Topics in Representation Theory: Borel Subgroups and Flag Manifolds

1 Borel and parabolic subalgebras

We have seen that to pick a single irreducible out of the space of sections $\Gamma(L_\lambda)$ we need to somehow impose the condition that elements of negative root spaces, acting from the right, give zero. To get a geometric interpretation of what this condition means, we need to invoke complex geometry. To even define the negative root space, we need to begin by complexifying the Lie algebra

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and then making a choice of positive roots $R^+ \subset R$

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in R^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

Note that the complexified tangent space to G/T at the identity coset is

$$T_{eT}(G/T) \otimes \mathbf{C} = \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and a choice of positive roots gives a choice of complex structure on $T_{eT}(G/T)$, with the holomorphic, anti-holomorphic decomposition

$$T_{eT}(G/T) \otimes \mathbf{C} = T_{eT}(G/T) \oplus \overline{T_{eT}(G/T)} = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

While $\mathfrak{g}/\mathfrak{t}$ is not a Lie algebra,

$$\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \text{ and } \mathfrak{n}^- = \sum_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

are each Lie algebras, subalgebras of $\mathfrak{g}_{\mathbf{C}}$ since

$$[\mathfrak{n}^+, \mathfrak{n}^+] \subset \mathfrak{n}^+$$

(and similarly for $\mathfrak{n}^-$). This follows from

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

The Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras, meaning

Definition 1 (Nilpotent Lie Algebra). *A Lie algebra $\mathfrak{g}$ is called nilpotent if, for some finite integer k, elements of $\mathfrak{g}$ constructed by taking k commutators are zero. In other words*

$$[\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]]] = 0$$

where one is taking k commutators.

The nilpotence of $\mathfrak{n}^+$ is clear since each time one takes a commutator one ends up in a root space with a more positive root, a process which must terminate in a finite number of steps due to the finite number of positive roots.

An important example of a nilpotent Lie algebra is the subalgebra of $\mathfrak{gl}(n, \mathbf{C})$ consisting of strictly upper triangular n by n matrices.

Another important Lie subalgebra is

$$\mathfrak{b} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{n}^-$$

This is called a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Note the choice of whether to use the negative or positive root space in this definition is a choice of convention. The choice of the negative root space makes some of the later discussion of the Borel-Weil theorem slightly simpler. The opposite choice, using the positive root space $\mathfrak{n}^+$ is perhaps somewhat more popular.

Our emphasis has been on the case of compact Lie groups G and their Lie algebras $\mathfrak{g}$, but one could instead emphasize general complex semi-simple lie algebras with $\mathfrak{g}_{\mathbf{C}}$ being special cases. In the general structure theory of complex semi-simple Lie algebras a Borel subalgebra is defined to be a maximal solvable subalgebra.

This requires a definition of a solvable Lie subalgebra. Among equivalent definitions, the one that is closest to the definition of a solvable group is

Definition 2 (Solvable Lie algebra). *A Lie algebra $\mathfrak{g}$ is solvable if there exists a sequence of Lie subalgebras*

$$0 \subset \cdots \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}_0 = \mathfrak{g}$$

such that for all i, $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and the quotient Lie algebra $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ is abelian.

One can develop the general structure theory of complex semi-simple Lie algebras using the fact that they have Borel subalgebras, all of which are conjugate, much in the same way that one develops the structure theory of compact Lie groups using maximal torii and the fact that they are all conjugate.

As an important example, note that the Borel subalgebra of $\mathfrak{sl}(n, \mathbf{C})$ consists of all lower-triangular elements of $\mathfrak{sl}(n, \mathbf{C})$.

We have a sequence of inclusions

$$\mathfrak{n}^- \subset \mathfrak{b} \subset \mathfrak{g}_{\mathbf{C}}$$

There are other interesting Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$ that contain the Borel subalgebra and can be constructed by adding to $\mathfrak{t}_{\mathbf{C}}$ not just the negative root space $\mathfrak{n}^-$, but in addition some subset of the set of the spaces $\mathfrak{g}_\alpha$ associated to positive simple roots α. More precisely

Definition 3. *A parabolic subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}_{\mathbf{C}}$ *is a Lie algebra satisfying*

$$\mathfrak{b} \subseteq \mathfrak{p} \subset \mathfrak{g}_{\mathbf{C}}$$

such that

$$\mathfrak{p} = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{\alpha \in T} \mathfrak{g}_{\alpha}$$

where T *consists of the negative roots and some subset (possibly empty) of the simple positive roots.*

The parabolic subalgebras of $\mathfrak{sl}(n, \mathbf{C})$ are determined by partitions of n and consist of "block lower-triangular" matrices. The size of the blocks is given by the partition, with the case of the Borel sub-algebra corresponding to the partition into blocks of size one.

