^i$$

where X_i is an orthonormal basis with respect to the Killing form and X^i is the dual basis. On any representation V, this gives a Casimir operator

$$\Omega_V = \sum_i \pi(X_i)\pi(X^i)$$

Note that, taking the representation V to be the space of functions $C^\infty(G)$ on the compact Lie group G, Ω_V is an invariant second-order differential operator, (minus) the Laplacian.

Ω is independent of the choice of basis, and belongs to $U(\mathfrak{g})^{\mathfrak{g}}$, the subalgebra of $U(\mathfrak{g})$ invariant under the adjoint action. It turns out that $U(\mathfrak{g})^{\mathfrak{g}} = Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. By Schur's lemma, anything in the center $Z(\mathfrak{g})$ must act on an irreducible representation by a scalar. One can compute the scalar for an irreducible representation (π, V) as follows:

Choose a basis $(H_i, X_\alpha, X_{-\alpha})$ of $\mathfrak{g}_{\mathbf{C}}$ with H_i an orthonormal basis of the Cartan subalgebra $\mathfrak{t}_{\mathbf{C}}$, and $X_{\pm\alpha}$ elements of $\mathfrak{n}^{\pm}$ in the $\pm\alpha$ root-spaces of $\mathfrak{g}_{\mathbf{C}}$, orthonormal in the sense of satisfying

$$(X_\alpha, X_{-\alpha}) = 1$$

Then one has the following expression for Ω:

$$\Omega = \sum_i H_i^2 + \sum_{+\ roots} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)$$

To compute the scalar eigenvalue of this on an irreducible representation (π, V_λ) of highest weight λ, one can just act on a highest weight vector $v \in V^\lambda = V^{\mathfrak{n}^+}$. On this vector the raising operators $\pi(X_\alpha)$ act trivially, and using the commutation relations

$$[X_\alpha, X_{-\alpha}] = H_\alpha$$

(H_α is the element of $\mathfrak{t}_{\mathbf{C}}$ satisfying $(H, H_\alpha) = \alpha(H)$) one finds

$$\Omega = \sum_i H_i^2 + \sum_{+roots} H_\alpha = \sum_i H_i^2 + 2H_\rho$$

where ρ is half the sum of the positive roots, a quantity which keeps appearing in this story. Acting on $v \in V^\lambda$ one finds

$$\Omega_{V_\lambda} v = (\sum_i \lambda(H_i)^2 + 2\lambda(H_\rho))v$$

Using the inner-product $< \cdot, \cdot >$ induced on $\mathfrak{t}^*$ by the Killing form, this eigenvalue can be written as:

$$< \lambda, \lambda > +2 < \lambda, \rho >= ||\lambda + \rho||^2 - ||\rho||^2$$

In the special case $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{g}_{\mathbf{C}} = \mathfrak{s}l(2, \mathbf{C})$, there is just one positive root, and one can take

$$H_1 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X_\alpha = e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ X_{-\alpha} = f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Computing the Killing form, one finds

$$(h, h) = 8, \ (e, f) = 4$$

and

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}(ef + fe) = \frac{1}{8}h^2 + \frac{1}{4}(h + 2fe)$$

On a highest weight vector Ω acts as

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}h = \frac{1}{8}h(h+2) = \frac{1}{2}(\frac{h}{2}(\frac{h}{2} + 1))$$

This is 1/2 times the physicist's operator $\mathbf{J} \cdot \mathbf{J}$, and in the irreducible representation V_n of spin $j = n/2$, it acts with eigenvalue $\frac{1}{2}j(j+1)$.

In the next section we'll discuss the Harish-Chandra homomorphism, and the question of how the Casimir acts not just on $V^{\mathfrak{n}^+} = H^0(\mathfrak{n}^+, V)$, but on all of the cohomology $H^*(\mathfrak{n}^+, V)$. After that, taking note that the Casimir is in some sense a Laplacian, we'll follow Dirac and introduce Clifford algebras and spinors in order to take its square root.

