Topics in Representation Theory: The Weyl Integral and Character Formulas

We have seen that irreducible representations of a compact Lie group G can be constructed starting from a highest weight space and applying negative roots to a highest weight vector. One crucial thing that this construction does not easily tell us is what the character of this irreducible representation will be. The character would tell us not just which weights occur in the representation, but with what multiplicities they occur (this multiplicity is one for the highest weight, but in general can be larger). The importance of the knowing the characters of the irreducibles is that, given an arbitrary representation, we can then compute its decomposition into irreducibles.

As a vector space, the character ring $R(G)$ has a distinguished basis given by the characters $\chi_i(g)$ of the irreducible representations. Recall that the orthogonality relations for characters are essentially the same as in the finite group case, with the sum over group elements replace by an integral

$$\int \overline{\chi_i(g)}\chi_j(g)dg = \delta^{ij}$$

where i and j are labels for irreducible representations and dg is the standard Haar measure, normalized so that the volume of g is 1. For an arbitrary representation V, once we know its character χ we can compute the multiplicities n_i of the irreducibles in the decomposition

$$V = \bigoplus_{i\in\hat{G}} n_i V_i$$

as

$$n_i = \int \overline{\chi_i(g)}\chi(g)dg$$

Another way of thinking of this is that as a vector space $R(G)$ has an inner product $< \cdot, \cdot >_G$ and the characters of the irreducible representations form an orthonormal basis.

1 The Weyl Integral Formula

We have seen that representations of G can be analyzed by considering their restrictions to a maximal torus T. On restricting to T, the character $\chi_V(g)$ of a representation V gives an element of $R(T)$ which is invariant under the Weyl group $W(G,T)$. We need to be able to compare the inner product $< \cdot, \cdot >_G$ on $R(G)$ to $< \cdot, \cdot >_T$, the one on $R(T)$. This involves finding a formula that will allow us to compute integrals of conjugation invariant functions on G in terms of integral over T, this will be the Weyl integral formula.

We'll be considering the map:

$$q : (gT, t) \in G/T \times T \to gtg^{-1} \in G$$

An element n of the Weyl group $W(G,T)$ acts on $G/T \times T$ by

$$(gT, t) \to gn^{-1}T, ntn^{-1}$$

and all points in a Weyl group orbit are mapped by q to the same point in G. It turns out that away from the singular points of T the map u is a covering map with fiber $W(G,T)$. One way to see this is to consider an element

$$x \in \mathfrak{t} \subset \mathfrak{g}$$

Under the adjoint action

$$x \to gxg^{-1}$$

T leaves x invariant, elements of G/T move it around in an orbit in $\mathfrak{g}$, one that intersects T at points corresponding to the elements of the Weyl group. For x regular these points are all distinct (one for each Weyl chamber).

The map q is a covering map of degree $|W|$ away from the singular elements of T, these are of codimension one in T. As long as a function f on G doesn't behave badly at the images of the singular elements, one can integrate a function over G by doing the integral over the pull-back function $u^*(f)$ on $G/T \times T$, getting $|W|$ copies of the answer one wants. The result of this calculation is the

Theorem 1 (Weyl Integral Formula). *For f a continuous function on a compact connected Lie group G with maximal torus T one has*

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T det((\mathbf{I} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})(\int_{G/T} f(gtg^{-1})d(gT))dt$$

and if f is a conjugation invariant function ($f(gtg^{-1}) = f(t)$), then

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T det((\mathbf{I} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})f(t)dt$$

For a detailed proof of this formula, see [2] Chapter IV.1, for a less detailed proof see [5] Chapter IV.3 or [3] page 443. The essence of the computation is computing the Jacobian of the map u. Using left invariance of the measures involved and translating the calculation to the origin of $G/T \times T$, in the decomposition $\mathfrak{g} = \mathfrak{g}/\mathfrak{t} + \mathfrak{t}$ the Jacobian is trivial on $\mathfrak{t}$, but picks up the non-trivial factor

$$J(t) = \det((\mathbf{I} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})$$

from the the $\mathfrak{g}/\mathfrak{t}$ piece.

