

Hyperfunctions and Imaginary Time Quantization *Under Construction*

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1 Introduction

Quantum theory, even in its relativistic versions, involves a profound asymmetry between spatial and time directions. While momentum eigenvalues can be arbitrarily positive or negative, energy eigenvalues go in one direction only, which by convention is that of positive energies. Having states supported only at non-negative energies implies (by Fourier transformation) that, as a function of time, states can be analytically continued in one complex half plane, not the other, and this is fundamental to the use of Euclidean methods in quantum theory. While states are conventionally described as square-integrable functions or as distributions, this analytic continuation property makes it natural to describe them in terms of hyperfunctions. Hyperfunctions generalize the notion of functions on the real number line by considering instead boundary values of

holomorphic functions on a complex half plane. Like distributions, they can be thought of as elements of a dual space to a space of well-behaved test functions, in this case the space of real analytic functions.

The use of hyperfunctions is natural for even the simplest quantum systems: the harmonic oscillator and free particles. A system of free particles is best described by a quantum field theory, which in some sense is simply a collection of harmonic oscillators. While none of the ideas involved are original, it seemed like a worthwhile project to write down explicitly the hyperfunction formulation of harmonic oscillators and free fields. For a more extensive discussion of hyperfunctions in this context, a good source is chapter 9 of Roger Penrose's *The Road to Reality* [4].

2 Hyperfunctions on \mathbf{R}

Solutions to wave equations are conventionally discussed using the theory of distributions, since even the simplest plane-wave solutions are delta-functions in energy-momentum space. Distributions are generalizations of functions that can be defined as elements of the dual space (linear functionals) of some well-behaved set of functions, for instance smooth functions of rapid decrease (Schwartz functions) for the case of tempered distributions. The theory of hyperfunctions gives a further generalization, providing a dual of an even more restricted set of functions, analytic functions. Working with analytic functions allows (unlike for distributions) the use of complex variable methods and analytic continuation. Two references which contain extensive discussions of the theory of hyperfunctions with applications are [3] and [2]. In this section we'll begin by describing the basic hyperfunction set-up, in the next show how to use it for the case of the harmonic oscillator.

To motivate the definition of a hyperfunction, consider the boundary values of a holomorphic function Φ_+ on the open upper half plane. These give a generalization of the usual notion of distribution, by considering the linear functional on analytic functions on \mathbf{R}

$$f \rightarrow \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \Phi_+(t + i\epsilon) f(t + i\epsilon) dt$$

Usual distributions are often written with a formal integral symbol denoting the linear functional. In the case of hyperfunctions, this is no longer formal, but becomes (a limit of) a conventional integral of a holomorphic function in the complex plane, so contour deformation and residue theorem techniques can be applied to its evaluation.

It is more convenient to have a definition involving symmetrically the upper and lower complex half-planes. The space $\mathcal{B}(\mathbf{R})$ of hyperfunctions on \mathbf{R} can be defined as equivalence classes of pairs of functions (Φ_+, Φ_-) , where Φ_+ is a holomorphic function on the open upper half-plane, Φ_- is a holomorphic function on the open lower half-plane. Pairs $(\Phi_{1,+}, \Phi_{1,-})$ and $(\Phi_{2,+}, \Phi_{2,-})$

are equivalent if

$$\Phi_{2,+} = \Phi_{1,+} + \psi, \quad \Phi_{2,-} = \Phi_{1,-} + \psi$$

for some globally holomorphic function ψ . We'll then write a hyperfunction as

$$\phi = [\Phi_+, \Phi_-]$$

The derivative ϕ' of a hyperfunction ϕ is given by taking the complex derivatives of the pair of holomorphic functions representing it

$$\phi' = [\Phi'_+, \Phi'_-]$$

As a linear functional on analytic functions, the hyperfunction ϕ is given by

$$f \rightarrow \oint_{-\infty}^{\infty} \phi(t) f(t) dt \equiv \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} (\Phi_+(t + i\epsilon) - \Phi_-(t - i\epsilon)) f(t) dt$$

We'll use coordinates t on \mathbf{R} , $z = t + i\tau$ on \mathbf{C} since our interest will be in physical applications involving functions of time t , as well as their analytic continuations to imaginary time τ .