9 The Harish-Chandra Homomorphism

The Casimir element discussed in the last section is a distinguished quadratic element of the center $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ (note, here $\mathfrak{g}$ is a complex semi-simple Lie algebra), but there are others, all of which will act as scalars on irreducible representations. The information about an irreducible representation V contained in these scalars can be packaged as the so-called *infinitesimal character* of V, a homomorphism

$$\chi_V : Z(\mathfrak{g}) \to \mathbf{C}$$

defined by $zv = \chi_V(z)v$ for any $z \in Z(\mathfrak{g})$, $v \in V$. Just as was done for the Casimir, this can be computed by studying the action of $Z(\mathfrak{g})$ on a highest-weight vector.

Note: this is not the same thing as the usual (or global) character of a representation, which is a conjugation-invariant function on the group G with Lie algebra $\mathfrak{g}$, given by taking the trace of a matrix representation. For infinite dimensional representations V, the character is not a function on G, but a distribution Θ_V. The link between the global and infinitesimal characters is given by

$$\Theta_V(zf) = \chi_V(z)\Theta_V(f)$$

i.e. Θ_V is a conjugation-invariant eigendistribution on G, with eigenvalues for the action of $Z(\mathfrak{g})$ given by the infinitesimal character. Knowing the infinitesimal character gives differential equations for the global character.

The Poincare-Birkhoff-Witt theorem implies that for a simple complex Lie algebra $\mathfrak{g}$ one can use the decomposition (here the Cartan subalgebra is $\mathfrak{h} = \mathfrak{t}_{\mathbf{C}}$)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

to decompose $U(\mathfrak{g})$ as

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}))$$

and show that If $z \in Z(\mathfrak{g})$, then the projection of z onto the second factor is in $U(\mathfrak{g})\mathfrak{n}^+ \cap \mathfrak{n}^- U(\mathfrak{g})$. This will give zero acting on a highest-weight vector. Defining $\gamma' : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{h})$ to be the projection onto the first factor, the infinitesimal character can be computed by seeing how $\gamma'(z)$ acts on a highest-weight vector.

Remarkably, it turns out that one gets something much simpler if one composes γ' with a translation operator

$$t_\rho : U(\mathfrak{h}) \rightarrow U(\mathfrak{h})$$

corresponding to the mysterious $\rho \in \mathfrak{h}^*$, half the sum of the positive roots. To define this, note that since $\mathfrak{h}$ is commutative, $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$, the symmetric algebra on $\mathfrak{h}$, which is isomorphic to the polynomial algebra on $\mathfrak{h}^*$. Then one can define

$$t_\rho(\phi(\lambda)) = \phi(\lambda - \rho)$$

where $\phi \in \mathbf{C}[\mathfrak{h}^*]$ is a polynomial on $\mathfrak{h}^*$, and $\lambda \in \mathfrak{h}^*$.

The composition map

$$\gamma = t_\rho \circ \gamma' : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$$

is also homomorphism, known as the Harish-Chandra homomorphism. One can show that the image is invariant under the action of the Weyl group, and the map is actually an isomorphism

$$\gamma : Z(\mathfrak{g}) \rightarrow \mathbf{C}[\mathfrak{h}^*]^W$$

It turns out that the ring $\mathbf{C}[\mathfrak{h}^*]^W$ is generated by *dim* $\mathfrak{h}$ independent homogeneous polynomials. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ these are of degree $2, 3, \cdots, n$ (where the first is the Casimir).

To see how things work in the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, where there is one generator, the Casimir Ω, recall that

$$\Omega = \frac{1}{8}h^2 + \frac{1}{4}(ef + fe) = \frac{1}{8}h^2 + \frac{1}{4}(h + 2fe)$$

so one has

$$\gamma'(\Omega) = \frac{1}{4}(h + \frac{1}{2}h^2)$$

Here $t_\rho(h) = h - 1$, so

$$\gamma(\Omega) = \frac{1}{4}((h-1) + \frac{1}{2}(h-1)^2) = \frac{1}{8}(h^2 - 1)$$

which is invariant under the Weyl group action $h \to -h$.

