Topics in Representation Theory: Homogeneous Vector Bundles and Induced Representations

1 Homogeneous Vector Bundles

We will begin by going over some terminology for those students who haven't seen some of these concepts before. The geometrical structures that appear in representation theory are very simple examples of basic geometrical structures that are normally studied in geometry and topology courses. They are as good a place as any to start learning some of this terminology if you haven't seen it before.

Given a Lie group G with a Lie subgroup H, one can form the space of right cosets G/H. This is a manifold and it has a transitive G action, an element g acts by left multiplication:

$$g_0H \rightarrow gg_0H$$

and any two points on G/H are related by such a mapping. Spaces like this are called "homogeneous spaces". At each point in G/H there is a subgroup of G such that left multiplication by elements in it leaves the point invariant, this is called the "isotropy group" of the point. At the identity $eH \in G/H$ this is just the subgroup H, at other points it will be a translation of H.

The space G/H is best thought of not just as a manifold, but as the "base-space" for a bundle. The total space of the bundle is G and the projection map from the bundle to the base-space is just the standard projection

$$\pi : g \in G \rightarrow gH \in G/H$$

The inverse image $\pi^{-1}(gH)$ of a point in G/H is the "fiber" above that point, it is a copy of H and one can think of G, the total space of the bundle, as a family of copies of H, parametrized by point in G/H.

This bundle is a "principal" H-bundle, meaning that there is a free H action from the right on the total space, with quotient the base-space. On the total space G there are two commuting actions, the right action of H which gives the bundle structure and the left action of G. A general principal H bundle does not have this transitive left G action, that is what makes the case we are considering quite special.

A map

$$s : G/H \rightarrow G$$

such that $\pi \circ s$ is the identity map on G/H is called a "section" of the bundle. In general the principal bundles we are considering do not have sections, at least not globally. One can define sections on local coordinate patches of G/H, but globally there is a topological obstruction to their existence. If there were a global section then as a topological space G would be the product space

$G/H \times H$. An important example to keep in mind is $G = SU(2) = S^3, H = T = U(1) = S^1, G/H = S^2$, the Hopf fibration. It is certainly not true that $S^3 = S^2 \times S^1$ and this is a bundle with no sections.

Given our principal H bundle G and a representation ρ of H on a vector space V we can construct a new "associated" bundle E_V as follows. Consider the product space $G \times V$ and quotient out by the following action of H (i.e. identify all points related by an element of $h \in H$ by the action).

$$(g, v) \in G \times V \rightarrow (gh, \rho(h^{-1})v)$$

We will call this quotient space

$$E_V = G \times_H V$$

It is an example of a "vector bundle" with a projection map

$$\pi(g, v) = gH$$

to the base-space G/H. The inverse image of each point in the base space is now not H (we've quotiented that out), but a copy of the vector space V. In general a vector bundle can be thought of as a family of vector spaces of the same dimension, parametrized by the base space. For the special case where the vector spaces are of dimension one, a vector bundle is called a "line bundle".

2 Induced Representations

Unlike principal bundles, vector bundles always have sections. In particular they always have a zero-section that maps every point in G/H to the origin of the vector space that is the fiber above that point. The space of sections $\Gamma(E_V)$ of the vector bundle E_V is given by the space

$$Map_H(G, V) = \{f : G \rightarrow V, \text{ such that } f(gh) = \rho(h^{-1})f(g) \;\; \forall\, g \in G, h \in H\}$$

this definition is equivalent to defining sections as maps

$$s : G/H \rightarrow E_V, \text{such that } \pi \circ s = Id$$

The equivalence of the two descriptions of $\Gamma(E_V)$ is given by associating to $f \in Map_H(G, V)$ the map $s : s(gH) = (g, f(g))$.

The space $\Gamma(E_V)$ can be thought of as a space of "twisted functions" on the base space, generalizing the standard space of functions which is what one gets for the special case of $V = \mathbf{C}$ and ρ the trivial representation. The vector bundles E_V are what is known as "homogeneous vector bundles", meaning that the transitive left action of G induces an action on E_V by

$$(g_0, v) \rightarrow (gg_0, v)$$

that identifies the vector space fibers over any two points on the base space. This left action also induces a representation of G on the the space of sections $\Gamma(E_V)$, explicitly, for each $g \in G$ we have a linear map $U(g)$ that takes

$$f(x) \in \Gamma(E_V) \to (U(g)f)(x) = f(g^{-1}x) \in \Gamma(E_V)$$

So we have started with a representation of H on V and have produced a representation of G, this is called an induced representation. The same construction works for finite groups and you may have first seen it in that context.