One way to get hyperfunctions is by choosing a function $\Phi(z)$ on \mathbf{C} , holomorphic away from the real axis \mathbf{R} , and taking

$$\phi = [\Phi|_{UHP}, \Phi|_{LHP}]$$

For example, consider the function

$$\Phi = \frac{i}{2\pi} \frac{1}{z - \alpha}$$

where $\alpha \in \mathbf{R}$. The corresponding hyperfunction will be the distribution given by the limit

$$\phi(t) = \lim_{\epsilon \rightarrow 0^+} \frac{i}{2\pi} \left(\frac{1}{t + i\epsilon - \alpha} - \frac{1}{t - i\epsilon - \alpha} \right) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \frac{1}{(t - \alpha)^2 + \epsilon^2}$$

The limit on the right-hand side is well-known as a way to describe the delta function distribution $\delta(t - \alpha)$ as a limit of functions. Using contour integration methods one finds that the hyperfunction version of the delta function behaves as expected since

$$\oint_{-\infty}^{\infty} \frac{i}{2\pi} \frac{1}{t - \alpha} f(t) dt = \text{Res}_\alpha \frac{1}{z - \alpha} f(z) = f(\alpha)$$

One would like to define a Fourier transform for hyperfunctions, with the same sort of definition as an integral in the usual case, so

$$\mathcal{F}(\phi)(E) = \tilde{\phi}(E) = \frac{1}{\sqrt{2\pi}} \oint_{-\infty}^{\infty} e^{iEt} \phi(t) dt$$

with the inverse Fourier transform defined by

$$\mathcal{F}^{-1}(\tilde{\phi})(t) = \phi(t) = \frac{1}{\sqrt{2\pi}} \oint_{-\infty}^{\infty} e^{-iEt} \tilde{\phi}(E) dE$$

(the sign convention is chosen to agree with the usual physical interpretation of the Fourier transform variable for time t to be energy). The problem with this though is that the Fourier transform and its inverse don't take functions holomorphic on the upper or lower half plane to functions with the same property.

One can however define a Fourier transform for hyperfunctions by taking advantage of the fact for functions $f(x)$ supported on $x > 0$ (respectively $x < 0$)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx$$

is holomorphic in the lower half (respectively upper half) ξ plane (since the exponential falls off there). The decomposition of a hyperfunction $\phi(t)$ into limits of functions Φ_+, Φ_- on the upper and lower half planes corresponds to decomposition of $\tilde{\phi}(E)$ into hyperfunctions $\tilde{\phi}_-(E), \tilde{\phi}_+(E)$ supported for negative and positive E respectively.

For an example, consider the hyperfunction version of a delta function supported at $E = \alpha, \alpha > 0$, then

$$\tilde{\phi}(E) = \tilde{\phi}_+(E) = \frac{i}{2\pi} \frac{1}{E - \alpha} \equiv \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \left(\frac{1}{E + i\epsilon - \alpha} - \frac{1}{E - i\epsilon - \alpha} \right)$$

This has as inverse Fourier transform the hyperfunction

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \oint_{-\infty}^{\infty} \frac{i}{2\pi} \frac{1}{E - \alpha} e^{-iEt} \tilde{\phi}(E) dE = \frac{1}{\sqrt{2\pi}} e^{-i\alpha t}$$

which has a representation as

$$\phi(t) = [0, -\frac{1}{\sqrt{2\pi}} e^{-i\alpha z}]$$

The Fourier transform of this will be, as expected

$$\begin{aligned} \tilde{\phi}(E) &= \frac{1}{\sqrt{2\pi}} \oint_{-\infty}^{\infty} e^{iEt} \phi(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iEt} \frac{1}{\sqrt{2\pi}} e^{-i\alpha t} dt \end{aligned}$$

but this needs to be interpreted as a sum of integrals for t negative and t positive

$$\begin{aligned} &= \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-\infty}^0 e^{i(E-i\epsilon-\alpha)t} dt + \int_0^{\infty} e^{i(E+i\epsilon-\alpha)t} dt \right) \\ &= \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} \left(\frac{1}{E + i\epsilon - \alpha} - \frac{1}{E - i\epsilon - \alpha} \right) \end{aligned}$$

The conjugation map for hyperfunctions is given by

$$\phi(t) = [\Phi_+, \Phi_-] \rightarrow \bar{\phi}(t) = [\bar{\Phi}_-, \bar{\Phi}_+]$$

or if given by a function Φ holomorphic away from the real axis, by

$$\Phi(z) \rightarrow \bar{\Phi}(\bar{z})$$

3 Hyperfunctions on the circle

For periodic functions, or equivalently, functions on the unit circle, the theory of hyperfunctions is simpler than in the case of the real number line. Instead of the upper half plane, the circle bounds the unit disk, and one gets hyperfunctions as boundary values of holomorphic functions on the open unit disk. One can take the circle to be the equator of a Riemann sphere, in which case the analog of $[\Phi_+, \Phi_-]$ is a pair of functions, one holomorphic on the open upper hemisphere, the other holomorphic on the open lower hemisphere.

