Topics in Representation Theory: Roots and Weights

1 The Representation Ring

Last time we defined the maximal torus T and Weyl group $W(G,T)$ for a compact, connected Lie group G and explained that our goal is to relate the representation theory of T to that of G. One aspect of the representation theory of T and of G for which there is a simple relation is that of their representation rings.

Consider representations π of a compact Lie group G on complex vector spaces and recall that a representation is characterized up to isomorphism by its character

$$\chi_\pi : g \in G \rightarrow Tr(\pi(g)) \in \mathbf{C}$$

and that character functions are conjugation invariant

$$\chi(hgh^{-1}) = \chi(g)$$

Since

$$\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$$

and

$$\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$$

the characters generate an interesting ring of functions on G. One can define

Definition 1 (Character Ring). *Let $R(G)$ be the free abelian group generated by characters of complex representations of G. This is called the character ring of G.*

The ring structure comes from the sum and product of representations. Very explicitly one can think of this ring as what one gets by adding, subtracting and multiplying character functions of irreducible representations. Elements of $R(G)$ are sometimes called virtual characters since, given any two representations π_1 and π_2, one has not only corresponding elements $[\pi_1]$ and $[\pi_2]$ in $R(G)$ corresponding to their characters, but also differences.

$$[\pi_1] - [\pi_2] \in R(G)$$

This kind of construction which starts with a semigroup (the irreducible representations with direct sum) and produces a group ($R(G)$) which also happens to be a ring appears in other contexts in mathematics. The simplest example is the construction of the ring of integers from the semigroup of natural numbers. A more sophisticated version of this is topological K-theory, which is the ring you get starting from the semigroup of isomorphism classes of vector bundles on a given topological space.

The Peter-Weyl theorem implies that for compact G any conjugation invariant function can by uniformly approximated by linear combinations of character functions.

Since we have seen that any element $g \in G$ can be conjugated into an element of T, a conjugation invariant function such as χ_π is determined by its values on T. Associated to the inclusion map

$$i : T \rightarrow G$$

there is a map on character rings induced by restriction of characters to T

$$i^* : R(G) \rightarrow R(T)$$

This map is injective and since the Weyl group $W(G,T)$ acts on T by conjugation, the image is in

$$R(T)^{W(G,T)} \subset R(T)$$

the $W(G,T)$ invariant character functions on T.

2 Weights

All the irreducible complex representations of T are one-dimensional. When we restrict an n-dimensional representation π of G to T, it will break up into n one-dimensional irreducible representations of T.

Definition 2 (Weights). *A weight is an irreducible representation of T. For any representation π of G, the weight space corresponding to a given weight is the subspace of the representation space of π that transforms under T according to the given weight.*

There are various ways of explicitly labelling weights, and one can do so in terms of either the group T or the Lie algebra $\mathfrak{t}$. This can be rather confusing, so we'll now belabor the rather trivial subject of the representation theory of T. As we'll see, there are many possible choices of convention. I'll be following to some extent [1].

A complex irreducible representation of S^1 can be chosen to be unitary and is given by a homomorphism

$$\theta : S^1 \rightarrow U(1)$$

taking the differential of this map gives a map

$$\theta_* : \mathbf{R} \rightarrow \mathbf{R}$$

on the Lie algebras. Such irreducible representations are labelled by an integer n and the θ corresponding to n is

$$\theta_n : [x] \in \mathbf{R}/\mathbf{Z} = S^1 \rightarrow e^{2\pi i n x}$$

The map on Lie algebras is

$$(\theta_n)_* : x \in \mathbf{R} \to nx \in \mathbf{R}$$

Note that $(\theta_n)_*$ is an element of the dual space to $\mathbf{R}$.

Now for $T = (S^1)^k$ the irreducible representations are homomorphisms

$$\theta : (S^1)^k \to U(1)$$

These correspond to a choice of k integers $\mathbf{n} = (n_1, \cdots, n_k)$ and

$$\theta_{\mathbf{n}} : ([x_1], \cdots, [x_k]) \in (\mathbf{R}/\mathbf{Z})^k \to e^{2\pi i(n_1x_1 + \cdots + n_kx_k)}$$

The corresponding Lie algebra maps are

$$(\theta_{\mathbf{n}})_* : (x_1, \cdots, x_k) \to n_1x_1 + \cdots + n_kx_k$$

and can be thought of as elements of the dual space to the Lie algebra of T, $\mathfrak{t}^*$. When we use the term weight, we will often mean an element of $\mathfrak{t}^*$ corresponding to an irreducible T representation. These elements of $\mathfrak{t}^*$ are those that take integer values on the integer lattice in $\mathbf{R}^k$. More abstractly, the integer lattice is the set of points

$$exp^{-1}(e) \subset \mathfrak{t}$$

So far we have been considering unitary representations of T on complex vector spaces. We also need to consider representations of T on real vector spaces. In this case irreducible representations are homomorphisms

$$\theta : T \to SO(2)$$

and there are two different sorts of irreducible representations:

1. The one-dimensional trivial representation on $\mathbf{R}$.
2. Non-trivial representations on $\mathbf{R}^2$ labelled by k integers, not all zero.

