Topics in Representation Theory: $SU(n)$, Weyl Chambers and the Diagram of a Group

1 Another Example: $G = SU(n)$

Last time we began analyzing how the maximal torus T of G acts on the adjoint representation, defining the roots of G as the non-trivial irreducibles that occur in this representation. The analysis proceeds by complexifying the Lie algebra, after which the adjoint actions of all $H \in \mathfrak{t}_{\mathbf{C}}$ are diagonalizable, with eigenvalues the complex roots and eigenvectors the root spaces. In general we have the commutation relations

$$[H, E_{\alpha_i}] = \alpha_i(H)E_{\alpha_i}$$

Here $H \in \mathfrak{t}_{\mathbf{C}}$, the complex root is a complex linear map

$$\alpha_i : \mathfrak{t}_{\mathbf{C}} \to \mathbf{C}$$

and

$$E_{\alpha_i} \subset \mathfrak{g}_{\mathbf{C}}$$

is one of the root spaces.

In the case $G = SU(2), \mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(2, \mathbf{C})$, we can take elements of the Cartan subalgebra $\mathfrak{t}_{\mathbf{C}}$ to be 2 by 2 diagonal matrices of the form

$$H_\lambda = \lambda\sigma_3 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

and the two root spaces consist of matrices with entries proportional to

$$X^+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with root

$$\alpha(H_\lambda) = 2\lambda$$

and

$$X^- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with root $-\alpha$.

The theory of roots and root spaces is one part of representation theory where the $SU(2)$ example is too trivial to be useful since here the rank is one, and there is only one root (and its conjugate). We need to begin looking at higher rank examples, and will begin with the case of $G = SU(n), \mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n, \mathbf{C})$. Note that while some formulas are simpler for the case of $G = U(n), \mathfrak{g}_{\mathbf{C}} = \mathfrak{gl}(n, \mathbf{C})$, this is not a semi-simple group and one needs to analyze its normal $U(1)$ subgroups separately. For this case T can be chosen to be the diagonal matrices D with entries

$$D_{jj} = e^{i\theta_j}, \quad \prod_{j=1,n} e^{i\theta_j} = 1$$

and the corresponding Cartan subalgebra consists of diagonal matrices H_λ with entries

$$(H_\lambda)_{jj} = \lambda_j, \quad \sum_{j=1}^{n} \lambda_j = 0$$

Here the subscript λ now refers to the entire set of n numbers λ_i. The root spaces are labelled by pairs $j, k, j \neq k$ of integers from 1 to n and correspond to matrices proportional to the matrix E_{jk} all of whose entries are 0 except for the j, k'th which is 1. Since

$$[H_\lambda, E_{jk}] = (\lambda_j - \lambda_k)E_{jk}$$

the complex roots are given by

$$\alpha_{jk}(H_\lambda) = \lambda_j - \lambda_k$$

The Weyl group in this case is the group S^n, acting on T by permuting the D_{jj}.

Note that roots come in pairs of opposite sign. Making a choice of one of each pair is a choice of which roots are "positive". One could for example choose in this example positive roots to be the

$$\alpha_{jk}, \quad j < k$$

but note that this choice is not invariant under the Weyl group. The Weyl group acts on the set of roots, but does not respect the decomposition into positive and negative roots.

A specific example to continually keep in mind, one where we can easily draw things, is the case of $G = SU(3)$. Here the rank is two, and G/T is a complex manifold of six real dimensions or three complex dimensions. There are a total of six roots

$$\alpha_{12}(H_\lambda) = \lambda_1 - \lambda_2$$
$$\alpha_{23}(H_\lambda) = \lambda_2 - \lambda_3$$
$$\alpha_{13}(H_\lambda) = \lambda_1 - \lambda_3$$

and their negatives. The three positive roots are not independent, they satisfy the relation

$$\alpha_{13} = \alpha_{12} + \alpha_{23}$$

We will draw these as vectors in a plane that can be identified with

$$i\mathfrak{t} \subset \mathfrak{t}_{\mathbf{C}}$$

Wel'll see a little later that the Killing form gives a natural inner product on this space, but for now we'll avoid using this, just dealing with the parts of the theory of roots that don't actually need the inner product.

2 Weyl Chambers and the Diagram of a Group

Recall that we defined the Weyl group $W(G, T)$ to be $N(T)/T$, this is a group of automorphisms of T that come from conjugation by an element of G.

We saw in our example that the Weyl group S^n of $SU(n)$ permutes its roots. In general the Weyl group acts on the set of roots since a root θ_α is a homomorphism

$$\theta_\alpha : T \to GL(\mathfrak{g})$$

and an element $w \in W(G, T)$ corresponds to a homomorphism

$$w : t \in T \to xtx^{-1} \in T$$

for some $x \in G$. Composing these two maps gives a new root $w(\alpha)$.

Elements $t \in T$ for which the group

$$N(t) = \{g \in G : gtg^{-1} = t\}$$

has dimension greater than $dim\ T$ are said to be singular, those with dimension equal to $dim\ T$ are said to be regular. In our example of $G = SU(n)$, the regular elements are those where all the diagonal elements in T are distinct, the singular ones are those where two or more diagonal elements are identical. To understand the structure of the set of singular points in T, for each root θ_α define

Definition 1. *For each root θ_α, there is a codimension one subgroup of T*

$$U_\alpha = ker(\theta_\alpha) = ker(\theta_{-\alpha})$$

The point $t \in U_\alpha$ is a singular points of T, since the root space $\mathfrak{g}_\alpha$ generates a one-parameter subgroup of G, not in T, of elements that commute with t, so the the dimension of $N(T)$ is greater than the dimension of T. Note that the intersection of all the U_α consists of elements that commute with all of G, the center $Z(G)$ of G.

