Topics in Representation Theory: $SU(2)$ Representations and Their Applications

We've so far been studying a specific representation of an arbitrary compact Lie group, the adjoint representation. The roots are the weights of this representation. We would now like to begin the study of arbitary representations and their weights. An arbitrary finite dimensionsional represesentation will have a direct sum decomposition

$$V = \bigoplus_{\alpha} V_\alpha$$

where the α are the weights of the representation labelled by elements of t^*, and V_α is the α-weight space, i.e. the vectors v in V satisfying

$$Hv = \alpha(H)v$$

for $H \in \mathfrak{t}$. The dimension of V_α is called the multiplicity of α. The problem we want to solve for each compact Lie group G is to identify the irreducible representations, computing their weights and multiplicities.

An important relation between roots and weights is the following:

Lemma 1. *If $X \in \mathfrak{g}_\beta$, then it maps*

$$X : V_\alpha \to V_{\alpha+\beta}$$

Proof:
If $v \in V_\alpha,\ H \in \mathfrak{t}$

$$HXv = XHv + [H, X]v = X\alpha(H)v + \beta(H)Xv = (\alpha(H) + \beta(H))Xv$$

so the roots act on the set of weights by translation.

We will begin with the simplest case, that of $G = SU(2)$. This case is of great importance both as an example of all the phenomena we want to study for higher rank cases, as well as playing a fundamental part itself in the analysis of the general case.

1 Review of $SU(2)$ Representations

One reason that $SU(2)$ representations are especially tractable is that there is a simple explicit construction of the irreducible representations. Consider the space V_2^n of homogeneous polynomials of two complex variables. An element of this space is of the form

$$f(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \cdots + a_n z_2^n$$

The group $SU(2)$ acts on V_2^n through the action of $U \in SU(2)$ as a linear transformation on the vector $\mathbf{z} = (z_1, z_2)$ as follows

$$\pi(U)f(\mathbf{z}) = f(U^{-1}\mathbf{z})$$

This is a group homomorphism since

$$\pi(U_1)(\pi(U_2)f)(\mathbf{z}) = (\pi(U_2)f)(U_1^{-1}\mathbf{z}) = f(U_2^{-1}U_1^{-1}\mathbf{z}) = \pi(U_1U_2)f(\mathbf{z})$$

The representation on V_2^n is of dimension $n+1$ and one can show that it is irreducible.

By differentiating the action of the group one can explicitly get the action of the Lie algebra and one finds that

$$\pi_*(H)f = -z_1\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}z_2$$

$$\pi_*(X^+)f = -z_2\frac{\partial f}{\partial z_1}$$

$$\pi_*(X^-)f = -z_1\frac{\partial f}{\partial z_2}$$

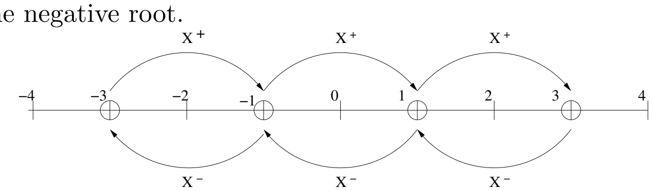
One can explicitly work out how the Lie algebra acts on V_2^n. Note that acting on the monomials we find

$$\pi_*(H)z_1^jz_2^k = (-j+k)z_1^jz_2^k$$

$$\pi_*(X^+)z_1^jz_2^k = -jz_1^{j-1}z_2^{k+1}$$

$$\pi_*(X^-)z_1^jz_2^k = -kz_1^{j+1}z_2^{k-1}$$

The monomials are eigenvectors of $\pi_*(H)$ with eigenvalues $-n, -n+2, \cdots, n-2, n$, these are the weights of the representation. The $\pi_*(X^+)$ are "raising operators" that increase this eigenvalue by 2, i.e, shift the weight by the one positive root, the $\pi_*(X^-)$ are "lowering operators" that decrease it by 2, i.e shift it by the negative root.



Weights, raising and lowering operators for $n = 3$ $SU(2)$ representation

Note that there is a "highest weight" n, corresponding to the monomial z_2^n. This is the one weight space on which $\pi_*(X^+)$ gives zero, and the rest of the representation can be constructed by applying $\pi_*(X^-)$ to a vector in this weight space. Such a vector is called a "highest weight vector".

Another important phenomenon one can see at work here is that of how $SO(3)$ representations are related to representations of its double-cover $SU(2)$. We will examine this in much more detail later when we come to the study of the Spin groups, but what happens here is that for n even these representations are also $SO(3)$ representations. The ones for n odd are only projective representations, i.e. the homomorphism property only holds up to a sign. We will examine the issue of projective representations also in much more detail later on.

One can calculate the Casimir operator for $SU(2)$ with the result

$$C = \frac{1}{2}(X^+X^- + X^-X^+) + \frac{H^2}{4}$$

and it is a second-order differential operator, with eigenvalue $\frac{n}{2}(\frac{n}{2}+1)$ on all elements of V_2^n.

If one takes the tensor product of two irreducible $SU(2)$ representations one gets one that is reducible. It decomposes into irreducibles as follows

$$V_2^n \otimes V_2^m = V_2^{n+m} \oplus V_2^{n+m-1} \oplus \cdots \oplus V_2^{|n-m|}$$

This decomposition of the tensor product goes under the name "Clebsch-Gordan" decomposition. The general issue of how the tensor product of two irreducibles decomposes is an important one we will study in general later. It is related to the representation theory of the symmetric group, by "Schur-Weyl duality".