As in the complex case a weight is one of these irreducible representations and is given by an element

$$(\theta_{\mathbf{n}})_* : (x_1, \cdots, x_k) \to n_1x_1 + \cdots + n_kx_k$$

of $\mathfrak{t}^*$ taking integer values on the integer lattice in $\mathbf{R}^k$. The corresponding T representation is explicitly

$$\theta_{\mathbf{n}} : ([x_1], \cdots, [x_k]) \in (\mathbf{R}/\mathbf{Z})^k \to \begin{pmatrix} \cos(2\pi(\theta_{\mathbf{n}})_*(x_1, \cdots, x_k)) & -\sin(2\pi(\theta_{\mathbf{n}})_*(x_1, \cdots, x_k)) \\ \sin(2\pi(\theta_{\mathbf{n}})_*(x_1, \cdots, x_k)) & \cos(2\pi(\theta_{\mathbf{n}})_*(x_1, \cdots, x_k)) \end{pmatrix}$$

except for the special case of $(\theta_{\mathbf{n}})_* = 0$ where the representation is the trivial one and not on $\mathbf{R}^2$, but $\mathbf{R}$.

Note that $\theta_{\mathbf{n}})_*$ and $-\theta_{\mathbf{n}})_*$ are isomorphic representations since you can go from the matrix for one to the matrix for the other by conjugation by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3 Roots

We'll be using some of the following terminology:

Definition 3. *A subalgebra* $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is an ideal if it satisfies* $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

If the Lie algebra $\mathfrak{h}$ of a subgroup $H \subset G$ is an ideal in $\mathfrak{g}$, then H will be a normal subgroup of G.

Definition 4. *A Lie algebra* $\mathfrak{g}$ *is called simple if it has no non-trivial proper ideals and if it is not one-dimensional. A Lie group is called simple if its Lie algebra is simple.*

Definition 5. *A Lie algebra* $\mathfrak{g}$ *is called semi-simple if has no non-trivial abelian ideals. A Lie group is called semi-simple if its Lie algebra is semi-simple.*

Equivalently, a semi-simple Lie group is one with no non-trivial abelian connected normal subgroups. Also equivalently, a semi-simple Lie algebra is isomorphic with a product of simple Lie algebras. Examples of non-semi-simple Lie groups and Lie algebras are $G = U(n)$ and $\mathfrak{g} = \mathfrak{u}(n)$. $U(1)$ is not a simple Lie group. Simple Lie groups and Lie algebras include $G = SU(n)$ and $\mathfrak{g} = \mathfrak{su}(n)$. Products such as $SU(n) \times SU(m)$ are semi-simple Lie groups, direct sums such as $\mathfrak{su}(n) \oplus \mathfrak{su}(n)$ are semi-simple Lie algebras.

The Lie algebras we will be concerned with are the Lie algebras of compact Lie groups. These are real vector spaces, but for some purposes it is convenient to complexify and consider complex Lie algebras. It turns out that the classification of the Lie algebras of semi-simple compact Lie groups is equivalent to the classification of complex semi-simple Lie algebras since each complex semi-simple Lie algebra has a unique associated real Lie algebra of a compact group. The general theory of complex semi-simple Lie algebras will not be considered in this course although we will be going a little more into the subject later on.

One important property of the real Lie algebra $\mathfrak{g}$ is that the adjoint representation is an orthogonal one

$$Ad : G \to SO(dim\ \mathfrak{g})$$

One can see this by observing that the adjoint representation has an invariant positive definite inner product.

We won't prove this, but it is true that for any semi-simple Lie algebra the Killing form is non-degenerate. For those corresponding to compact groups it is negative definite and its negative gives a positive definite inner product. Next week we will examine in more detail the properties of the Killing form.

Definition 6 (Cartan Subalgebra). *For a compact Lie group* G*, a Cartan sub-algebra is a Lie subalgebra whose Lie group is a maximal torus* T *of* G.

Note: One can equivalently define a Cartan sub-algebra as a maximal abelian sub-algebra. There is a more general notion of Cartan subalgebra for general complex semi-simple Lie algebras.

Given the adjoint representation *Ad* of G on the real vector space $\mathfrak{g}$, one can ask the question of what the weights of the T action are in this case. T acts trivially on the Cartan subalgebra, so the trivial weight will appear with multiplicity $rank(G)$. The part of $\mathfrak{g}$ orthogonal to the Cartan subalgebra will break up into non-trivial two-dimensional irreducible orthogonal representations of T, these are the roots:

Definition 7 (Roots). *The roots of* G *are the non-trivial weights of the adjoint representation on the real vector space* $\mathfrak{g}$*. More explicitly, the roots are non-zero elements of* $\alpha \in \mathfrak{t}^*$*, taking integer values on the integer lattice.*

Next week we will be studying further properties of roots, and classifying all possible systems of roots, giving a classification of all compact, connected Lie groups.

References

[1] Adams, J. Frank, *Lectures on Lie Groups,* University of Chicago Press, 1969.