The Stiefel diagram of a group is the set consisting of all $exp^{-1}U_\alpha$ and will consist of infinite sets of parallel hyperplanes in $\mathfrak{t}$.

If we pass to the Lie algebra (now α will be thought of as elements of $\mathfrak{t}^*$) and just look at those that go through zero we can define

Definition 2 (Diagram of G). *The infinitesimal diagram or diagram of G consists of the hyperplanes*

$$Lie(U_\alpha) = ker\ \alpha \subset \mathfrak{t}$$

i.e. those elements $v \in \mathfrak{t}$ such that $\alpha(v) = 0$.

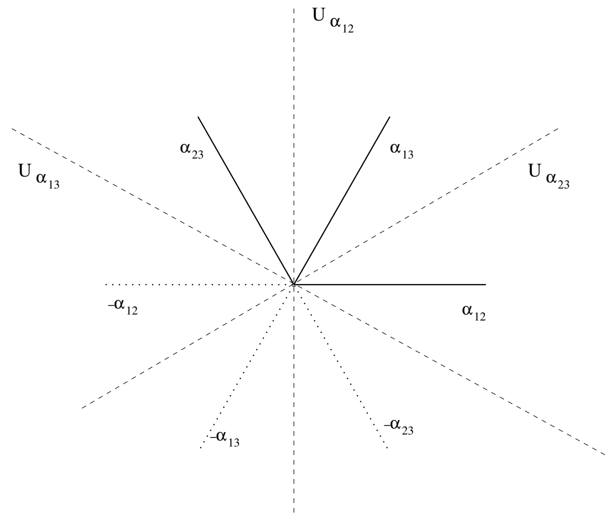


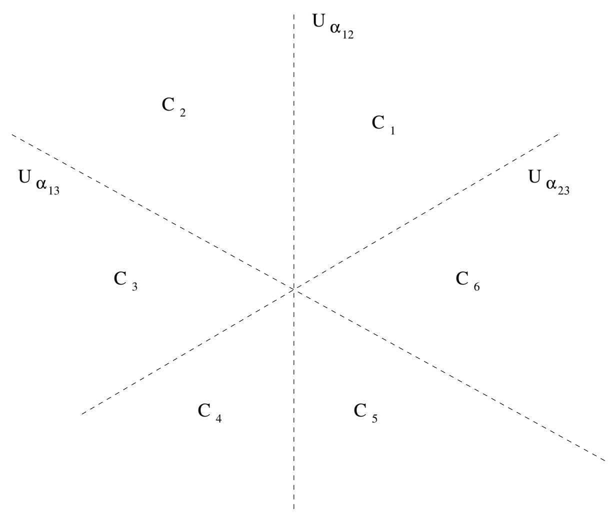
Diagram for $SU(3)$

The hyperplanes of the diagram divide $\mathfrak{t}$ into finitely many convex regions

Definition 3 (Weyl Chambers). *Given a choice of positive roots, a Weyl chamber is a set of the form*

$$\{v \in \mathfrak{t} : \epsilon_i \alpha_i(v) > 0\}$$

for all positive roots α_i *and some choice of signs* $\epsilon_i = \pm 1$.



Weyl Chambers for $SU(3)$

The roots are not independent, we can express some in terms of others. A set of roots that form a basis are called simple roots

Definition 4 (Simple Roots). *A subset S of the set R of roots is called a system of simple roots if the roots in S are linearly independent and every root $\beta \in R$ can be written as*

$$\beta = \sum_{\alpha \in S} m_\alpha \cdot \alpha$$

with m_α integers all ≥ 0 or ≤ 0.

In the case of $SU(3)$, a set of simple roots will have two elements. One possible choice of the set S of simple roots is

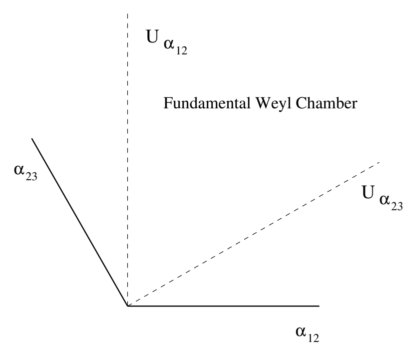
$$S = \{\alpha_{12}, \alpha_{23}\}$$

A choice of a set S of simple roots is equivalent to a choice of positive roots. Positive roots are those where all m_α are positive, negative roots ones where all m_α are negative. Corresponding to each choice of S there is a distinguished Weyl chamber

Definition 5. *The fundamental Weyl chamber corresponding to a set S of simple roots is*

$$K(S) = \{v \in \mathfrak{t} : \alpha_i(v) > 0\}$$

for all α_i in S.



Simple Roots and Fundamental Weyl Chamber for $G = SU(3)$

We will not go into much more detail about the general theory of the Weyl group and how it acts on the diagram of G, referring to extensive discussions in the texts [1],[2],[3]. but there are two basic facts that will be of importance.

1. The Weyl group acts simply transitively on the set possible choices of S.
2. The Weyl group is generated by reflections in the hyperplanes corresponding to simple roots in S.

References

[1] Adams, J. Frank, *Lectures on Lie Groups,* University of Chicago Press, 1969.

[2] Bröcker, T. and tom Dieck, T.,*Representations of Compact Lie Groups,* Springer-Verlag, 1985.

[3] Simon, B., *Representations of Finite and Compact Groups,*American Mathematical Society, 1996.