Topics in Representation Theory: Spin Groups

1 Spin Groups

One of the most important aspects of Clifford algebras is that they can be used to explicitly construct groups called $Spin(n)$ which are non-trivial double covers of the orthogonal groups $SO(n)$. For a more exhaustive discussion of this than the simplified one that will follow, there are many possible sources to read, but two in particular that may be useful are [2] Chapters I.2 and I.6, and [1] Chapter I.6.

There are several equivalent possible ways to go about defining the $Spin(n)$ groups as groups of invertible elements in the Clifford algebra.

1. One can define $Spin(n)$ in terms of invertible elements $\tilde{g}$ of $C_{\text{even}}(n)$ that leave the space $V = \mathbf{R}^n$ invariant under conjugation

$$\tilde{g}V\tilde{g}^{-1} \subset V$$

2. One can show that, for $v, w \in V$,

$$v \rightarrow wvw^{-1}$$

is reflection in the hyperplane perpendicular to w. Then $Pin(n)$ is defined as the group generated by such reflections with $||w||^2 = 1$. $Spin(n)$ is the subgroup of $Pin(n)$ of even elements. Any rotation can be implemented as an even number of reflections (Cartan-Dieudonné) theorem.

3. One can define the Lie algebra of $Spin(n)$ in terms of quadratic elements of the Clifford algebra. This is what we will do here.

The Lie algebra of $SO(n)$ consists of n by n antisymmetric real matrices. A basis for these is given by

$$L_{ij} = E_{ij} - E_{ji}$$

for $i < j$. The L_{ij} generate rotations in the $i - j$ plane. They satisfy the commutation relations

$$[L_{ij}, L_{kl}] = \delta_{il}L_{kj} - \delta_{ik}L_{lj} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}$$

The generators of e_i of the Clifford algebra $C(n)$ satisfy the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

and one can use these to show that the $\frac{1}{2}e_i e_j$ satisfy the same commutation relations as the L_{ij}.

$$[\frac{1}{2}e_i e_j, \frac{1}{2}e_k e_l] = \delta_{il}(\frac{1}{2}e_k e_j) - \delta_{ik}(\frac{1}{2}e_l e_j) + \delta_{jl}(\frac{1}{2}e_i e_k) - \delta_{jk}(\frac{1}{2}e_i e_l)$$

This shows that the vector space spanned by quadratic elements of $C(n)$ of the form $e_i e_j$, $i < j$, together with the operation of taking commutators, is isomorphic to the Lie algebra $\mathfrak{so}(n)$. To get the group $Spin(n)$, we can exponentiate these quadratic elements of $C(n)$. Since (to show this, just use the defining relation of $C(n)$)

$$\left(\frac{1}{2} e_i e_j\right)^2 = -\frac{1}{4}$$

one can calculate these exponentials to find

$$e^{\theta\left(\frac{1}{2} e_i e_j\right)} = \cos\left(\frac{\theta}{2}\right) + e_i e_j \sin\left(\frac{\theta}{2}\right)$$

As θ goes from 0 to 4π this gives a $U(1)$ subgroup of $Spin(n)$. One can check that, acting on vectors by

$$v \to e^{\theta\left(\frac{1}{2} e_i e_j\right)} v \left(e^{\theta\left(\frac{1}{2} e_i e_j\right)}\right)^{-1}$$

rotates the vector v by an angle $\frac{\theta}{2}$ in the $i - j$ plane. As we go around this circle in $Spin(n)$ once, we go around the the circle of $SO(n)$ rotations in the $i - j$ plane twice. This is a reflection of the fact that $Spin(n)$ is a double-covering of the group $SO(n)$.