Once one has the Harish-Chandra homomorphism γ, for each$\lambda \in \mathfrak{h}^*$ one has a homomorphism

$$\chi_\lambda : z \in Z(\mathfrak{g}) \to \chi_\lambda(z) = \gamma(z)(\lambda) \in \mathbf{C}$$

and the infinitesimal character of an irreducible representation of highest weight λ is $\chi_{\lambda+\rho}$.

9.1 The Casselman-Osborne Lemma

We have computed the infinitesimal character of a representation of highest weight λ by looking at how $Z(\mathfrak{g})$ acts on $V^{\mathfrak{n}^+} = H^0(\mathfrak{n}^+, V)$. On $V^{\mathfrak{n}^+}, z \in Z(\mathfrak{g})$ acts by

$$z \cdot v = \chi_V(z)v$$

This space has weight λ, so $U(\mathfrak{h}) = \mathbf{C}[\mathfrak{h}^*]$ acts by evaluation at λ

$$\phi \cdot v = \phi(\lambda)v$$

These two actions are related by the map $\gamma' : Z(\mathfrak{g}) \to U(\mathfrak{h})$ and we have

$$\chi_V(z) = (\gamma'(z))(\lambda) = (\gamma(z))(\lambda + \rho)$$

It turns out that one can consider the same question, but for the higher cohomology groups $H^k(\mathfrak{n}^+, V)$. Here one again has an action of $Z(\mathfrak{g})$ and an action of $U(\mathfrak{h})$. $Z(\mathfrak{g})$ acts on k-cochains $C^k(\mathfrak{n}^+, V) = Hom_{\mathbf{C}}(\Lambda^k\mathfrak{n}^+, V)$ just by acting on V, and this action commutes with d so is an action on cohomology. $U(\mathfrak{h})$ acts simultaneously on $\mathfrak{n}^+$ and on V, again in a way that descends to cohomology. The content of the Casselman-Osborne lemma is that these two actions are again related in the same way by the Harish-Chandra homomorphism. If μ is a weight for the $\mathfrak{h}$ action on $H^k(\mathfrak{n}^+, V)$, then

$$\chi_V(z) = (\gamma'(z))(\mu) = (\gamma(z))(\mu + \rho)$$

Since $\chi_V(z) = (\gamma(z))(\lambda + \rho)$, one can use this equality to show that the weights occurring in $H^k(\mathfrak{n}^+, V)$ must satisfy

$$(\mu + \rho) = w(\lambda + \rho)$$

and thus

$$\mu = w(\lambda + \rho) - \rho$$

for some element $w \in W$. Non zero elements of $H^k(\mathfrak{n}^+, V)$ can be constructed with these weights, and the Casselman-Osborne lemma used to show that these are the only possible weights. This gives the computation of $H^k(\mathfrak{n}^+, V)$ as an $\mathfrak{h}$ - module referred to earlier in these notes, which is known as Kostant's theorem (the algebraic proof was due to Kostant[5], an earlier one using geometry and sheaf cohomology was due to Bott[4]).

For more details about this and a proof of the Casselman-Osborne lemma, see Knapp's *Lie Groups, Lie Algebras and Cohomology*[3], where things are worked out for the case of $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{C})$ in chapter VI.

9.2 Generalizations

So far we have been considering the case of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and its orthogonal complement with a choice of splitting into two conjugate subalgebras, $\mathfrak{n}^+ \oplus \mathfrak{n}^-$. Equivalently, we have a choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. At the group level, this corresponds to a choice of Borel subgroup $B \subset G$, with the space G/B a complex projective variety known as a flag manifold. More generally, much of the same structure appears if we choose larger subgroups $P \subset G$ containing B such that G/P is a complex projective variety of lower dimension. In these cases *Lie* $P = \mathfrak{l} \oplus \mathfrak{u}^+$, with $\mathfrak{l}$ (the Levi subalgebra) a reductive algebra playing the role of the Cartan subalgebra, and $\mathfrak{u}^+$ playing the role of $\mathfrak{n}^+$.

