Topics in Representation Theory: The Spinor Representation

As we have seen, the groups $Spin(n)$ have a representation on $\mathbf{R}^n$ given by identifying $v \in \mathbf{R}^n$ as an element of the Clifford algebra $C(n)$ and having $\tilde{g} \in Spin(n) \subset C(n)$ act by

$$v \to \tilde{g}v\tilde{g}^{-1}$$

This is also a $SO(n)$ representation, the fundamental representation on vectors. Replacing v in this formula by an arbitrary element of $C(n)$ we get a representation of $Spin(n)$ (and also $SO(n)$) on $C(n)$ which can be identified with its representation on $\Lambda^*(\mathbf{R}^n)$. This representation is reducible, the decomposition into irreducibles is just the decomposition of $\Lambda^*(\mathbf{R}^n)$ into the various $\Lambda^k(\mathbf{R}^n)$ for $k = 0, \cdots, n$. The $k = 0$ and $k = n$ cases give the trivial representation, but we get fundamental irreducible representations for $k = 1, \cdots, n-1$.

The fundamental irreducible representations that are missed by this construction are called the spinor representations. They are true representations of $Spin(n)$, but only representations up to sign (projective representations) of $SO(n)$. For the even case of $Spin(2n)$, we will see that there are two different irreducible half-spinor representations of dimension 2^{n-1} each, for the odd dimensional case there is just one irreducible spinor representation, of dimension 2^n.

In terms of the Dynkin diagrams of these groups, for $SO(2n+1)$, the spinor representation corresponds to the node at one end connected to the rest by a double bond. For $SO(2n)$ the two half-spin representations correspond to the two nodes that branch off one end of the diagram.

For small values of n, special phenomena occur. Here are some facts about the first few spin groups, the ones that behave in a non-generic way.

- $Spin(2)$ is a circle, double-covering the circle $SO(2)$.
- $Spin(3) = SU(2)$, and the spin representation is the fundamental representation of $SU(2)$. The Dynkin diagram is a single isolated node.
- $Spin(4) = SU(2) \times SU(2)$, and the half-spin representations are the fundamental representations on the two copies of $SU(2)$. The Dynkin diagram is two disconnected nodes.
- $Spin(5) = Sp(2)$, and the spin representation on $\mathbf{C}^4$ can be identified with the fundamental $Sp(2)$ representation on $\mathbf{H}^2$. The Dynkin diagram has two nodes connected by a double bond.
- $Spin(6) = SU(4)$, and each of the half-spin representations on $\mathbf{C}^4$ can be identified with the fundamental $SU(4)$ representation on $\mathbf{C}^4$. The Dynkin diagram has three nodes connected by two single bonds.

- The Dynkin diagram for *Spin*(8) has three nodes, each connected to a fourth central node. The representations associated to the three notes are all on $\mathbf{C}^8$ and correspond to the two half-spin representations and the representation on vectors. There is a "triality" symmetry that permutes these representations, this is a 3! element group of outer automorphisms of *Spin*(8).

The geometric picture of spin representations is unfortunately not well explained in standard textbooks. Some of the best places to look for more details from the point of view used here are [3] Chapters I.5 and IV.9, [1] Chapter 20, and [2] Chapter 12.

1 Spinors

We will be constructing spinor representations on complex vector spaces using the Clifford algebra and our first step is to consider what happens when one complexifies the Clifford algebra. The complex Clifford algebras will turn out to have a much simpler structure than the real ones, with a periodicity of degree 2 rather than degree 8 as in the real case.

Definition 1 (Complex Clifford Algebra)**.** *The complex Clifford algebra* $C^{\mathbf{C}}(V,Q)$ *is the Clifford algebra constructed by starting with the complexified vector space* $V \otimes_{\mathbf{R}} \mathbf{C}$*, extending* Q *to this by complex-linearity, then using the same definition as in the real case. If we start with a real vector space* V *of dimension* n*, this will be denoted* $C^{\mathbf{C}}(n)$*.*

One can easily see that $C^{\mathbf{C}}(n) = C(n) \otimes \mathbf{C}$. The construction of the spin representation as invertible elements in $C(n)$ can also be complexified, producing a construction of $Spin(n, \mathbf{C})$ (the complexification of $Spin(n)$) as invertible elements in $C^{\mathbf{C}}(n)$.

