Quantum Theory and Group Representations

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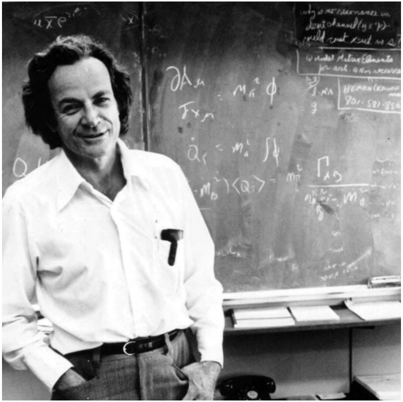
"No One Understands Quantum Mechanics"

"I think it is safe to say that no one understands quantum mechanics"
Richard Feynman
The Character of Physical Law, 1967

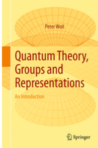


Outline

Today would like to

- Explain how Lie groups and unitary representations are related to quantum mechanics, providing some sort of "understanding" of the structure of the subject.
- Advertise a recently completed book

Quantum Theory, Groups and Representations: An Introduction



Available any day now from Springer, or at
`http://www.math.columbia.edu/~woit/QM/qmbook.pdf`

- Explain some points about the relations between quantum theory and mathematics that became clear to me while writing the book, may be similarly enlightening to others.

What We Really Don't Understand About Quantum Mechanics

While representation theory gives insight into the basic structure of the quantum mechanics formalism, a mystery remains

The mystery of classical mechanics

We don't understand well at all how "classical" behavior emerges when one considers macroscopic quantum systems.

This is the problem of "measurement theory" or "interpretation" of quantum mechanics. Does understanding this require some addition to the fundamental formalism? **Nothing to say today about this.**

What is Quantum Mechanics?

Three Basic Axioms of Quantum Mechanics

- The states of a quantum system are given by vectors $\psi \in \mathcal{H}$ where $\mathcal{H}$ is a complex vector space with a Hermitian inner product. In the finite-dimensional case, for vectors $\mathbf{v}, \mathbf{w} \in \mathbf{C}^n$, the inner product is

$$\mathbf{v} \cdot \mathbf{w} = \overline{v_1} w_1 + \overline{v_2} w_2 + \cdots + \overline{v_n} w_n$$

- Observables correspond to self-adjoint linear operators on $\mathcal{H}$. In the finite dimensional case these are matrices M satisfying $M = M^\dagger = \overline{M^T}$.
- There is a distinguished observable, the Hamiltonian H, and time evolution of states ψ is given by the Schrödinger equation ($\hbar = 1$)

$$i \frac{d}{dt} \psi = H \psi$$

Where (what mathematical structure) do these axioms come from?

Our claim is that a

unitary representation of the Lie algebra $\mathfrak{g}$ of a Lie group G

gives exactly these mathematical structures. To explain this mathematical structure need to explain

- What is a Lie group?
- What is a Lie algebra?
- What is a unitary representation of a Lie algebra?

What is a Lie group?

For our purposes, best to think of a Lie group G as a group of matrices, with product the matrix product. Some examples are

- The group $SO(2)$ of rotations of the plane. This is isomorphic to $U(1)$, the group of rotations of the complex plane, by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2) \leftrightarrow e^{i\theta} \in U(1)$$

- The group $U(n)$ of unitary transformations of $\mathbf{C}^n$. These are matrices U with $U^\dagger = U^{-1}$.
- The group $GL(n, \mathbf{R})$ of all invertible linear transformations of $\mathbf{R}^n$
- The group $SO(3)$ of rotations of $\mathbf{R}^3$.
- The additive group $\mathbf{R}$, which can be written as matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

If G is a group of matrices M, near the identity matrix we can write such group elements using the exponential as

$$M = e^X = \mathbf{1} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

The X are the elements of $\mathfrak{g}$.
The possible non-commutativity of the group elements M is reflected in the non-zero commutator of elements of the Lie algebra. One has (Baker-Campbell-Hausdorff formula):

$$e^{X_1}e^{X_2} = e^{X_1+X_2+\frac{1}{2}[X_1,X_2]+\cdots}$$

where

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

is the commutator, and the higher order terms above can be written as iterated commutators of X_1 and X_2.