Just as the adjoint action of the Lie algebra of $Spin(n)$ on itself is given by taking commutators, the Lie algebra representation on vectors is also given by taking commutators in the Clifford algebra. One can check that an infinitesimal rotation in the $i - j$ plane of a vector v is given by

$$v \to [e_i e_j, v]$$

This is the infinitesimal version of the representation at the group level

$$v \to \tilde{g} v (\tilde{g})^{-1}$$

where $\tilde{g} \in Spin(n)$ is gotten by exponentiating $e_i e_j$.

2 Maximal Tori

For the even-dimensional case of $Spin(2n)$, one can proceed as follows to identify its maximal torus, which we'll call $\tilde{T}$. Fixing an identification $\mathbf{C}^n = \mathbf{R}^{2n}$, we have

$$T \subset U(n) \subset SO(2n)$$

where T is a maximal torus of both $U(n)$ and $SO(2n)$. T can be taken to be the group of diagonal n by n complex matrices with k-th diagonal entry $e^{i\theta_k}$. As an element of $SO(2n)$ these become 2 by 2 block diagonal real matrices with blocks

$$\begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$$

These blocks rotate by an angle θ_k in the $(2k-1)-2k$ plane, and all commute. For the odd-dimensional case of $SO(2n+1)$, which is of the same rank, the same T can be used, but one has to add in another diagonal entry, 1 as the $2n+1$'th entry, to embed this in $2n+1$ by $2n+1$ real matrices.

The double covering of $U(n)$ that is the restriction of the double covering of $SO(2n)$ by $Spin(2n)$ can be described in various ways. One is to define it as

$$\tilde{U}(n) = \{(A,u) \in U(n) \times S^1 : u^2 = \det A\}$$

The maximal torus $\tilde{T}$ of $Spin(2n)$ can be given explicitly in terms of n angles $\tilde{\theta}_k$ as

$$\prod_k (\cos(\tilde{\theta}_k) + e_{2k-1}e_{2k}\sin(\tilde{\theta}_k))$$

and is a double cover of the group T.

3 $Spin^c(n)$

A group related to $Spin(n)$ that has turned out to be of great interest in topology is the group $Spin^c(n)$. This can be defined as

$$Spin(n) \times_{\{\pm 1\}} S^1$$

i.e. by considering pairs $(A,u) \in Spin(n) \times S^1$ and identifying (A,u) and $(-A,-u)$. This group can also be defined as the subgroup of invertible elements in the complexified Clifford algebra $C(n) \otimes \mathbf{C}$ generated by $Spin(n)$ and $S^1 \subset \mathbf{C}$.

A Riemannian manifold M of dimension $2n$ comes with a bundle of orthonormal frames. This is a principal bundle with group $SO(2n)$. Locally it is possible to choose a double-cover of this bundle such that each fiber is the $Spin(2n)$ double cover, but globally there can be a topological obstruction to the continuous choice of such a cover. When such a global cover exists M is said to have a spin-structure. If M is Kähler, its frame bundle can be chosen to be a $U(n)$ bundle, but such an M will often not have a spin structure and one can't consider the spinor geometry of M. However, one reason for the importance of considering $Spin^c(2n)$ is that Kähler manifolds will always have a $Spin^c$ geometry, i.e the obstruction to a spin-structure can be unwound within $Spin^c(2n)$. This is because while there is no homomorphism

$$U(n) \subset SO(2n) \rightarrow Spin(2n)$$

there is a homomorphism

$$f : U(n) \rightarrow Spin^c(2n)$$

that covers the inclusion

$$A \in U(n) \rightarrow (A, \det A) \in SO(2n) \times U(1)$$

given on diagonal matrices in T by

$$f(\mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})) = \prod_k (\cos(\frac{\theta_k}{2}) + e_{2k-1}e_{2k}\sin(\frac{\theta_k}{2})) \times e^{i\sum_k \frac{\theta_k}{2}}$$

References

[1] Brocker, T. and tom Dieck, T., *Representations of Compact Lie Groups,* Springer-Verlag, 1985.

[2] Lawson, B. and Michelson, M., *Spin Geometry,* Princeton University Press, 1989.