We will study the structure of the algebras $C^{\mathbf{C}}(n)$ by an inductive argument. To begin the induction, recall that

$$C(1) = \mathbf{C}, \quad C(2) = \mathbf{H}$$

so

$$C^{\mathbf{C}}(1) = C(1) \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C} \oplus \mathbf{C}$$

and

$$C^{\mathbf{C}}(2) = C(2) \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} = M(2, \mathbf{C})$$

Theorem 1.

$$C^{\mathbf{C}}(n+2) = C^{\mathbf{C}}(n) \otimes_{\mathbf{C}} C^{\mathbf{C}}(2) = C^{\mathbf{C}}(n) \otimes_{\mathbf{C}} M(2, \mathbf{C})$$

Using the cases $n = 1, 2$ to start the induction, one finds

Corollary 1. *If* $n = 2k$

$$C^{\mathbf{C}}(2k) = M(2, \mathbf{C}) \otimes \cdots \otimes M(2, \mathbf{C}) = M(2^k, \mathbf{C})$$

where the product has k *factors, and if* $n = 2k + 1$*,*

$$C^{\mathbf{C}}(2k+1) = C^{\mathbf{C}}(1) \otimes M(2^k, \mathbf{C}) = M(2^k, \mathbf{C}) \oplus M(2^k, \mathbf{C})$$

Proof of Theorem: Choose generators h_1, h_2 of $C(2)$, $f_1, \cdots, f_n$ of $C(n)$ and $e_1, \cdots e_{n+2}$ of $C(n+2)$. Then the isomorphism of the theorem is given by the following map of generators

$$e_1 \to 1 \otimes h_1$$
$$e_2 \to 1 \otimes h_2$$
$$e_3 \to if_1 \otimes h_1 h_2$$
$$\cdots$$
$$e_{n+2} \to if_n \otimes h_1 h_2$$

One can check that this map preserves the Clifford algebra relations and is surjective, thus an isomorphism of algebras.

From now we'll concentrate on the even case $n = 2k$. In this case we have seen that the complexified Clifford algebra is the algebra of 2^k by 2^k complex matrices. A spinor space S will be a vector space that these matrices act on:

Definition 2 (Spinors)**.** *A spinor module* S *for the Clifford algebra* $C^{\mathbf{C}}(2k)$ *is given by a choice of a* 2^k *dimensional complex vector space* S*, together with and identification* $C^{\mathbf{C}}(2k) = End(S)$ *of the Clifford algebra with the algebra of linear endomorphisms of* S*.*

So a spinor space is a complex dimensional vector space S, together with a choice of how the $2k$ generators e_i of the Clifford algebra act as linear operators on S.

To actually construct such an S, together with appropriate operators on it, we will use exterior algebra techniques. We'll begin by considering the real exterior algebra $\Lambda^*(\mathbf{R}^n)$. Associated to any vector v there is an operator on $\Lambda^*(\mathbf{R}^n)$ given by exterior multiplication by v

$$v\wedge : \alpha \to v \wedge \alpha$$

If one has chosen an inner product on $\mathbf{R}^n$ there is an induced inner product on $\Lambda^*(\mathbf{R}^n)$ (the one where the wedge products of orthonormal basis vectors are orthonormal). With respect to this inner product, the operation $v\wedge$ has an adjoint operator, $v\lfloor$.

