Topics in Representation Theory: The Moment Map and the Orbit Method

The orbit method in representation theory uses the fact that G orbits in $\mathfrak{g}^*$ are naturally symplectic manifolds with a transitive G action that preserves the symplectic structure. The quantization of the corresponding classical mechanical system will be a quantum mechanical system whose Hilbert space carries a representation of G. The orbit method (sometimes known as the Kirillov correspondence), tries to identify co-adjoint orbits with irreducible unitary representations. This can be done successfully for a wide range of groups, but a complete understanding of why this method works (and of why it sometimes fails) still does not seem to exist. For an expository article about the orbit method, see [3]. More details about the moment map and co-adjoint orbits can be found in [1] and [2].

1 Hamiltonian G-actions and the Moment Map

Last time we saw that for a (connected) symplectic manifold with symplectic G action there is an exact sequence of Lie algebra homomorphisms

$$0 \to \mathbf{R} \to C^\infty(M) \to Vect(M,\omega) \to H^1(M,\mathbf{R}) \to 0$$

where the middle map is given by

$$f \in C^\infty(M) \to X_f \in Vect(M,\omega)$$

where X_f satisfies

$$\omega(X_f,\cdot) = -df$$

If $H^1(M,\mathbf{R}) \neq 0$, there exist vector fields in $Vect(M,\omega)$ that don't correspond to a function f. These are vector fields X such that $\omega(X,\cdot)$ is closed but not exact. From now on, we'll assume that $H^1(M,\mathbf{R}) = 0$, so there is an exact sequence

$$0 \to \mathbf{R} \to C^\infty(M) \to Vect(M,\omega) \to 0$$

Given a symplectic action of G on M, its differential is a Lie algebra homomorphism

$$\mathfrak{g} \to Vect(M,\omega)$$

and one may or may not be able to lift this map. If so

Definition 1 (Poisson Action). *A G action on M is said to be a Poisson action if one can choose a Lie algebra homomorphism*

$$g : \mathfrak{g} \to C^\infty(M)$$

such that composing with the Lie algebra homomorphism

$$C^\infty(M) \to Vect(M,\omega)$$

one recovers the infinitesimal action of G.

We won't go into the Lie algebra cohomology necessary to establish this result, but it turns out that the obstruction to finding a Poisson action lies in the Lie algebra cohomology group $H^2(\mathfrak{g})$. For many Lie algebras (e.g. finite semi-simple ones), this is zero, but for others (e.g. the Lie algebras of loop groups that we will consider later) it is non-trivial. $H^2(\mathfrak{g})$ classifies central extensions, and when it is non-zero, the method of quantization naturally produces projective representations that are true representations only of the central extension.

When the G action is Poisson, the Lie algebra map

$$g : \mathfrak{g} \rightarrow C^\infty(M)$$

satisfies

$$-dg_\zeta = \omega(X_\zeta, \cdot)$$

where $g_\zeta = g(\zeta)$ is the function on M corresponding to an element $\zeta \in \mathfrak{g}$ and X_ζ is the corresponding vector field on M. If one doesn't worry about the Lie algebra homomorphism property, one can easily choose a g satisfying this equation, but note that there is an $\mathbf{R}$ ambiguity in the solution to this equation. The $H^2(\mathfrak{g})$ constraint comes from requiring that this ambiguity can be fixed in a way that satisfies the Lie algebra homomorphism property.

The information contained in the map g can be packaged differently, as follows:

Definition 2 (The Moment Map). *The moment map associated to a Poisson action is the map*

$$\mu : M \rightarrow \mathfrak{g}^*$$

that satisfies

$$< \mu(m), \zeta >= g_\zeta(m)$$

where $m \in M$, $< \cdot, \cdot >$ *is the pairing between* $\mathfrak{g}^*$ *and* $\mathfrak{g}$, *and* g_ζ *is the map defined earlier.*