Definition 1 (Induced Representation)**.** *The representation of G on $\Gamma(E_V) = Map_H(G, V)$ is called the representation induced from the representation V and sometimes written $Ind_H^G(V)$.*

The induction map from representations of H to representations of G induces a map of representation rings. Denoting the inclusion map

$$i : H \to G$$

this map is often denoted

$$i_* : R(H) \to R(G)$$

There is an obvious map in the other direction given by restriction, this is often written

$$i^* : R(G) \to R(H)$$

Recall that there is a natural inner product on representation rings for which the irreducible representations form an orthonormal basis. One can define this by integration of characters, but also more abstractly as

$$< \chi_{V_1}, \chi_{V_2} >_G = \dim \mathrm{Hom}_G(V_1, V_2)$$

From this point of view, orthonormality of characters of irreducible representations is just Schur's lemma, one form of which is that for V_1, V_2 irreducible representations

$$\dim \mathrm{Hom}_G(V_1, V_2) = \begin{cases} 0 & V_1 \neq V_2 \\ 1 & V_1 = V_2 \end{cases}$$

The induced representation is generally not irreducible, how it decomposes into irreducibles is determined by the following theorem

Theorem 1 (Frobenius Reciprocity)**.** *For W a representation of G and V a representation of a subgroup H we have*

$$< W, i_* V >_G = < i^* W, V >_H$$

Idea of proof:
In one direction, given

$$F : W \to i_* V$$

map this to $ev_1 \circ F$, where ev_1 is the evaluation map at the identity

$$ev_1 : f \in i_* V \to f(1) \in V$$

In the other direction, given

$$f : W \to V$$

define

$$F : w \in W \to F(w)(g) = f(g^{-1}w)$$

To complete the proof, one just needs to show that these maps are G and H maps respectively, and are inverses.

Our study of the representation theory of G for G a compact Lie group has shown the importance of the maximal torus subgroup T. Representations of T are weights and are on one complex dimensional vector spaces. Associated to each weight λ of T there is a homogeneous line bundle L_λ over the base space G/T. The space of sections $\Gamma(L_\lambda)$ is the induced G representation. $\Gamma(L_\lambda)$ will be an infinite dimensional vector space and Frobenius reciprocity tells us that

$$\dim\ \mathrm{Hom}_G(V_\gamma, \Gamma(L_\lambda)) = \dim\ \mathrm{Hom}_T(i^* V_\gamma, \lambda)$$

In words, this says that the multiplicity in $\Gamma(L_\lambda)$ of an irreducible representation V_γ with highest weight γ is given by the multiplicity of the weight λ in V_γ. For any given weight λ, the induced representation $\Gamma(L_\lambda)$ will decompose into an infinity of irreducibles, all those that contain the weight λ.

A simple example of this phenomenon is behind the theory of functions on the 2-sphere called "spherical harmonics. These are often written using the notation

$$Y^l_m(\phi, \theta)$$

where ϕ and θ are angles parametrizing the sphere, l is a non-negative integer and m is an integer taking on the $2l+1$ values $-m, -m+1, \cdots, m-1, m$. Recall our standard example $G = SU(2),\ T = U(1),\ G/T = S^2$. The representations of T are labelled by an integer n and the representations V_k of $SU(2)$ correspond to non-negative values of k (dominant weights in this case) where k is the largest integer that occurs when you restrict V_k to be a $U(1)$ representation. Frobenius reciprocity tells us that

$$\dim\ \mathrm{Hom}_{SU(2)}(V_k, \Gamma(L_n)) = \dim\ \mathrm{Hom}_{U(1)}(V_k, n)$$

where the right hand side is the multiplicity of the weight n in V_k. Since we know what the weights of V_k are ($-k, -k+2, \cdot, k-2, k$, all with multiplicity one), we can see that the decomposition of $\Gamma(L_n)$ into irreducibles will include all V_k where k is the same parity as n and $k \geq n$.

Spherical harmonics correspond to the case $n = 0$, in this case L_0 is the trivial $\mathbf{C}$ bundle and $\Gamma(L_0)$ is just the space of complex-valued functions on S^2. All V_k with k even will occur in $\Gamma(L_0)$, each with multiplicity one. The V_k with k odd do not occur since they do not include the weight 0. This gives spherical functions that are $SU(2)$ representations, these are also $SO(3)$ representations. V_k, k even corresponds to standard spherical harmonics $Y^l_m(\phi, \theta)$ with $l = k/2$.