Physicists have a useful notation for these operations, in which

$$a^\dagger(v) = v\wedge, \quad a(v) = v\lfloor$$

and

$$a^\dagger(e_i) = a_i^\dagger, \quad a(e_i) = a_i$$

The algebra satisfied by these operators is called the algebra of Canonical Anticommutation Relations (CAR)

Definition 3 (CAR)**.** *For each positive integer n, there is an algebra called the algebra of Canonical Anticommutation Relations, with $2n$ generators $a_i, a_i^\dagger$, $i = 1, \cdots, n$ satisfying the relations*

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$$

$$\{a_i, a_j^\dagger\} = \delta_{ij}$$

As we have seen, this algebra can be represented as operators on $\Lambda^*(\mathbf{R}^n)$. One can identify this algebra with the Clifford algebra $C(n)$ as follows

$$v \in C(n) \to a^\dagger(v) - a(v)$$

since one can check that

$$v^2 = (a^\dagger(v) - a(v))^2 = (a^\dagger(v))^2 + (a(v))^2 - \{a^\dagger(v), a(v)\} = -||v||^2\mathbf{1}$$

For the case $n = 2k$, one can complexify $\Lambda^*(\mathbf{R}^n)$ and get a complex representation of $C^{\mathbf{C}}(2k)$ on this space. This representation is of dimension 2^{2k}, so it is not the 2^k dimensional irreducible representation on a spinor space S. Some way must be found to pick out a single irreducible representation S from the reducible representation on $\Lambda^*(\mathbf{R}^n)\otimes\mathbf{C}$. This problem is a bit like the one faced in the Borel-Weil approach to the representations of compact Lie groups, where we used a complex structure to pick out a single irreducible representation. Here we will have to use a similar trick.

If we pick a complex structure J on $V = \mathbf{R}^{2k}$, we have a decomposition

$$V \otimes \mathbf{C} = W_J \oplus \bar{W}_J$$

Where W_J is the $+i$ eigenspace of J, $\bar{W}_J$ is the $-i$ eigenspace. We will pick an orthogonal complex structure J, i.e. one satisfying

$$< Jv, Jw >=< v, w >$$

Definition 4 (Isotropic Subspace)**.** *A subspace W of a vector space V with inner product $< \cdot, \cdot >$ is an isotropic subspace of V if*

$$< w_1, w_2 >= 0 \ \ \forall \ w_1, w_2 \in W$$

and one has

Claim 1. *The subspaces W_J and $\bar{W}_J$ are isotropic subspaces of $V \otimes \mathbf{C}$.*

This is true since, for $w_1, w_2 \in W_J$,

$$< w_1, w_2 >=< Jw_1, Jw_2 >=< iw_1, iw_2 >= - < w_1, w_2 >$$

Since W_J is isotropic, the Clifford subalgebra $C(W_J) \subset C^{\mathbf{C}}(2k)$ generated by elements of W_J is actually the exterior algebra $\Lambda^*(W_J)$ since on W_J the quadratic form coming from the inner product is zero, and the Clifford algebra for zero quadratic form is just the exterior algebra. In a similar fashion $C(\bar{W}_J)$ can be identified with $\Lambda^*(\bar{W}_J)$. There are various ways of setting this up, but what we plan to do is to construct a spinor space S, as, say $\Lambda^*(W_J)$. This has the right dimension (2^k) and we just need to show that $C^{\mathbf{C}}(2k)$ can be identified with the algebra of endomorphisms of this space.

Later on we'll examine the dependence of this whole set-up on the choice of complex structure J, but for now we will simply pick the standard choice of complex structure, identifying $w_j = e_{2j-1} + ie_{2j}$ for $j = 1, \cdots, k$.