In the general case, parabolic subalgebras correspond to choices of sets of nodes on the Dynkin diagram of the group, since each such node corresponds to a simple positve root.

2 Flag manifolds

So far we have been discussing the complex picture at the level of Lie algebras, but there is a corresponding story at the level of Lie groups. This involves the notion of the complexification $G_{\mathbf{C}}$ of a compact Lie group G. This is a Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. For a more extensive discussion of the complexification of a compact group, see [1], Chapter 12.

There are subgroups N^-, N^+, B, P corresponding to the nilpotent, Borel and parabolic subalgebras. Since $\mathfrak{n}^-$ is an ideal in $\mathfrak{b}$, N^- is a normal subgroup of G with quotient group $B/N^- = T_{\mathbf{C}}$.

For the case of $G = U(n)$, $G_{\mathbf{C}} = GL(n, \mathbf{C})$, and N^- consists of the lower triangular unipotent matrices (those with 1 on the diagonal), and B is the group of lower triangular invertible matrices. The parabolic subgroups P are groups of "block lower triangular invertible matrices.

Recall from linear algebra the procedure of Gram-Schmidt orthonormalization. This takes an arbitrary basis $\{\mathbf{v}_1, \cdots \mathbf{v}_n\}$ of $\mathbf{C}^n$ and produces an orthonormal basis $\{\mathbf{u}_1, \cdots \mathbf{u}_n\}$. It is expressed by a sequence of formulas

$$\begin{aligned} \mathbf{v}_1 &= \lambda_{11}\mathbf{u}_1 \\ \mathbf{v}_2 &= \lambda_{21}\mathbf{u}_1 + \lambda_{22}\mathbf{u}_2 \\ \cdots &= \cdots \end{aligned}$$

Thinking of the basis $\{\mathbf{v}_1, \cdots \mathbf{v}_n\}$ as, say the columns of an element of $GL(n, \mathbf{C})$, the Gram-Schmidt formulas equate such an element in a well-defined way with a product of a lower-triangular matrix (the λ_{ij}) and the columns of a unitary matrix (the $\{\mathbf{u}_1, \cdots \mathbf{u}_n\}$). So we have the following

Theorem 1. *Any $g \in GL(n, \mathbf{C})$ can be factorized uniquely as*

$$g = bu$$

where $u \in U(n)$ and b is a lower triangular matrix with positive real diagonal entries.

Note that $B \cap U(n) = T$, the subgroup of diagonal unitary matrices, so we have an identification

$$U(n)/T = GL(n, \mathbf{C}/B$$

This identification of $U(n)/T$ as a quotient of two complex Lie groups shows that the space G/T has a complex structure. This space appears in many contexts in mathematics and is known as a "flag manifold" $Fl(n, \mathbf{C})$ for the following reason.

Definition 4. *A flag in $\mathbf{C}^n$ is a sequence of linear subspaces of $\mathbf{C}^n$ such that*

$$L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset \mathbf{C}^n$$

with $dim L_i = i$.

Given the standard basis $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$ of $\mathbf{C}^n$ one can define a standard flag by taking L_1 to be all linear combinations of $\mathbf{e}_1$, L_2 to be all linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$, etc. Then $U(n)$ acts transitively on the set of all flags, with the subgroup T not changing the flag. Thus $U(n)/T$ is the space of all flags.

For general compact G, the space G/T continues to be referred to as a flag manifold, and can be identified with the quotient of complex groups $G_{\mathbf{C}}/B$. The $G_{\mathbf{C}}/P$ for parabolic subgroups P are called "partial flag manifolds". An important special case is

$$Gr(k, n) = \frac{U(n)}{U(k) \times U(n-k)}$$

the Grassmanian of complex k-planes in $\mathbf{C}^n$.

References

[1] Carter, R., Segal, G., and MacDonald, I., *Lectures on Lie Groups and Lie Algebras* Cambridge University Press, 1995.