In this more general setting, there is a generalization of the Harish-Chandra homomorphism, now taking $Z(\mathfrak{g})$ to $Z(\mathfrak{l})$. This acts on the cohomology groups $H^k(\mathfrak{u}^+, V)$, with a generalization of the Casselman-Osborne lemma determining what representations of $\mathfrak{l}$ occur in this cohomology. The Dirac cohomology formalism to be discussed later generalizes this even more, to cases of a reductive subalgebra $\mathfrak{r}$ with orthogonal complement that cannot be given a complex structure and split into conjugate subalgebras. It also provides a compelling explanation for the continual appearance of ρ, as the highest weight of the spin representation.

10 Clifford Algebras

Clifford algebras are well-known to physicists, in the guise of matrix algebras generated by the γ -matrices first used in the Dirac equation. They also have a more abstract formulation, which will be the topic of this posting. One way to think about Clifford algebras is as a "quantization" of the exterior algebra, associated with a symmetric bilinear form.

Given a vector space V with a symmetric bilinear form $(\cdot,\cdot)$, the associated Clifford algebra $Cliff(V,(\cdot,\cdot))$ can be defined by starting with the tensor algebra $T^*(V)$ ($T^k(V)$ is the k-th tensor power of V), and imposing the relations

$$v\otimes w + w\otimes v = -2(v,w)1$$

where $v,w \in V = T^1(V),\ 1\in T^0(V)$. Note that many authors use a plus instead of a minus sign in this relation. The case of most interest in physics is $V=\mathbf{R}^4, (\cdot,\cdot)$ the Minkowski inner product of signature (3,1). The theory of Clifford algebras for real vector spaces V is rather complicated. Here we'll stick to complex vector spaces V, where the theory is much simpler, partially because over $\mathbf{C}$ there is, up to equivalence, only one non-degenerate symmetric bilinear form. We will suppress mention of the bilinear form in the notation, writing $Cliff(V)$ for $Cliff(V,(\cdot,\cdot))$.

For a more concrete definition, one can choose an orthonormal basis e_i of V. Then $Cliff(V)$ is the algebra generated by the e_i, with multiplication satisfying the relations

$$e_i^2=-1,\quad e_ie_j=-e_je_i\quad (i\neq j)$$

One can show that these complex Clifford algebras are isomorphic to matrix algebras, more precisely

$$Cliff(\mathbf{C}^{2n})\simeq M(\mathbf{C},2^n),\quad Cliff(\mathbf{C}^{2n+1})\simeq M(\mathbf{C},2^n)\oplus M(\mathbf{C},2^n)$$

10.1 Clifford Algebras and Exterior Algebras

The exterior algebra $\Lambda^*(V)$ is the algebra of anti-symmetric tensors, with product the wedge product $\wedge$. This is also exactly what one gets if one takes the Clifford algebra $Cliff(V)$, with zero bilinear form. Multiplying a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ by a parameter t gives for non-zero t a Clifford algebra $Cliff(V,t(\cdot,\cdot))$ that can be thought of as a deformation of the exterior algebra $\Lambda^*(V)$. Thinking of the exterior algebra on V of dimension n as the algebra of functions on n anticommuting coordinates, the Clifford algebra can be thought of as a "quantization" of this, taking functions (elements of $\Lambda^*(V)$) to operators (elements of $Cliff(V)$, matrices in this case).