A standard example from basic physics takes as symplectic manifold $M = \mathbf{R}^6$, the phase space for a particle moving in $\mathbf{R}^3$. The coordinates on M consist of three position coordinates q^1, q^2, q^3 and three momentum coordinates p^1, p^2, p^3, and the symplectic form is

$$\omega = \sum_{i=1}^{3} dp^i \wedge dq^i$$

$G = \mathbf{R}^3$ acts on M by translation with ζ^i a coordinate of an element of G acting by

$$q^i \rightarrow q^i + \zeta^i$$

and the corresponding map

$$g : \mathbf{R}^3 \rightarrow C^\infty(\mathbf{R}^6)$$

is given by

$$g_\zeta(\mathbf{p},\mathbf{q}) = p^1\zeta^1 + p^2\zeta^2 + p^3\zeta^3$$

and the moment map is just the linear momentum map

$$\mu(\mathbf{p},\mathbf{q})(\zeta) = \mathbf{p}\cdot\zeta$$

The group $G = SO(3)$ also acts symplectically by rotating $\mathbf{q}$. Now taking ζ to be an element of $\mathfrak{so}(3) = \mathbf{R}^3$, the corresponding moment map is the angular momentum map

$$\mu(\mathbf{p},\mathbf{q})(\zeta) = (\mathbf{p}\times\mathbf{q})\cdot\zeta$$

As we have seen before, the symplectic group $G = Sp(2n,\mathbf{R})$ acts symplectically on $\mathbf{R}^{2n}$. Choosing the standard complex structure on $\mathbf{R}^{2n}$, there is a distinguished subgroup $U(1) \subset Sp(2n,\mathbf{R})$ or overall phase rotations. For this action of G, the corresponding functions g are the quadratic polynomials in the p and q coordinates. For the $U(1)$ subgroup the function is

$$\frac{1}{2}(||\mathbf{p}||^2 + ||\mathbf{q}||^2)$$

This is the Hamiltonian function for a harmonic oscillator in n coordinates.

2 Co-adjoint Orbits

One can check that the moment map will be a G-equivariant map from M to $\mathfrak{g}^*$, the dual of the Lie algebra of G. The action of G on $\mathfrak{g}^*$ is called the co-adjoint action. Recall that G acts on $\mathfrak{g}$ by the adjoint action

$$Ad(g) : \mathfrak{g} \to \mathfrak{g}$$

The co-adjoint action of G on $\mathfrak{g}^*$

$$K(g) : \mathfrak{g}^* \to \mathfrak{g}^*$$

is defined on an element $F \in \mathfrak{g}^*$ by

$$< K(g)F, X > = < F, Ad(g^{-1})X >$$

The derivative of this is

$$K_*(X) : \mathfrak{g}^* \to \mathfrak{g}^*$$

given by

$$< K_*(X)F, Y > = < F, -ad(X)Y > = < F, [Y, X] >$$

Note that for the Lie algebras of simple compact Lie groups where we have a non-degenerate Killing form, it can be used to relate $\mathfrak{g}$ and its dual. In this case there is no real difference between the adjoint and co-adjoint action, but this is not true in general.

Definition 3 (Co-adjoint Orbit). *Given a $F \in \mathfrak{g}^*$, the co-adjoint orbit $\mathcal{O}(F)$ is the image of the map*

$$g \in G \to K(g)F \in \mathfrak{g}^*$$

Such orbits $\mathcal{O}(F)$ will have the form $G/Stab(F)$ where $Stab(F)$ is that subgroup of G that stabilizes F.

For each $F \in \mathfrak{g}^*$, define

$$\Omega_F(X,Y) =< F, [X,Y] >$$

This has the following properties:

Claim 1.

$$\Omega_F(X,Y) = -\Omega_F(Y,X)$$

Claim 2. *The kernel of Ω_F is $stab(F)$, the Lie algebra of $Stab(F)$.*

This follows from the calculation

$$\begin{aligned} ker\ \Omega_F(X,Y) &= \{X \in \mathfrak{g} : \Omega_F(X,Y) = 0\ \forall\ Y \in \mathfrak{g}\} \\ &= \{X \in \mathfrak{g} :< K_*(X)F, Y >= 0\ \forall\ Y \in \mathfrak{g}\} \\ &= \{X \in \mathfrak{g} : K_*F = 0\} \\ &= stab(F) \end{aligned}$$

Claim 3. *Ω_F is invariant under the action of $Stab(F)$*

This means

$$\begin{aligned} \Omega_F(Ad(g)X, Ad(g)Y) &= < F, [Ad(g)X, Ad(g)Y] > \\ &= < F, Ad(g)[X,Y] > \\ &= < K(g)^{-1}F, [X,Y] > \\ &= < F, [X,Y] >= \Omega_F(X,Y) \end{aligned}$$

for $g \in Stab(F)$.