Finally, one can calculate the character of the irreducible representation. For V_2^n the character as a function on the maximal torus is

$$\begin{aligned} \chi_n(\theta) &= e^{-in\theta} + e^{-i(n+2)\theta} + \cdots + e^{in\theta} \\ &= \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)} \end{aligned}$$

This character formula is the simplest example of a the more general Weyl character formula that we will prove later.

2 Applications to Physics

The group $SU(2)$ appears in two guises in physics. One is as the spin double cover of the rotation group $SO(3)$, the other is as an "internal" symmetry relating types of particles and generalizing the notion of charge.

2.1 Angular Momentum and Spin

At the beginning of this course I explained that in quantum mechanics the state of the world is given by a vector in a complex vector space $\mathcal{H}$, the "Hilbert space" of the theory. There is a unitary representation of any symmetry group G of the theory on the complex vector space $\mathcal{H}$ (which in general will be infinite-dimensional. We can decompose $\mathcal{H}$ into irreducible representations of G. Many physical situations are invariant under rotations in three dimensions and in these situations $\mathcal{H}$ will be an $SO(3)$ representation. We can decompose it into irreducibles

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4 \cdots$$

where $\mathcal{H}_0$ consists of some multiplicity of the trivial representation of dimension 1, $\mathcal{H}_0$ consists of some multiplicity of the representation of weight 2, of dimension 3, etc.

Early on in the theory of atomic spectra it was observed that the possible energy states of a hydrogen atom were organized into groups with the same energy corresponding exactly to the dimensions of the representations noted above. These were labelled, by an integer l corresponding to half of our weight label. The $l = 0$ states were called "s states" and came in singlets, the $l = 1$ states were called "p states" and came in triplets with the same energy, the $l = 2$ states were called "d states" and came in energy multiplets of dimension 5, etc.

In classical mechanics there is a conserved quantity called angular momentum whose existence follows from rotational symmetry. In quantum mechanics the states are said to have "quantized" angular momentum, taking on the value l. Such quantum mechanical states are sometimes thought of as "spinning", with the amount of spinning quantized and given by the integer l.

You can see the multiplicity of the different states by putting the system in a magnetic field. Such a magnetic field is not rotation invariant, so the Hamiltonian operator that generates time translations no longer commutes with the operators that generate rotations. As a result, the states that were in an irreducible representation of the rotation group, all with the same energy in the rotation invariant case, now have slightly different energies and their multiplicity can be observed.

By 1925 it was discovered that certain unexplained features of atomic spectra could be explained if one assumed that the electrons in an atom carried their own intrinsic quantized angular momentum, with the corresponding representation being the fundamental two-dimensional $n = 1$ or angular momentum $l = \frac{1}{2}$ representation of $SU(2)$, which is a projective representation of $SO(3)$. In other words, the Hilbert space had to be changed as follows

$$\mathcal{H} \to \mathcal{H} \otimes V_{\frac{1}{2}}$$

doublings its dimension. The Clebsch-Gordan decomposition is then used to analyze the behavior of the new Hilbert space as an $SU(2)$ representation in terms of the behavior of the two factors. This is part of a story discussed extensively in any quantum mechanics textbook under the title "addition of angular momenta".

2.2 Isotopic Spin

Physicists studying atomic spectra became very familiar with the use of $SU(2)$ representation theory to study the "spin" of particles, corresponding to their behavior under spatial rotations. During the 1930s and 40s the study of nuclear physics began, dealing with the question of how protons and neutrons interact via a strong force to bind together into nuclei. It was soon discovered that the strong force had an $SU(2)$ invariance and this $SU(2)$ was named “isospin symmetry” with its irreducible representations labelled by “isospin” $\frac{1}{2}, 1, \cdots$. The proton and neutron states formed a two-dimensional isospin $\frac{1}{2}$ representation, new particles called “pions” were discovered that fit into a three-dimensional isospin 1 representation.

2.3 $SU(3)$ and the Eightfold-Way

By the early 1960s a large array of new strongly interacting particles had been discovered and there were many attempts to classify them. Physicists at the time were generally unaware of the subject of representation theory beyond the case of $SU(2)$. Gell-Mann finally made the discovery that strongly interacting particles could be organized into irreducible representations of $SU(3)$, with the isospin $SU(2)$ as a subgroup. Since one of the main representations he was using was the eight dimensional adjoint representation, he referred to this as the “Eightfold Way”. His discovery came a year after he had spent a sabbatical in Paris, having lunch daily with Serre, but never talking to him about what he was working on since he was convinced that whatever mathematicians were doing, it had nothing to do with the problem he was working on.

After a couple years it became clear that the $SU(3)$ symmetry Gell-Mann had found was due to existence of three relatively light particles called quarks, which fit into the fundamental three-dimensional representation of $SU(3)$. Quarks are bound together in the states that are observed, which thus live in the tensor product of several copies of the fundamental representation. The decomposition of these tensor products into irreducibles gives the observed representations, including the adjoint.

In the modern “Standard Model” of particles and their interactions, Gell-Mann’s $SU(3)$ plays no fundamental role. It is an approximate symmetry of the theory due to the relatively small masses of three of the six known types of quarks. Another $SU(3)$ symmetry does play an important role, but this one corresponds to the fact that each quark comes in three completely identical states, called “colors”.

Various attempts have been made to produce “Grand Unified” theories of the weak, strong and electromagnetic forces. These mostly use Lie groups of rank at least four. Standard possibilities that have been studied include $SU(5), SO(10)$ and E_6, but to this day there is no experimental support for these theories. For a description of representation theory as used by modern-day particle theorists, a good reference is [1].

References

[1] Georgi, H., *Lie Algebras in Particle Physics,* Benjamin Cummings 1982.