The Fourier transform now takes a hyperfunction on S^1 to the coefficients a_n , $n \in \mathbf{Z}$ of a Fourier series. Boundary values of functions holomorphic on the upper hemisphere correspond to Fourier coefficients satisfying $a_n = 0$ for $n < 0$, those with $a_n = 0$ for $n > 0$ correspond to boundary values of functions holomorphic on the lower hemisphere. One sees the same phenomenon as in the case of the real line: the analog of positive energy hyperfunctions is those that have Fourier coefficients satisfying $a_n = 0$ for $n < 0$. The global holomorphic functions in this case are just the constants, those with only a_0 non-zero.

In the usual theory of Fourier series for functions, different classes of functions imply different sorts of fall-off behavior on the Fourier coefficients. For real analytic functions, the coefficients a_n must fall off faster than $e^{-c|n|}$ for some $c > 0$. Hyperfunctions allow one to make sense of a very large class of Fourier series, just requiring that the Fourier coefficients grow at less than exponential rate as $n \rightarrow \pm\infty$.

The discrete series representations of the non-compact Lie group $SL(2, \mathbf{R})$ can naturally be constructed using hyperfunctions on the circle. The group $SL(2, \mathbf{R})$ acts on the Riemann sphere, with orbits the upper hemisphere, the lower hemisphere, and the equator. The discrete series representations are hyperfunctions on the equator, boundary values of holomorphic sections of a line bundle on either the upper or lower hemisphere. For more about this, see section 10.1 of [1]. For a more general discussion of hyperfunctions on the circle and their relation to hyperfunctions on \mathbf{R} , see the previously mentioned chapter 9 of [4].

4 Hyperfunction solutions of the harmonic oscillator equation

The classical one-dimensional harmonic oscillator equation

$$\left(\frac{d^2}{dt^2} + \alpha^2\right)\phi = 0 \tag{1}$$

(for $\phi(t)$ a real-valued function and α a positive real number) has solutions

$$\phi(t) = A \cos(\alpha t) + B \sin(\alpha t)$$

We'll denote this two real dimensional space of solutions by \mathcal{M} , where \mathcal{M} can be thought of as the phase space of the system (the usual canonical variables parametrizing \mathcal{M} are the initial value data $\phi(0) = A$, $\dot{\phi}(0) = \alpha B$).

Quantization of the harmonic oscillator proceeds by first complexifying \mathcal{M} and choosing a decomposition

$$\mathcal{M} \otimes \mathbf{C} = \mathcal{M}_+ \oplus \mathcal{M}_-$$

of complex solutions into positive and negative energy solutions. Elements of \mathcal{M}_+ are of the form

$$\phi(t) = C_+ e^{-i\alpha t}$$

and have positive energy α since $i\frac{d}{dt}$ is the Hamiltonian or energy operator ($\hbar = 1$). Elements of \mathcal{M}_- are negative energy solutions, given by

$$\phi(t) = C_- e^{i\alpha t}$$

The Fourier transform

$$\phi(t) \rightarrow (\mathcal{F}\phi)(E) = \tilde{\phi}(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iEt} \phi(t) dt$$

turns the differential equation for $\phi(t)$ into the equation

$$(-E^2 + \alpha^2)\tilde{\phi}(E) = 0 \tag{2}$$

This equation has no functions as solutions, but does have distributional solutions supported at $E = \pm\alpha$. The complex solutions of equation 2 are given by the distributions

$$\tilde{\phi}(E) = \frac{1}{\sqrt{2\pi}} (C_+ \delta(E - \alpha) + C_- \delta(E + \alpha))$$

Again the space of complex solutions breaks up into positive energy (proportional to $\delta(E - \alpha)$) and negative energy (proportional to $\delta(E + \alpha)$) subspaces.

4.1 Hyperfunction solutions

Instead of working with distributional solutions to equations 1 or 2, one can instead consider hyperfunction solutions $\phi(t)$ or their Fourier transforms $\tilde{\phi}(E)$. If we consider functions on the complex energy plane, with complex coordinate $s = E + i\sigma$, then equation 2 becomes

$$(-s^2 + \alpha^2)\Psi(s) = 0$$

where $\Psi(s)$ is a function on the complex energy plane, holomorphic away from the real energy line, with corresponding hyperfunction solution

$$\tilde{\phi}(E) = [\Psi|_{UHP}, \Psi|_{LHP}] = \lim_{\epsilon \rightarrow 0^+} (\Psi(E + i\epsilon) - \Psi(E - i\epsilon))$$