Examples of Lie algebras

The Lie algebras corresponding to our examples of Lie groups are

- For $G = SO(2)$ the Lie algebra is $\mathfrak{g} = \mathbf{R}$, which can be identified with the rotation angle.
- For $G = U(n)$ elements of the Lie algebra $\mathfrak{u}(n)$ are n by n complex matrices X that are "skew-adjoint": $X^\dagger = -X$. Exponentiating these gives unitary matrices.
- For $G = GL(n, \mathbf{R})$ elements of the Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ are all n by n real matrices.
- For $G = SO(3)$ elements of the Lie algebra $\mathfrak{so}(3, \mathbf{R})$ are 3 by 3 antisymmetric real matrices.
- For $G = \mathbf{R}$ the Lie algebra is again $\mathbf{R}$, identified with matrices

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

What is a unitary representation of a Lie group?

If you think of groups as "symmetries" of some object, the group is just the set of possible transformations. The object acted on together with the action of the group on it is the representation. We will be interested in "linear representations", where the object is a vector space and the transformations are linear transformations. In particular we will take our vector spaces to be complex ($\mathbf{C}^n$ in the finite dimensional case). "Unitary" means that the transformations preserve the Hermitian inner product on $\mathbf{C}^n$, so are in $U(n)$. We have the following abstract definition:

Unitary representation of a Lie group

A unitary representation π on $\mathbf{C}^n$ of a Lie group G is a homomorphism $\pi : G \to U(n)$. This means that for every $g \in G$ we have a unitary n by n matrix $\pi(g) \in U(n)$, and these satisfy

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$$

Examples of unitary representations of Lie groups

If $G = U(n)$, taking π to be the identity map gives a unitary representation on $\mathbf{C}^n$, the "defining representation".

There are many more possibilities. We will see that quantum mechanics produces more examples.

What is a unitary representation of a Lie algebra?

Given a unitary representation π of a Lie group G we can define a unitary representation π' of the Lie algebra $\mathfrak{g}$ to be the derivative at the identity:

$$\pi'(X) = \frac{d}{dt}\pi(e^{tX})_{t=0}$$

These will give a map

$$\pi' : X \in \mathfrak{g} \to \pi'(X) \in \mathfrak{u}(n)$$

satisfying the so-called Lie algebra homomorphism property

$$\pi'([X_1, X_2]) = [\pi'(X_1), \pi'(X_2)]$$

These linear approximations to the representations π are much easier to work with than the non-linear maps π

What is a unitary representation of a Lie algebra?

A unitary representation of a Lie algebra is giving us a set of linear operators $\pi'(X)$, one for each element X of the Lie algebra. These act in the finite dimensional case on a complex vector space $\mathbf{C}^n$ with Hermitian inner product. Since they are in the Lie algebra $\mathfrak{u}(n)$ the $\pi(X)$ are skew-adjoint

$$(\pi(X))^\dagger = -\pi(X)$$

but if we multiply by i we get self-adjoint operators

$$(i\pi(X))^\dagger = (i\pi(X))$$

We see that we have self-adjoint operators acting on a complex vector space with Hermitian inner product, the same structure that appears in quantum mechanics.

Relating Quantum Mechanics and Representations

Basic Principle

Quantum mechanical systems carry unitary representations π of various Lie groups G on their state spaces $\mathcal{H}$. The corresponding Lie algebra representations π' give the operators for observables of the system.

Significance for physicists

Identifying observables of one's quantum system as coming from a unitary representation of a Lie group allows one to use representation theory to say many non-trivial things about the quantum system.

Significance for mathematicians

Whenever physicists have a physical system with a Lie group G acting on its description, the state space $\mathcal{H}$ and the operators for observables should provide a unitary representation of G. This is a fertile source of interesting unitary representations of Lie groups.

$G = U(1)$

If a physical system has a group $G = U(1)$ acting on it, we expect the quantum system to be a unitary representation of $U(1)$. It turns out that all unitary representations of $U(1)$ can by decomposed into one-dimensional ones π_N, classified by an integer N. The π_N are given by

$$\pi_N(e^{i\theta}) = e^{iN\theta}$$

The corresponding Lie algebra representation is given by

$$\pi'_N(\theta) = iN\theta$$

Multiplying by i to get a self-adjoint operator, we get $i(iN)\theta = -N\theta$. These self-adjoint operators are characterized by an integer N. So, for physical systems with $U(1)$ acting, we expect states to be characterized by integers.

The “quantum” comes from $U(1)$

Where “quantum” comes from

In a very real sense, this is the origin of the name “quantum”: many physical systems have a $U(1)$ group acting on them, so states are characterized by an integer, thus we get “quantization” of some observables.