While $\Lambda^*(V)$ is a $\mathbf{Z}$ graded algebra, $Cliff(V)=Cliff^{even}(V)\oplus Cliff^{odd}(V)$ is only $\mathbf{Z}_2$-graded, since the Clifford product does not preserve degree but can change it by two when multiplying generators. The Clifford algebra is filtered by a $\mathbf{Z}$ degree, taking $F_p(Cliff(V))\subset Cliff(V)$ to be the subspace of elements that can be written as sums of $\leq p$ generators. The exterior algebra is naturally isomorphic to the associated graded algebra for this filtration

$$\Lambda^p(V)\simeq F_p(Cliff(V))/F_{p-1}(Cliff(V))$$

and $\Lambda^*(V)$ and $Cliff(V)$ are isomorphic as vector spaces. One choice of such an isomorphism is given by composing the skew-symmetrization map

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p = \frac{1}{p!}\sum_{s\in S_p} sgn(s) v_{s(1)} \otimes v_{s(2)} \otimes \cdots \otimes v_{s(p)}$$

with the projection $T^*(V) \to Cliff(V)$.

Denoting this map by q, it is sometimes called the "quantization map". Using an orthonormal basis e_i, q acts as

$$q(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = e_{i_1} e_{i_2} \cdots e_{i_p}$$

The inverse $\sigma = q^{-1} : Cliff(V) \to \Lambda^*(V)$ is sometime called the "symbol map".

This identification as vector spaces is known as the "Chevalley identification". Using it, one can think of the Clifford algebra as just an exterior algebra with a different product.

10.2 Clifford Modules and Spinors

Given a Clifford algebra, one would like to classify the modules over such an algebra, the Clifford modules. Such a module is given by a vector space M and an algebra homomorphism

$$\pi : Cliff(V) \to End(M)$$

To specify π, we just need to know it on generators, and see that it satisfies

$$\pi(v)\pi(w) + \pi(w)\pi(v) = -2(v,w)Id$$

One such Clifford module is $M = \Lambda^* V$, with

$$\pi(v)\omega = v \wedge \omega - i_v \omega$$

where i_v is contraction by v. This gives the inverse to the quantization map (the symbol map σ) as

$$\sigma : a \in Cliff(V) \to \pi(a)1 \in \Lambda^*(V)$$

$\Lambda^*(V)$ is not an irreducible Clifford module, and we would like to decompose it into irreducibles. For $dim_{\mathbf{C}} V = 2n$ even, there will be a single such irreducible S, of dimension 2^n, and the module map $\pi : Cliff(V) \to End(S)$ is an isomorphism. In the rest of this posting we'll stick to the this case, for the odd dimensional case see the references mentioned at the end.

To pick out an irreducible module $S \subset \Lambda^*(V)$, one can begin by choosing a linear map $J : V \to V$ such that $J^2 = -1$ and J is orthogonal ($(Jv, Jw) = (v, w)$). Then let $W_J \subset V$ be the subspace on which J acts by $+i$, $\overline{W}_J$ be the subspace on which J acts by $-i$. Note that V is a complex vector space, and now has two linear maps on it that square to -1, multiplication by i, and multiplication by J. W_J is an isotropic subspace of V, since

$$(v_1, v_2) = (Jv_1, Jv_2) = (iv_1, iv_2) = -(v_1, v_2)$$

for any $v_1, v_2 \in W_J$. We now have a decomposition $V = W_j \oplus \overline{W}_J$ into two isotropic subspaces. Since the bilinear form is zero on these subspaces, we get two subalgebras of the Clifford algebra, $\Lambda^*(W_J)$ and $\Lambda^*(\overline{W_J})$. It turns out that one can choose $S \simeq \Lambda^*(W_J)$.

One can make this construction very explicit by picking a particular J, for instance the one that acts on the element of an orthonormal basis by $Je_{2j-1} = e_{2j}$, $Je_{2j} = -e_{2j-1}$ for $j = 1, \cdots n$. Letting $w_j = e_{2j-1} + ie_{2j}$ we get a basis of W_J. To get an explicit representation of S as a $Cliff(V)$ module isomorphic to $\Lambda^*(\mathbf{C}^n)$, we will use the formalism of fermionic annihilation and creation operators. These are the operators on an exterior algebra one gets from wedging by or contracting by an orthonormal vector, operators a_i^+ and a_i for $i = 1, \cdots, n$ satisfying

