

TOPICS IN REPRESENTATION THEORY: THE KILLING FORM, REFLECTIONS AND CLASSIFICATION OF ROOT SYSTEMS

1 Roots and the Killing Form

So far we have just used the combinatorial structure coming from the roots and the action of the Weyl group. To go further with this theory, in particular to define explicitly the reflections that generate the Weyl group, we need to fix an inner product on \mathfrak{t}^* . This can be done using the Killing form, and since this provides an adjoint-invariant inner product, it will turn out that the Weyl group acts by isometries.

Recall that we defined the Killing form on \mathfrak{g} as

$$K(X, Y) = \text{Tr}(ad(X) \circ ad(Y))$$

and it is invariant under the adjoint action of G on \mathfrak{g} . If \mathfrak{g} has an abelian ideal, then there will be directions on which ad is zero and the bilinear form will be degenerate. One can show that for semi-simple Lie algebras the Killing form is non-degenerate.

For semi-simple Lie groups the Killing form can be used to define an inner product which in general may not be positive definite. Non-compact groups such as $SL(2, \mathbf{R})$ have an indefinite Killing form. For compact groups the Killing form will be negative definite since

Theorem 1. *For \mathfrak{g} the Lie algebra of a compact semisimple Lie group, the Killing form K is strictly negative definite on \mathfrak{g} , i.e. for any $X \neq 0$, $K(X, X) < 0$.*

Proof:

If \mathfrak{g} is the Lie algebra of a semi-simple compact Lie group G , one can choose a G -invariant inner product on (\cdot, \cdot) on \mathfrak{g} by averaging an arbitrary one. Then Ad is an orthogonal representation and elements $ad(X)$ of its Lie algebra are skew-symmetric ($ad(X)^t = -ad(X)$). So

$$K(X, X) = \text{Tr}(ad(X)ad(X)) = -\text{Tr}((ad(X))^t ad(X))$$

But for any real matrix

$$\text{Tr}(A^t A) = \sum_{i,j} (a_{ij})^2 > 0$$

for $A \neq 0$.

The Killing form K can be extended to a bilinear form on $\mathfrak{g}_{\mathbf{C}}$. Recall that the real elements of \mathfrak{g} sit inside $\mathfrak{g}_{\mathbf{C}}$ as the purely imaginary elements. Thus K

will be positive definite on these elements and we can define the inner product we want to use as

$$\langle \cdot, \cdot \rangle = K(\cdot, \cdot)$$

This inner product will be Ad-invariant and positive-definite for non-zero real elements of $\mathfrak{g}_{\mathbf{C}}$. If we restrict it to $\mathfrak{t}_{\mathbf{C}}$ and use the fact that for $H \in \mathfrak{t}_{\mathbf{C}}$, $ad(H)$ is diagonal with eigenvalues given by the roots $\alpha_i(H)$ we see that

$$\langle H, H \rangle = \sum_{i \in R} (\alpha_i(H))^2$$

where R is the set of roots, containing both the positive and negative ones. Note that these are now the complex roots, and that we will be interested in $H \in i\mathfrak{t} \subset \mathfrak{t}_{\mathbf{C}}$, and on these the inner product is positive.

For the example of $G = SU(n)$, recall that $\mathfrak{t}_{\mathbf{C}}$ consists of the diagonal matrices of trace 0, and the roots are

$$\alpha_{ij}(H_\lambda) = \lambda_i - \lambda_j$$

for $i \neq j$, so

$$\begin{aligned} \langle H_\lambda, H_\lambda \rangle &= \sum_{i \neq j} (\lambda_i - \lambda_j)^2 \\ &= 2(n-1) \sum_i \lambda_i^2 + 2 \sum_{i \neq j} \lambda_i \lambda_j \\ &= (2(n-1) + 2) \sum_i \lambda_i^2 + 2 \left(\sum_i \lambda_i \right)^2 \\ &= 2n \sum_i \lambda_i^2 \end{aligned}$$

We see that the Killing form gives a positive definite inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{t} , and this can be used to identify \mathfrak{t} and \mathfrak{t}^* in the standard way

$$v \in \mathfrak{t} \rightarrow \langle v, \cdot \rangle \in \mathfrak{t}^*$$

With this identification, we can also interpret the inner product as an inner product on \mathfrak{t}^* .