Some examples are

- Spatial rotations about some chosen axis, e.g. the z-axis. We find that atomic energy levels are classified by an integer: “angular momentum in z-direction.”
- Electromagnetic systems have a “gauge” symmetry given by a $U(1)$ action. Such states are classified by an integer: the electric charge.
- The classical phase space of a harmonic oscillator has a $U(1)$ symmetry. The quantized harmonic oscillator has states classified by an integer: the number of “quanta”.

Example: time translations, $G = \mathbf{R}$

Physical systems all come with an action of the group $G = \mathbf{R}$ by time translations. In quantum mechanics we expect to have a unitary representation of the Lie algebra of this group (the Lie algebra of $\mathbf{R}$ is also $\mathbf{R}$). Such a unitary representation is given by a skew-adjoint operator which we write as $-iH$ where H is self-adjoint. This is just the Hamiltonian operator, and the Schrödinger equation

$$\frac{d}{dt}\psi = -iH\psi$$

is just the statement that this operator gives the infinitesimal action of time translations.

Unitary representations of $G = \mathbf{R}$ can, like those of $U(1)$, be decomposed into one-dimensional representations. In this case these representations are characterized by an arbitrary real number rather than an integer. For the case of the Hamiltonian, this real number will be the energy.

Example: translations in space, $G = \mathbf{R}^3$

Physics takes place in a space $\mathbf{R}^3$. One can consider the Lie group $G = \mathbf{R}^3$ of spatial translations.

The quantum state space $\mathcal{H}$ will provide a unitary representation of this group. The Lie algebra representation operators are called the "momentum operators"

$$P_j, \quad j = 1, 2, 3$$

and states can be decomposed into one-dimensional representations, which now will be characterized by an element of $\mathbf{R}^3$, the momentum vector. When states ψ are taken to be functions on space ("wavefunctions"), the momentum operators are related to infinitesimal translation by

$$\frac{\partial}{\partial x_j}\psi = iP_j\psi$$

Example: Rotations, $G = SO(3)$

The group $G = SO(3)$ acts on physical space $\mathbf{R}^3$ by rotations about the origin.

Unitary representations of $SO(3)$ break up into direct sums of irreducible components π_l on $\mathbf{C}^{2l+1}$, where $l = 0, 1, 2, \ldots$. Physicists call these the "angular momentum l" representations. It is a major part of any course in quantum mechanics to discuss the angular momentum operators L_1, L_2, L_3. These are the Lie algebra representation operators coming from the fact that the quantum mechanical state space has a unitary representation of $SO(3)$. Unlike the P_j, the operators L_j do not commute, providing a much more non-trivial example of a Lie algebra and its representations.

Example: The Euclidean group of translations and rotations of $\mathbf{R}^3$

One can put together the two previous examples and consider the group $G = E(3)$ of all translations and rotations of $\mathbf{R}^3$. Studying the possible unitary representations of this group, one recovers essentially the usual quantum theory of a free particle moving in $\mathbf{R}^3$.
This generalizes to the relativistic case of four-dimensional space-time, where the symmetry group is the Poincaré group. Its unitary representations can be decomposed into pieces which correspond to the possible quantum mechanical systems of relativistic free particles.

Lie Group Representations and Symmetries

When the action of a Lie group G on a quantum system commutes with the Hamiltonian operator, G is said to be a "symmetry group" of the system, acting as "symmetries" of the quantum system. Then one has

Conservation Laws

Since the observable operators $\mathcal{O}$ corresponding to Lie algebra elements of G commute with H, which gives infinitesimal time translations, if a state is an eigenstate of $\mathcal{O}$ with a given eigenvalue at a given time, it will have the same property at all times. The eigenvalue will be "conserved."

Degeneracy of Energy Eigenstates

Eigenspaces of H will break up into irreducible representations of G. One will see multiple states with the same energy eigenvalue, with dimension given by the dimension of an irreducible representation of G.

Lie Group Representations Are Not Always Symmetries

When G acts on a classical system, the state space of a corresponding quantum system will be a unitary representation of G, even when this action of G on the state space is not a symmetry, i.e. does not commute with the Hamiltonian.

The basic structure of quantum mechanics involves a unitary group representation in a much more fundamental way than the special case where there are symmetries. This has to do with a group (and its Lie algebra) that already is visible in classical mechanics. This group does not commute with any non-trivial Hamiltonian, but it plays a fundamental role in both the classical and quantum theories.