One can explicitly calculate the Jacobian factor $J(t)$ in terms of the roots, since they are the eigenvalues of Ad on $\mathfrak{g}/\mathfrak{t}$ with the result

$$J(t = e^H) = \prod_{\alpha \in R}(1 - e^{-\alpha(H)}) = \prod_{\alpha \in R^+}(1 - e^{\alpha(H)})(1 - e^{-\alpha(H)})$$

where R is the set of roots, R^+ is the set of positive roots. From this formula we can see that we can in some sense take the square root of $J(t)$, one choice of phase for this is to set

$$\delta(e^H) = e^{\rho}(H) \prod_{\alpha \in R^+} (1 - e^{-\alpha(H)}) = \prod_{\alpha \in R^+} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})$$

where ρ is half the sum of the positive roots. Then

$$J(t) = \delta\overline{\delta}$$

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2 The Weyl Character Formula

The Weyl integral formula was used by Weyl to derive a general formula for character of an irreducible G representation. In this section we'll give this character formula and outline its proof. This proof is rather indirect and there are at least two other more direct ways of deriving the formula. We'll see one of them later when we discuss the Borel-Weil-Bott theorem. Another important one derives the formula using a fixed point argument. For the fixed point argument, see [4], Chapter 14.2, or [1].

Weyl's derivation of the character formula uses a surprising and hard to motivate trick. A character of a G representation gives an element of $R(T)$ (explicitly a complex-valued function on T), one that is invariant under the Weyl group $W(G,T)$. The trick is to carefully analyze not the Weyl-symmetric character functions, but the antisymmetric characters, those elements of $R(T)$ that are antisymmetric under Weyl reflections. These are functions on T that change sign when one does a Weyl reflection in the hyperplane perpendicular to a simple root, i.e.

$$f(wt) = sgn(w)f(t)$$

where $sgn(w)$ is $+1$ for elements of $w \in W(G,T)$ built out of an even number of simple reflections, -1 for an odd number. The character function we want will turn out to be the ratio of two such antisymmetric characters.

Theorem 2 (Weyl Character Formula). *The character of the irreducible G representation V_λ with highest weight $\lambda \in \mathfrak{t}^*$ is*

$$\chi_{V_\lambda}(e^H) = \frac{\sum_{w \in W} sgn(w) e^{(\lambda+\rho)(wH)}}{\delta}$$

where $H \in \mathfrak{t}$, ρ is one-half the sum of the positive roots and δ is the function on T defined in the discussion of the Weyl integral formula

Outline of Proof:

Weyl's proof begins by showing that the antisymmetric characters

$$A_\lambda(e^H) = \sum_{w \in W} sgn(w) e^{\lambda(wH)}$$

for λ a weight in the dominant Weyl chamber form an integral basis for the vector space of all antisymmetric characters. Furthermore they satisfy the orthogonality relations

$$< A_\gamma, A_\lambda >_T = \int_T \overline{A}_\gamma A_\lambda = \pm|W|, A_\gamma = \pm A_\lambda$$

and 0 if $\gamma \neq \lambda$.

The irreducibility of V implies that $< \chi_V, \chi_V >_G = \int_G \overline{\chi_V}\chi_V = 1$ and the Weyl integral formula implies this is

$$1 = \frac{1}{|W|}\int_T \overline{\chi_V}\chi_V \overline{\delta}\delta dt = \frac{1}{|W|}\int_T \overline{(\chi_V\delta)}(\chi_V\delta)dt$$

Now $\chi_V\delta$ is an antisymmetric character since one can show that δ is antisymmetric under Weyl reflections and thus must be of the form

$$\chi_V\delta = \sum_\beta n_\beta A_\beta$$

where β are weights in the dominant Weyl chamber and n_β are integers. So we must have

$$\frac{1}{|W|}\int_T \overline{(\sum_\beta n_\beta A_\beta)}(\sum_\beta n_\beta A_\beta) = 1$$

but by the orthogonality relations this is

$$1 = \sum_\beta (n_\beta)^2$$

so only one value of β can contribute, with coefficient ± 1. Finally one can show that for an irreducible representation with highest weight λ, $A_{\lambda+\delta}$ must occur in $\chi\delta$ with coefficient 1.

This finishes the outline of the proof, the details are in many textbooks, see for instance [2].