This has solutions

$$\Psi(s) = \frac{g(s)}{s^2 - \alpha^2}$$

where $g(s)$ is holomorphic in a neighborhood of the real number line. But then, as a hyperfunction

$$\Psi(s) = \frac{1}{2\alpha} \left(\frac{1}{s - \alpha} - \frac{1}{s + \alpha} \right) g(s) = \frac{1}{2\alpha} \left(g(\alpha) \frac{1}{s - \alpha} - g(-\alpha) \frac{1}{s + \alpha} \right)$$

As in the distributional case, in the hyperfunction case there is a two-dimensional space of solutions, which breaks up into positive and negative energy subspaces. Basis elements for the hyperfunction solutions are:

•

$$\frac{i}{2\pi} \frac{1}{s - \alpha}$$

which is supported at positive energy α and is the linear functional $\delta(E - \alpha)$ on analytic functions of E .

Its Fourier transform is the hyperfunction

$$[0, -e^{-i\alpha z}]$$

•

$$\frac{i}{2\pi} \frac{1}{s + \alpha}$$

which is supported at negative energy $-\alpha$ and is the linear functional $\delta(E + \alpha)$ on analytic functions of E .

Its Fourier transform is the hyperfunction

$$[e^{i\alpha z}, 0]$$

4.2 Hyperfunctions and Euclidean methods

The advantage of describing the space of complexified solutions of equation 1 using hyperfunctions is that it gives a new way of dealing with analytic continuation from time t to imaginary time τ . The space \mathcal{M}_+ of positive energy solutions is given by hyperfunction solutions of the form

$$[\Phi_+, 0]$$

where Φ_+ is a holomorphic function on the open upper half plane. These can be characterized either in terms of their dependence on t (as a boundary value along the real axis) or their dependence on τ (as a function of τ along the positive imaginary axis).

5 Quantization of free fields in Minkowski and Euclidean space

Free field theories can be thought of as infinite collections of decoupled harmonic oscillators, and quantized in much the same way as we have seen in the harmonic oscillator case. Instead of the harmonic oscillator equation ?? there will be a field equation given by a higher dimensional partial differential equations. The complexified space of solutions will again break up as

$$\mathcal{M} \otimes \mathbf{C} = \mathcal{H}_1 + \overline{\mathcal{H}}_1$$

except that now \mathcal{H}_1 will be an infinite dimensional complex vector space with positive Hermitian inner product, rather than the one-dimensional complex space \mathbf{C} with its usual Hermitian inner product. The quantized free field will be constructed treating time in the same way as we have in the harmonic oscillator case, using hyperfunction methods.

5.1 Klein-Gordon fields

The free relativistic scalar field theory in $d + 1$ space-time dimensions will correspond to the case of the Klein-Gordon wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi = 0 \tag{3}$$

for $\phi(t, \mathbf{x})$ depending on a time coordinate t and spatial coordinates $\mathbf{x} = (x_1, x_2, \dots, x_d)$. Here Δ is the Laplacian

$$\Delta = \frac{\partial^1}{\partial x_1^2} + \frac{\partial^1}{\partial x_2^2} + \dots + \frac{\partial^1}{\partial x_d^2}$$

Fourier transforming in the spatial coordinates

$$\tilde{\phi}(t, \mathbf{p}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbf{R}^d} \phi(t, \mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{p}} d^d \mathbf{x}$$

equation 3 becomes

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\tilde{\phi}(t, \mathbf{p}) = \left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right)\tilde{\phi}(t, \mathbf{p}) = 0$$

where

$$\omega_{\mathbf{p}}^2 = |\mathbf{p}|^2 + m^2$$

Also Fourier transforming in the t coordinate, equation 3 becomes

$$(-E^2 + \omega_{\mathbf{p}}^2)\tilde{\phi}(E, \mathbf{p}) = 0$$

As a function of \mathbf{x} or \mathbf{p} , ϕ and the Fourier transforms $\tilde{\phi}$ will be (tempered) distributions, as a function of t or E they will be treated as hyperfunctions. For a fixed value of \mathbf{p} , equation 3 becomes exactly the harmonic oscillator equation, with $\alpha = \omega_{\mathbf{p}}^2$.

References

- [1] R. J. Baston and M.G. Eastwood, *The Penrose transform: its interaction with representation theory*, Oxford, 1989.
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- [3] Isao Imai, *Applied hyperfunction theory*, Mathematics and its applications, vol. 8, Kluwer Academic Publishers, 1992.
- [4] Roger Penrose, *The road to reality*, Jonathan Cape, 2004.