Classical (Hamiltonian) Mechanics

The theory of classical mechanical systems, in the Hamiltonian form, is based on the following structures

- An even dimensional vector space $\mathbf{R}^{2n}$, called the "phase space" M, with coordinate functions that break up into position coordinates $q_1, \cdots, q_n$ and momentum coordinates $p_1, \cdots, p_n$.
- The "Poisson bracket", which takes as arguments two functions f, g on M and gives a third such function

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)$$

A state is a point in M, observables are functions on M. There is a distinguished function, h, the Hamiltonian, and observables evolve in time by

$$\frac{df}{dt} = \{f, h\}$$

First-year Physics Example

A classical particle of mass m moving in a potential $V(q_1, q_2, q_3)$ in $\mathbf{R}^3$ is described by the Hamilton function h on phase space $\mathbf{R}^6$

$$h = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$

where the first term is the kinetic energy, the second the potential energy. Calculating Poisson brackets, one finds

$$\frac{dq_j}{dt} = \{q_j, h\} = \frac{p_j}{m} \implies p_j = m\frac{dq_j}{dt}$$

and

$$\frac{dp_j}{dt} = \{p_j, h\} = -\frac{\partial V}{\partial q_j}$$

where the first equation says momentum is mass times velocity, and the next is Newton's second law ($F = -\nabla V = m\mathbf{a}$).

The Lie algebra of functions on phase space

The Poisson bracket has the right algebraic properties to play the role of the commutator in the matrix case, and make the functions on phase space a Lie algebra (infinite dimensional). This Lie algebra (and its corresponding Lie group) are responsible for much of the structure of Hamiltonian mechanics.

There are finite-dimensional subalgebras, including (for the case $n = 1$)

- The functions with basis $1, q, p$, since
$$\{q, p\} = 1$$
and all other Poisson brackets amongst these vanish. This is the "Heisenberg Lie algebra". Note that this Lie algebra already appears in classical mechanics, while it usually first appears in discussion of quantum mechanics.
- The functions with basis q^2, p^2, qp. The Poisson bracket relations of these are the same as the commutator relations of a basis of the Lie algebr of $SL(2, \mathbf{R})$ (invertible matrices with determinant 1).

Heisenberg Commutation Relations

The quantum theory of a single particle includes not just momentum operators P, but also position operators Q, satisfying the Heisenberg commutator relation

$$[Q, P] = i\mathbf{1}$$

Soon after the discovery (1925) by physicists of this relation, Hermann Weyl realized that it is the Lie bracket relation satisfied by a certain Lie algebra, the Heisenberg Lie algebra mentioned above (it's isomorphic to the Lie algebra of strictly upper-triangular 3 by 3 matrices). There's a corresponding group, the Heisenberg group (sometimes called the "Weyl group" by physicists).

Dirac and Quantization

Dirac noticed the similarity of the Poisson bracket relation $\{q, p\} = 1$ and the Heisenberg operator relation $[Q, P] = i$ and proposed the following method for "quantizing" any classical mechanical system

Dirac Quantization

To functions f on phase space, quantization takes

$$f \to \mathcal{O}_f$$

where $\mathcal{O}_f$ are operators satisfying the relations

$$\mathcal{O}_{\{f,g\}} = -i[\mathcal{O}_f, \mathcal{O}_g]$$

Quantization and Symplectomorphisms

Dirac's proposal can be stated very simply in terms of representation theory, it just says

Dirac quantization

A quantum system is a unitary representation of the Lie algebra of functions on phase space.

The Lie algebra representation is a homomorphism taking functions on phase space to operators, with the Poisson bracket going to the commutator. The factor of i appears because of the difference between the physicist's self-adjoint and the mathematicians skew-adjoint unitary representation operators. Unfortunately it turns out this doesn't work....

Bad News and Good News

It turns out that if one tries to follow Dirac's suggestion one finds

Bad News, Groenewold-van Hove

No-go theorem: there is a representation that quantizes polynomial functions on phase space of degree up to two, but this can't be done consistently for higher degrees.

but also

Good News, Stone-von Neumann

The quantization of polynomials of degree ≤ 2 is unique, there is only one possible unitary representation of this Lie algebra (fixing $\hbar$, and that integrates to a representation of the group). This is what physicists know and love as "canonical quantization".

Summary

Representation theory is a subject that brings together many different areas of mathematics, providing an often surprising “unification” of mathematics.

It is remarkable that exactly this sort of mathematics underlies and gives us some understanding of our most fundamental physical theory, quantum mechanics.

There likely is much more to be learned about the relation of fundamental physics and mathematics, with representation theory like to play a major role.

Thanks for your attention!