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0$$

$$\{a_i, a_j^+\} = \delta_{ij}$$

In terms of these operators on $\Lambda^*(\mathbf{C}^n)$, $Cliff(n)$ acts by

$$e_{2j-1} = a_j^+ - a_j$$

$$e_{2j} = -i(a_j^+ + a_j)$$

10.3 The Spin Representation

The group that preserves $(\cdot, \cdot)$ is $O(n, \mathbf{C})$, and its connected component of the identity $SO(n, \mathbf{C})$ has compact real form $SO(n)$. $SO(n)$ has a non-trivial double cover, the group $Spin(n)$. One can construct $Spin(n)$ explicitly as invertible elements in $Cliff(V)$ for $V = \mathbf{R}^n$, and its Lie algebra using quadratic elements of $Cliff(V)$, with the Lie bracket given by the commutator in the Clifford algebra.

For the even case, a basis for the Cartan subalgebra of *Lie Spin*$(2n)$ is given by the elements

$$\frac{1}{2}e_{2j-1}e_{2j}$$

These act on the spinor module $S \simeq \Lambda^*(\mathbf{C}^n)$ as

$$\frac{1}{2}e_{2j-1}e_{2j} = -i\frac{1}{2}(a_j^+ - a_j)(a_j^+ + a_j) = i\frac{1}{2}[a_j, a_j^+]$$

with eigenvalues $(\pm\frac{1}{2}, \cdots, \pm\frac{1}{2})$. S is not irreducible as a representation of $Spin(2n)$, but decomposes as $S = S^+ \oplus S^-$ into two irreducible half-spin representations, corresponding to the even and odd degree elements of $\Lambda^*(\mathbf{C}^n)$.

With a standard choice of positive roots, the highest weight of S^+ is

$$(+\frac{1}{2}, +\frac{1}{2} \cdots, +\frac{1}{2}, +\frac{1}{2})$$

and that of S^- is

$$(+\frac{1}{2}, +\frac{1}{2} \cdots, +\frac{1}{2}, -\frac{1}{2})$$

Note that the spinor representation is not a representation of $SO(2n)$, just of $Spin(2n)$. However, if one restricts to the $U(n) \subset SO(2n)$ preserving J, then the $\Lambda^*(W_J)$ are the fundamental representations of this $U(n)$. These representations have weights that are 0 or 1, shifted by $+\frac{1}{2}$ from those of the spin representation. One can't restrict from $Spin(2n)$ to $U(n)$, but one can restrict to $\tilde{U}(n)$, a double cover of $U(n)$. On this double cover the notion of $\Lambda^n(\mathbf{C}^n)^{\frac{1}{2}}$ makes sense and one has, as $\tilde{U}(n)$ representations

$$S \otimes \Lambda^n(\mathbf{C}^n)^{\frac{1}{2}} \simeq \Lambda^*(\mathbf{C}^n)$$

So, projectively, the spin representation is just $\Lambda^*(\mathbf{C}^n)$, but the projective factor is a crucial part of the story.

The above has been a rather quick sketch of a long story. For more details, a good reference is the book *Spin Geometry*[7] by Lawson and Michelsohn. Chapter 12 of Segal and Pressley's *Loop Groups*[8] contains a very geometric version of the above material, in a form suitable for generalization to infinite dimensions. My notes for my graduate class also have a bit more detail, see http://www.math.columbia.edu/~woit/notes19.pdf.

11 Clifford Algebras and Lie Algebras

When a Lie group with Lie algebra $\mathfrak{g}$ acts on a manifold M, one gets two sorts of actions of $\mathfrak{g}$ on the differential forms $\Omega^*(M)$. For each $X \in \mathfrak{g}$ one has operators:

- $\mathcal{L}_X : \Omega^k(M) \rightarrow \Omega^k(M)$, the Lie derivative along the vector field on M corresponding to X.
- $i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$, contraction by the vector field on M corresponding to X.