Claim 2.

$$C^{\mathbf{C}}(2k) = End(\Lambda^*(\mathbf{C}^k))$$

Using the creation and annihilation operator notation for operators on $\Lambda^*(\mathbf{C}^k)$, we can identify Clifford algebra generators with generators of the CAR algebra as follows (for $j = 1, k$)

$$e_{2j-1} = a_j^\dagger - a_j$$

$$e_{2j} = -i(a_j^\dagger + a_j)$$

One can check that the CAR algebra relations imply the Clifford algebra relations

$$\{e_i, e_j\} = -2\delta_{ij}$$

With this explicit model $S = \Lambda^*(\mathbf{C}^k)$ and the explicit Clifford algebra action on it, one can see how elements of $\mathfrak{spin}(2k)$ act on S, and thus compute the character of S as a $Spin(2k)$ representation. The k commuting elements of $\mathfrak{spin}(2k)$ that generate the maximal torus are the $\frac{1}{2}e_{2j-1}e_{2j}$. In the representation on $\Lambda^*(\mathbf{C}^k)$ they are given by

$$\frac{1}{2}e_{2j-1}e_{2j} = -i\frac{1}{2}(a_j^\dagger - a_j)(a_j^\dagger + a_j) = i\frac{1}{2}[a_j, a_j^\dagger]$$

The eigenvalues of $[a_j, a_j^\dagger]$ on $\Lambda^*(\mathbf{C}^k)$ are ± 1, depending on whether or not the basis vector e_j is in the string of wedge products that makes up the eigenvector. The weights of the the representation S are sets of k choices of $\pm\frac{1}{2}$

$$(\pm\frac{1}{2}, \cdots, \pm\frac{1}{2})$$

In this normalization, representations of $Spin(2k)$ that are actually representations of $SO(2k)$ have integral weights, the ones that are just representations of $Spin(2k)$ have half-integral weights.

The decomposition into half-spin representations S^+ and S^- corresponds to the decomposition into weights with an even or odd number of minus signs. With a standard choice of positive roots, the highest weight of one half-spin representation is

$$(+\frac{1}{2}, +\frac{1}{2}, \cdots, +\frac{1}{2}, +\frac{1}{2})$$

and for the other it is

$$(+\frac{1}{2},+\frac{1}{2},\cdots,+\frac{1}{2},-\frac{1}{2})$$

Note that these weights are not the weights of the representation of $U(k)\subset SO(2k)$ on $\Lambda^*(\mathbf{C}^k)$. If one looks a how the Lie algebra of the maximal torus there acts on $\Lambda^*(\mathbf{C}^k)$, one finds that the weights are all 0 or 1. So the weights of $\Lambda^*(\mathbf{C}^k)$ are the same as the weights of S, but shifted by an overall factor

$$(-\frac{1}{2},-\frac{1}{2},\cdots,-\frac{1}{2})$$

In other words, as a representation it is best to think of S as not $\Lambda^*(\mathbf{C}^k)$, but

$$S=\Lambda^*(\mathbf{C}^k)\otimes(\Lambda^k(\mathbf{C}^k))^{-\frac{1}{2}}$$

This overall factor disappears if you just consider the projective represen-taion on $P(S)$ (complex lines in S), since then

$$P(S)=P(\Lambda^*(\mathbf{C}^k))$$

This is a reflection of the general point that by trying to construct S only knowing $End(S)=C^{\mathbf{C}}(2k)$, we can only canonically construct $P(S)$, not S itself. Multiplying S by a scalar doesn't affect $End(S)$.

2 Complex Structures and Borel-Weil for Spinors

The construction of S given above used a specific choice of complex structure J. In general, a choice of J allows us to write

$$\mathbf{R}^{2k}\otimes\mathbf{C}=W_J\oplus\bar{W}_J$$

and construct S as $\Lambda^*(W_J)$. Recall that to understand its transformation properties under the maximal torus of $Spin(n)$, it is better to think of

$$S=\Lambda^*(W_J)\otimes(\Lambda^k(W_J))^{-\frac{1}{2}}$$

In the language of the CAR algebra, one can think of S as being generated by the operators $a_j^\dagger$ and a_j acting on a "vacuum vector" Ω_J. Ω_J depends on the choice of complex structure, and transforms under the maximal torus of $Spin(2k)$, transforming as $(\Lambda^k(\bar{W}_J))^{\frac{1}{2}}$.