The Weyl character formula has a wide range of corollaries and applications, for example

Corollary 1 (Weyl Denominator Formula). *Applying the Weyl character formula to the case of the trivial representation gives*

$$\delta(e^H) = \sum_{w\in W} e^{\rho(wH)}$$

By evaluating the character formula at $H = 0$ one can compute the dimension of the irreducible representations giving

Corollary 2 (Weyl Dimension Formula). *The dimension of the irreducible representation with highest weight λ is*

$$dim\ V_\lambda = \frac{\prod_{\alpha\in R^+} < \alpha, \lambda+\rho >}{\prod_{\alpha\in R^+} < \alpha, \rho >}$$

Here R^+ is the set of positive roots.

Proof:

This calculation is a little bit tricky since the Weyl character formula gives the character at $H = 0$ as the quotient of two functions that vanish to several orders there and so one has to use l'Hôpital's rule. We'll compute the values of the numerator and denominator of the Weyl character formula at $s\rho \in \mathfrak{t}$ (we need the inner product to identify ρ as an element of $\mathfrak{t}$) and then take the limit as s goes to 0.

$$\begin{aligned}
A_{\lambda+\rho}(s\rho) &= \sum_{w\in W} sgn(w)e^{s<\lambda+\rho,w\rho>} \\
&= A_\rho(s(\lambda+\rho)) \\
&= \delta(s(\lambda+\rho)) \\
&= \prod_{\alpha\in R^+}(e^{\frac{s}{2}<\alpha,\lambda+\rho>} - e^{-\frac{s}{2}<\alpha,\lambda+\rho>}) \\
&= \prod_{\alpha\in R^+} s<\alpha,\lambda+\rho> +O(s^{n+1})
\end{aligned}$$

where n is the number of positive roots. Taking the ratio of this to the same expression with $\lambda = 0$ and taking the limit as s goes to zero gives the corollary.

3 Some Examples

For the case of $G = U(n)$, the Weyl integration formula gives a very explicit formula for how to integrate a function of unitary matrices that only depends on their eigenvalues over the space of all unitary matrices. In this case an element of T is a set of n angles $e^{i\theta_i}$ and the Jacobian factor in this case is

$$\begin{aligned}
J(\theta_1,\cdots,\theta_n) &= \prod_{\alpha\in R}(1-e^{-\alpha}) \\
&= \prod_{j\neq k}(1-e^{i(\theta_k-\theta_j)}) \\
&= \prod_{j<k}(1-e^{i(\theta_j-\theta_k)})(e^{i\theta_k}e^{-i\theta_k})(1-e^{i(\theta_k-\theta_j)}) \\
&= \prod_{j<k}(e^{i\theta_k}-e^{i\theta_j})(e^{-i\theta_k}-e^{-i\theta_j}) \\
&= \prod_{j<k}|e^{i\theta_k}-e^{i\theta_j}|^2
\end{aligned}$$

Note that this factor suppresses contributions to the integral when two eigenvalues become identical. The full integration formula becomes

$$\int_{U(n)} f = \frac{1}{n!}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1,\cdots,\theta_2)\prod_{j<k}|e^{i\theta_k}-e^{i\theta_j}|^2\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}$$

One can check that the Weyl character formula gives the character formula for $SU(2)$ representations previously shown to hold.

$$\chi_n(\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

If an irreducible representation with highest weight ρ exists, the Weyl dimension formula implies that its dimension will be

$$\dim V_\rho = 2^m$$

where m is the number of positive roots. For $G = SU(3)$, this is the adjoint representation, which has dimension $8 = 2^3$ as predicted.

References

[1] Bott, R., On Induced Representations, in *The Mathematical Heritage of Hermann Weyl, Proceedings of Symposia in Pure Mathematics, vol.48,* American Mathematical Society, 1988.

[2] Brocker, T. and tom Dieck, T., *Representations of Compact Lie Groups,* Springer-Verlag, 1985.

[3] Fulton, W., Harris, J., *Representation Theory: A First Course,* Springer-Verlag, 1991.

[4] Pressley, A., and Segal, G., *Loop Groups,* Oxford University Press, 1986.

[5] Simon, B., *Representations of Finite and Compact Groups,* American Mathematical Society, 1996.