2 Weyl Group Elements as Reflections

We have seen that elements of the Weyl group $W(G, T)$ act on the roots by permuting them. It turns out that associated to each root α there is a distinguished element $s_\alpha \in W(G, T)$. s_α acts on T leaving invariant the space U_α , and on the Lie algebra \mathfrak{t} leaving invariant the diagram of the group. Since Weyl group elements come from elements in $N(T)$ acting by conjugation, the action is an isometry when we use the Ad-invariant inner product defined using the Killing form.

Using the inner product to identify \mathfrak{t} and \mathfrak{t}^* we will study the action of the s_α as reflection transformations in \mathfrak{t}^* , where s_α is a reflection in the hyperplane orthogonal to the root α and takes the root α to the root $-\alpha$. For a proof that these reflection maps are actually elements of the Weyl group, see [3] Theorem VIII.8.1. We need this theorem to know that these reflection maps always take roots to roots. The theorem tells us also that the element of $N(T)$ that gives the reflection s_α is

$$\exp\left(\frac{\pi}{2}(X_\alpha - X_{-\alpha})\right)$$

where X_α is an element of \mathfrak{g}_α , and $X_{-\alpha}$ a conjugate element in $\mathfrak{g}_{-\alpha}$.

If we want to reflect an element $x \in \mathfrak{t}^*$ in the hyperplane perpendicular to α , the formula for the reflection map is

$$s_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

Since the Weyl group acts on roots by permuting them, if β and α are simple roots, then

$$s_\alpha(\beta) = \beta - 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

is a root and must be expressible as a linear combination of α and β with integer coefficients. So

$$2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

must be an integer.

Defining

Definition 1 (Cartan Matrix). For a rank r group, the r by r matrix A of integers

$$n_{\alpha\beta} = 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is called the Cartan matrix and its entries are called the Cartan numbers.

Note that the diagonal entries $n_{\alpha\alpha}$ of the Cartan matrix must be equal to 2.

This integrality condition puts very tight constraints on the relative configurations of two roots. Using the formula for the inner product of two vectors in terms of their lengths and the angle between them

$$n_{\alpha\beta} = 2 \frac{\sqrt{\langle \alpha, \alpha \rangle} \sqrt{\langle \beta, \beta \rangle} \cos(\theta_{\alpha\beta})}{\langle \alpha, \alpha \rangle}$$

where $\theta_{\alpha\beta}$ is the angle between the roots α and β , we have

$$n_{\alpha\beta} n_{\beta\alpha} = 4 \cos^2(\theta_{\alpha\beta})$$

and since this must be integral it can only take on the values 0, 1, 2, 3, 4. The case 4 correspond to collinear roots, and there are a total of 7 possible choices of ways to satisfy this integrality condition for two non-collinear roots.

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\theta_{\alpha\beta}$	$\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$
0	0	$\pi/2$	
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

3 An Example: $SU(3)$

We will work out in detail what happens for perhaps the simplest non-trivial case, that of $G = SU(3)$. Here the rank is two and \mathfrak{t}^* is spanned by two simple roots

$$\alpha_{12}(H_\lambda) = \lambda_1 - \lambda_2 \quad \text{and} \quad \alpha_{23}(H_\lambda) = \lambda_2 - \lambda_3$$