Given the spinor representation S, one can think of Ω_J as a vector in S satisfying

$$\bar{w}\cdot\Omega_J=0\ \ \forall\bar{w}\in\bar{W}_J$$

This relation defines Ω_J up to a scalar factor and sets up an identification between orthogonal complex structures J and lines in S. Not all elements of $P(S)$ correspond to possible J's. The ones that do are said to be the lines generated by "pure spinors". This correspondance between pure spinors and orthogonal complex structures J could equivalently be thought of as a correspondance betwen pure spinors and isotropic subspaces of $\mathbf{R}^{2k} \otimes \mathbf{C}$ of maximal dimension (the $\bar{W}_J$).

Once one has made a specific choice of J, say J_0, this defines a model for the spinor space $\Lambda^*(W_{J_0})$ and in this model $\Omega_{J_0} = 1$. If one changes complex structure from J_0 to a J such that

$$W_J = \text{graph}(S : W_{J_0} \to \bar{W}_{J_0})$$

then one has identified J with a map $S \in W_{J_0} \times \bar{W}^*_{J_0}$. Using the inner product and the fact that W_J is isotropic one can show that $S \in \Lambda^2(W_{J_0})$. One can then show that taking

$$\Omega_J = e^{\frac{1}{2}S}$$

solves the equation

$$\bar{w} \cdot \Omega_J = 0 \quad \forall \bar{w} \in \bar{W}_J$$

for this model of S. For more details about this construction, see [2] Chapter 12.

The space of all orthogonal complex structures on $\mathbf{R}^{2k}$ can be identified with the homogeneous space $O(2k)/U(k)$. An orthogonal complex structure J is explicitly represented by an orthogonal map $J \in O(2k)$. This space has two components, corresponding to a choice of orientation. We'll mainly consider the component $J \in SO(2k)$. For any such J, the subgroup that preserves the decompostion of $\mathbf{R}^{2k} \otimes \mathbf{C}$ into eigenspaces of J (and thus preserves that complex structure) is a subgroup isomorphic to $U(k)$.

The space of orientation-preserving orthogonal complex structures can be thought of as

$$SO(2k)/U(k) = Spin(2k)/\tilde{U}(k)$$

There is a complex line bundle L over this space whose fiber above J is the complex line in $P(S)$ generated by Ω_J. The fact that Ω_J transforms as $(\Lambda^k(W_J))^{-\frac{1}{2}}$ corresponds to the global fact that as line bundles

$$L \otimes L = \det{}^{-1}$$

where det is the line bundle whose fiber over J is $\Lambda^k(W_J)$.

The space $Spin(2k)/\tilde{U}(k)$ is one of the generalized flag manifolds that occurs if one develops the Borel-Weil picture for the compact group $Spin(2k)$. It is a complex manifold, with a description as a quotient of complex Lie groups

$$SO(2k)/U(k) = Spin(2k)/\tilde{U}(k) = Spin(2k, \mathbf{C})/P$$

where P is a parabolic subgroup of $Spin(2k, \mathbf{C})$. The line bundle L is a holomorphic line bundle, and corresponds to the Borel-Weil construction of a line bundle with highest weight

$$(+\frac{1}{2}, +\frac{1}{2}, \cdots, +\frac{1}{2}, +\frac{1}{2})$$

This is the holomorphic line bundle corresponding to one of the half-spinor representations S^+ and the Borel-Weil theorem gives a construction of S^+ as

$$S^+ = \Gamma_{hol}(L)$$

Unlike the construction in terms of the complex exterior algebra, this construction doesn't depend upon a fixed choice of complex structure, but uses the global geometry of the space of all complex structures.

References

[1] Fulton, W., Harris, J., *Representation Theory: A First Course,* Springer-Verlag, 1991.

[2] Segal, G., and Pressley, A., *Loop Groups,* Oxford University Press, 1986.

[3] Lawson, B. and Michelson, M., *Spin Geometry,* Princeton University Press, 1989.