Topics in Representation Theory: Decomposition of the Induced Representation

1 The Peter-Weyl theorem, a review

The Peter-Weyl theorem can be formulated in many equivalent ways. One of these is the statement that the matrix entries $\pi^{\gamma}_{i,j}$ of irreducible representations of a compact Lie group G on a vector space V_γ (γ is a representation label) are functions on G that satisfy the Schur orthogonality relations

$$(\pi^{\gamma}_{i,j}, \pi^{\beta}_{k,l}) = \int_G \overline{\pi^{\gamma}_{i,j}(g)}\pi^{\beta}_{k,l}(g)dg = \frac{1}{\dim(V_\gamma)}\delta^{\gamma\beta}\delta^{ik}\delta^{jl}$$

and that furthermore matrix entries form a complete set, i.e. any $f \in L^2(G)$ has an L^2 convergent expansion

$$f = \sum_{\gamma,ij} \dim(V_\gamma)(f, \pi^{\gamma}_{i,j})\pi^{\gamma}_{i,j}$$

More abstractly, one can restate this as

Theorem 1 (Peter-Weyl). *For compact G, $L^2(G)$ is an orthogonal L^2-sum of the subspaces $M(V_\gamma)$ spanned by the matrix coefficients of the irreducible representations π^γ of G (the functions $\pi^{\gamma}_{i,j}(g)$), i.e.*

$$L^2(G) = \widehat{\bigoplus}_{\gamma} M(V_\gamma)$$

The left and right regular representations give a representation of $G_L \times G_R$ (G_L, G_R are just two copies of G) acting on the space $L^2(G)$ by

$$(g_L, g_R) \cdot f(g) = f(g_L^{-1} g g_R)$$

One can restrict attention to $M(V_\gamma)$, the part of $L^2(G)$ spanned by matrix coefficients of the irreducible representation on V_γ and see how $G_L \times G_R$ acts on this space. It is convenient to think of $M(V_\gamma)$ as $V_\gamma \otimes V^*_\gamma$, using the identification

$$(v, \alpha) \in V_\gamma \otimes V^*_\gamma \to \alpha(\pi_\gamma(g^{-1})v) \in M(V_\gamma)$$

Using this identification, G_L acts on V_γ as

$$v \to \pi_\gamma(g_L)v$$

and G_R acts on V^*_γ by the dual representation

$$\alpha \to \pi^{\vee}_{\gamma}(g_R)\alpha$$

This dual representation is defined by

$$(\pi_\gamma^\vee(g_R)\alpha)(v) = \alpha(\pi_\gamma(g_R^{-1})v)$$

and is sometimes known as the "contra-gredient" representation.

What we have shown is that the Peter-Weyl theorem can be interpreted as saying that

$$L^2(G) = \widehat{\bigoplus}_\gamma V_\gamma \otimes V_\gamma^*$$

with the left regular representations of G on $L^2(G)$ corresponding to the action on the factor $V\gamma$ and the right regular representation corresponding to the dual action on the factor V_γ^*. Under the left regular representation $L^2(G)$ decomposes into irreducibles as a sum over all irreducibles, with each one occuring with multiplicity dim V_γ (which is the dimension of V_γ^*).

2 Decomposition of the Induced Representation

Given a weight λ of T, we have seen that we can form the induced representation of G on $\Gamma(L_\lambda)$. The space $\Gamma(L_\lambda)$ is a space of functions on G and is acted on by $g_L \in G_L$ by

$$f(g) \rightarrow g_L f(g) = f(g_L^{-1} g)$$

$\Gamma(L_\lambda)$ is thus a sub-representation of the left-regular representation. What picks out the subspace

$$\Gamma(L_\lambda) \subset L^2(G)$$

is the condition

$$f(gt) = \rho_\lambda(t^{-1})f(g)$$

Here the representation of T is one dimensional so if $t = e^H$

$$\rho_\lambda(t^{-1}) = e^{-\lambda(H)}$$

This condition says that under the action of the subgroup $T_R \subset G_R$, $\Gamma(L_\lambda)$ is the subspace of $L^2(G)$ that has weight $-\lambda$. In other words

$$\Gamma(L_\lambda) = \widehat{\bigoplus}_\gamma V_\gamma \otimes (V_\gamma^*)_{-\lambda}$$

where $(V_\gamma^*)_{-\lambda}$ is the $-\lambda$ weight space of V_γ^*.

Our induced representation $\Gamma(L_\lambda)$ thus breaks up into irreducibles as a sum over all irreducibles V_γ, with multiplicity given by the dimension of the weight $-\lambda$ in V_γ^*. We can get a single irreducible if we impose the condition that $-\lambda$ be a lowest weight since $-\lambda$ will be the lowest weight for just one irreducible representation. So if we impose the condition on $\Gamma(L_\lambda)$ that infinitesimal right translation by an element of a negative root space give zero, we will finally have a construction of a single irreducible, it will be the irreducible with highest weight λ.