The reflection $s_{\alpha_{12}}$ is just the Weyl group transformation interchanging λ_1 and λ_2 . so

$$s_{\alpha_{12}}(\alpha_{23}) = \alpha_{13} = \alpha_{12} + \alpha_{23}$$

and the reflection $s_{\alpha_{23}}$ similarly interchanges λ_2 and λ_3 , so

$$s_{\alpha_{23}}(\alpha_{12}) = \alpha_{13} = \alpha_{12} + \alpha_{23}$$

This shows that

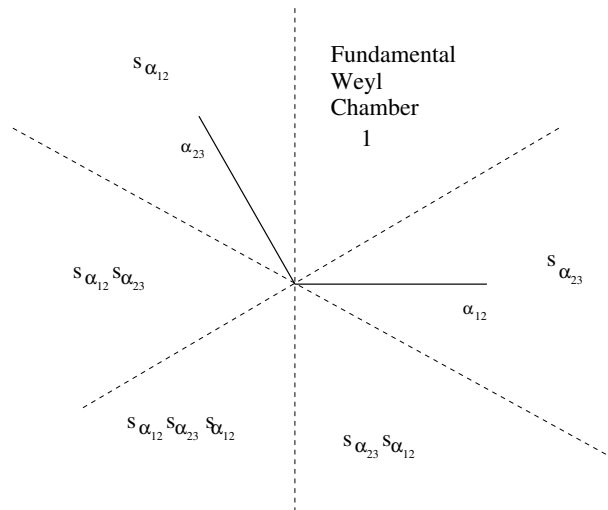
$$n_{\alpha_{12}\alpha_{23}} = n_{\alpha_{23}\alpha_{12}} = -1$$

which implies that the Cartan matrix for $SU(3)$ is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the angle between α_{12} and α_{23} is $2\pi/3$.

We can thus draw the configuration of the simple roots as follows



Configuration of Simple Roots for $SU(3)$

Note that the two reflection maps $s_{\alpha_{12}}$ and $s_{\alpha_{23}}$ generate the entire Weyl group S_3 of six elements as follows

$$1, s_{\alpha_{12}}, s_{\alpha_{23}}, s_{\alpha_{23}}s_{\alpha_{12}}, s_{\alpha_{12}}s_{\alpha_{23}}, s_{\alpha_{12}}s_{\alpha_{23}}s_{\alpha_{12}}$$

The six Weyl chambers can be labelled by the sequence of reflections necessary to take the fundamental Weyl chamber to the given Weyl chamber.

Diagram 2: Weyl Chambers and Reflections for $SU(3)$

4 Dynkin Diagrams and the Classification of Semi-simple Lie Algebra

The system of roots for a simple complex Lie algebra can be generated from the simple roots by acting with the reflections that generate the Weyl group. The possible configurations of systems of simple roots are limited by the constraints derived in the previous section. One can construct a list of all possible systems of simple roots, how this is done is outlined in [3] and in [2] (see chapter 2 of the article by Carter). We won't go through this derivation, but the end result is that one can associate to each possible system of simple root a diagram and then classify all possible diagrams.

Definition 2 (Dynkin Diagram). *The Dynkin diagram of a system of simple roots is a graph with one node for each simple root α . The nodes corresponding to two different simple roots α and β are joined together by $n_{\alpha\beta}n_{\beta\alpha}$ bonds. If α and β are of different length, the bond between them contains an arrow pointing to the longer one.*

The Dynkin diagram for $SU(3)$ contains two nodes, connected by a single bond. The full classification theorem shows that there are four infinite families of Dynkin diagrams that correspond to the families of classical groups $SU(n)$, $SO(2n)$, $SO(2n + 1)$, $Sp(n)$, and five exceptional cases called G_2 , F_4 , E_6 , E_7 , E_8 , where the subscript is the rank of the group. The group G_2 can be interpreted as the automorphism group of the octonions, the others have no simple geometrical interpretation.

For more details on the exceptional groups, see [1]. For more details on the root systems of the classical groups, see [3]. We have already seen a little bit about $SU(n)$ and will study the orthogonal groups $SO(n)$ in detail later on in the course using Clifford algebras and the spin double covers $Spin(n)$.

References

- [1] Adams, J. Frank, *Lectures on Exceptional Lie Groups*, University of Chicago Press, 1996.
- [2] Carter, R., Segal, G., and MacDonal, I., *Lectures on Lie Groups and Lie Algebras* Cambridge University Press, 1995.
- [3] Simon, B., *Representations of Finite and Compact Groups*, American Mathematical Society, 1996.