

# TOPICS IN REPRESENTATION THEORY: OTHER TOPICS

There are several other topics I hoped to have time to cover in this course but could not get to. They are:

- Topology of  $G/T$ : Flag manifolds  $G/T$  have cohomology only in even degrees, with Euler characteristic the degree of the Weyl group. The Euler characteristic can be calculated by a Lefschetz fixed point argument (see Adams [1], pgs. 90-92 for this). Identifying the flag manifold with a co-adjoint orbit, there is a Morse theory calculation of the cohomology that goes back to Bott, for an outline of the story, see [2]. This can also be thought of as a simple example of a torus action on a Kähler manifold, exhibiting the phenomenon of the moment map giving a perfect Morse function.
- Borel-Weil-Bott theorem: Using Lie algebra cohomology, one can show that for non-dominant weights  $\lambda$ , irreducible representations occur not in

$$\Gamma_{hol}(L_\lambda) = H^0(G/T, L_\lambda)$$

but in higher cohomology. The crucial step here is “Kostant’s theorem”, which computes  $H^*(\mathfrak{n}, V_\lambda)$ . This calculation seems to be best understood in terms of spinors and is related to more recent work of Kostant on the so-called “Kostant Dirac operator”. For Kostant’s original paper, see [3]. For the relation to spinors and the Dirac operator, see [4].

- Theta functions: For an exposition of the theory of theta functions from the point of view of the Heisenberg algebra and the metaplectic group, see [5].
- Finally it would be nice to say a bit about automorphic forms and their relation to the representation theory of  $SL(2, \mathbf{R})$ , but this is perhaps way too ambitious for a course like this.

## References

- [1] Adams, J. Frank, *Lectures on Lie groups*, University of Chicago Press, 1969.
- [2] Bott, R., The geometry and representation theory of compact Lie groups, in *Representation Theory of Lie Groups*, Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, Oxford, 1977. Cambridge University Press, 1979.
- [3] Kostant, B., Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.* **74** (1961), 329-387.

- [4] Kostant, B., A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank groups, *Duke Mathematical Journal* **100** (1999), 447-501.
- [5] Mumford, D. (with Nori M. and Norman, P.), *Tata Lectures on Theta, volume III*, Birkhäuser, 1991.