These operators satisfy the relation $di_X + i_X d = \mathcal{L}_X$ where d is the de Rham differential $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$, and the operators $d, i_X, \mathcal{L}_X$ are (super)-derivations. In general, an algebra carrying an action by operators satisfying the same relations satisfied by $d, i_X, \mathcal{L}_X$ will be called a $\mathfrak{g}$-differential algebra. It will turn out that the Clifford algebra $Cliff(\mathfrak{g})$ of a semi-simple Lie algebra $\mathfrak{g}$ carries not just the Clifford algebra structure, but the additional structure of a $\mathfrak{g}$-differential algebra, in this case with $\mathbf{Z}_2$, not $\mathbf{Z}$ grading.

Note that in this section the commutator symbol will be the supercommutator in the Clifford algebra (commutator or anti-commutator, depending on the $\mathbf{Z}_2$ grading). When the Lie bracket is needed, it will be denoted $[\cdot, \cdot]_{\mathfrak{g}}$.

To get a $\mathfrak{g}$-differential algebra on $Cliff(\mathfrak{g})$ we need to construct super-derivations i_X^{Cl}, $\mathcal{L}_X^{Cl}$, and d^{Cl} satisfying the appropriate relations. For the first of these we don't need the fact that this is the Clifford algebra of a Lie algebra, and can just define

$$i_X^{Cl}(\cdot) = [-\frac{1}{2}X, \cdot]$$

For $\mathcal{L}_X^{Cl}$, we need to use the fact that since the adjoint representation preserves the inner product, it gives a homomorphism

$$\widetilde{ad} : \mathfrak{g} \to \mathfrak{spin}(\mathfrak{g})$$

where $\mathfrak{spin}(\mathfrak{g})$ is the Lie algebra of the group $Spin(\mathfrak{g})$ (the spin group for the inner product space $\mathfrak{g}$), which can be identified with quadratic elements of $Cliff(\mathfrak{g})$, taking the commutator as Lie bracket. Explicitly, if X_a is a basis of $\mathfrak{g}$, X_a^* the dual basis, then

$$\widetilde{ad}(X) = \frac{1}{4}\sum_a X_a^*[X, X_a]_{\mathfrak{g}}$$

and we get operators acting on $Cliff(\mathfrak{g})$

$$\mathcal{L}_X^{Cl}(\cdot) = [\widetilde{ad}(X), \cdot]$$

Remarkably, an appropriate d^{Cl} can be constructed using a cubic element of $Cliff(\mathfrak{g})$. Let

$$\gamma = \frac{1}{24}\sum_{a,b} X_a^* X_b^*[X_a, X_b]_{\mathfrak{g}}$$

then

$$d^{Cl}(\cdot) = [\gamma, \cdot]$$

$d^{Cl} \circ d^{Cl} = 0$ since γ^2 is a scalar which can be computed to be $-\frac{1}{48}tr\Omega_{\mathfrak{g}}$, where $\Omega_{\mathfrak{g}}$ is the Casimir operator in the adjoint representation.

The above constructions give $Cliff(\mathfrak{g})$ the structure of a filtered $\mathfrak{g}$-differential algebra, with associated graded algebra $\Lambda^*(\mathfrak{g})$. This gives $\Lambda^*(\mathfrak{g})$ the structure of a $\mathfrak{g}$-differential algebra, with operators $i_X, \mathcal{L}_X, d$. The cohomology of this differential algebra is just the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbf{C})$.

$Cliff(\mathfrak{g})$ can be thought of as an algebra of operators corresponding to the quantization of an anti-commuting phase space $\mathfrak{g}$. Classical observables are anti-commuting functions, elements of $\Lambda^*(\mathfrak{g}^*)$. Corresponding to $i_X, \mathcal{L}_X, d$ one has both elements of $\Lambda^*(\mathfrak{g}^*)$ and their quantizations, the operators in $Cliff(\mathfrak{g})$ constructed above.

For more details about the above, see [9],[10],[11], and [12]

12 Equivariant Cohomology and the Weil Algebra

13 The Quantum Weil Algebra and the Kostant Dirac Operator

14 Dirac Cohomology

15 Semi-infinite Cohomology

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