The map

$$X \in \mathfrak{g} \to K_*(X)F \in T_F(\mathcal{O}(F))$$

identifies $\mathfrak{g}/(stab(F))$ with the tangent space to $\mathcal{O}(F)$ at F. The claims above imply that the following definition yields a non-degenerate, G-invariant symplectic two-form on $\mathcal{O}(F)$, making it into a symplectic manifold with symplectic G action. The moment map in this case is just the inclusion map.

Definition 4 (Kirillov-Kostant-Souriau form). *The Kirillov-Kostant-Souriau two-form on the co-adjoint orbit $\mathcal{O}(F)$ is*

$$\omega_F(K_*(X), K_*(Y)) = \Omega_F(X,Y)$$

3 Examples

The orbit method is based on the idea of associating co-adjoint orbits $\mathcal{O}(F)$ to irreducible representations. The representation is supposed to come from quantizing the classical mechanical system with G-symmetry determined by the symplectic manifold $(\mathcal{O}(F), \omega_F)$. We'll consider two basic examples of such co-adjoint orbits.

The first is for the Heisenberg group. Let H_1 be the group of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Its Lie algebra is $\mathfrak{h}_1$, matrices of the form

$$\begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

Using the inner product

$$< X, Y >= Tr(XY)$$

the dual space $\mathfrak{h}_1^*$ can be identified with matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix}$$

and the co-adjoint action on this space acts by

$$(x, y, z) \rightarrow (x - az, y + bz, z)$$

The co-adjoint orbits come in two kinds

1. For each value of $z \neq 0$, one gets a copy of $\mathbf{R}^2$, with symplectic form proportional to the area form $dx \wedge dy$.

2. For $z = 0$, for each pair of coordinates (x, y), there is a point orbit.

These orbits correspond to the known unitary representations of H_1 as follows. The first case gives the metaplectic representation, recall that the Stone-von Neumann theorem says that this is the unique representation one gets once one chooses the value of the action of the central element (z in this case).

The second case corresponds to the standard one-dimensional representations of the commutative group $\mathbf{R}^2$.

The second example we'll consider will be that of $G = SU(2)$. In this case $\mathbf{g} = \mathbf{R}^3$ and the Killing form identifies $\mathbf{g}$ and $\mathbf{g}^*$. Co-adjoint orbits are spheres S^2 inside $\mathbf{R}^3$.

We know from the Borel-Weil theorem that unitary representations can be constructed in a way that involves the geometry of S^2, but more specifically that of line bundles over S^2. To make the orbit method work in this case, one needs to impose the extra condition that not all co-adjoint orbits be considered, but just those for which the symplectic form ω_F is the curvature two-form of a line bundle L. One also has to use a complex structure on the co-adjoint orbit, and then the representation is on

$$\mathcal{H} = \Gamma_{hol}(L)$$

.

This same construction generalizes to the case of an arbitrary compact Lie group G. The co-adjoint orbits are just the generalized flag manifolds we have previously discussed. Quantizable orbits correspond to possible highest weights.

References

[1] Bryant, R., An Introduction to Lie Groups and Symplectic Geometry, in *Geometry and Quantum Field Theory,* Freed, D., and Uhlenbeck, K., eds., American Mathematical Society, 1995.

[2] Guillemin, V. and Sternberg, S., *Symplectic Techniques in Physics,* Cambridge University Press, 1984.

[3] Kirillov, A., Merits and demerits of the orbit method, *Bull. Amer. Math. Soc.* **36** (1999), 413-432.