PHYSICS 253a

Gauge theories
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Assignment 1

√1. In class we saw that the Hamiltonian for a gas of interacting Fermi particles could be written as

$$H = \int d^3\vec{r}\ \psi^\dagger(\vec{r})\left(\frac{-\hbar^2\nabla^2}{2m}\right)\psi(\vec{r}) + \tfrac{1}{2}\int d^3r\,d^3r'\,\psi^\dagger(\vec{r})\psi^\dagger(\vec{r}')V(\vec{r}-\vec{r}')\psi(\vec{r}')\psi(\vec{r})$$

Show that for a gas of spin - $\frac{1}{2}$ particles subject to periodic boundary conditions in a box of volume V, and interacting through a spin-independent potential, the Hamiltonian takes the form

$$H = \sum_{\vec{k}\lambda} \frac{\hbar^2\vec{k}^2}{2m} a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} + \frac{1}{2V}\sum_{\vec{q}} \tilde{V}(\vec{q}) \sum_{\substack{\vec{k}\lambda,\\ \vec{p}\lambda'}} a^\dagger_{\vec{k}+\vec{q}\lambda} a^\dagger_{\vec{p}-\vec{q}\lambda'} a_{\vec{p}\lambda'} a_{\vec{k}\lambda}$$

where λ, λ', are spin indices, and $\tilde{V}(\vec{q}) = \int d^3r e^{-i\vec{q}\cdot\vec{r}} V(\vec{r})$.
[Note: The normalized single-particle wavefunctions are $\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}\ u_\lambda$, with $u_\uparrow = \binom{1}{0}$ and $u_\downarrow = \binom{0}{1}$].

✓ 2. At zero temperature, a non-interacting Fermi gas is described by the many-body state vector $|F\rangle$ which has all single particle states with wave vectors of magnitude less than the Fermi wavevector $k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$ occupied, and all other states empty. The ground state energy of the non-interacting Fermi gas is the familiar

$$E_0 = \tfrac{3}{5}N\varepsilon_F \qquad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

√(a) Show, using first-order stationary state perturbation theory, that the correction to the ground state energy of the Fermi gas arising from two-body interactions is

$$E_1 = \frac{N(N-1)}{2V}\tilde{V}(\vec{q}=0) - \sum_{\vec{q}\neq 0}\tilde{V}(\vec{q})\ I(\vec{q})$$

where $I(\vec{q}) \equiv \frac{1}{2V}\sum_{\vec{k}\lambda}\theta(k_F-|\vec{k}|)\,\theta(k_F-|\vec{k}+\vec{q}|) =$

$$\frac{2}{2}\int\frac{d^3k}{(2\pi)^3}\,\theta(k_F-|k|)\,\theta(k_F-|\vec{k}+\vec{q}|)$$

0 for k>kF 0 for k>

(b) Evaluate $I(\vec{q})$.

3. A simple model of a plasma is the "jellium" model in which the positive ions are treated as an immobile uniform background charge, and only the dynamics of the ionized electrons are studied. A careful analysis leads to the expression

$$\tilde{V}(q) = \frac{4\pi e^2}{q^2}$$

for the Fourier transform of the potential. The effect of the uniform background is to make $\tilde{V}(\vec{q}=0)$ identically zero.

(a) Using this information and the results of problem 2 calculate the ground state energy of a large self-interacting "jellium" plasma. Please express your answer solely in terms of $N, e, a_0 = \hbar^2/me^2$ (the Bohr radius), and the average dimensionless interparticle separation $r_s \equiv \left(\frac{3V}{4\pi N}\right)^{1/3}/a_0$ (and any numerical factors that are necessary).

(b) Make a rough sketch of the ground state energy as a function of r_s. What is the equilibrium value of r_s? For a sodium crystal $E/N = -1.13\text{eV}$ and $r_s = 3.96$. Compare these experimental values with your theoretical results.

4. The cross section for the scattering of slow neutrons by a fluid consisting of many identical atoms is proportional to the Fourier transform $S(\vec{q},\omega)$ of the density-density correlation function; i.e.

$$S(\vec{q},\omega) \equiv \int e^{i(\omega t-\vec{q}\cdot\vec{r})} \langle\psi|n(\vec{r}t)\; n(00)\;|\psi\rangle d\vec{r}\; dt$$

where $n(\vec{r},t) = \psi^\dagger(\vec{r}t)\;\psi(\vec{r}t)$ is the density operator, $\hbar\omega$ and $\hbar\vec{q}$ are the energy and momentum transfers to the fluid respectively. If the energy spectrum of outgoing neutrons is not broad in relation to the incident energy and the detector does not discriminate between different energies, the angular distribution measured for the scattered neutrons will be proportional to the static form factor

$$S(\vec{q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\vec{q}\omega)\, d\omega$$

$$= \int e^{-i\vec{q}\cdot\vec{r}} \langle\psi| n(\vec{r},0)\; n(00)\; |\psi\rangle\, d\vec{r}$$

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(The same function multiplied by an atomic form factor typically describes X-ray scattering as well.)

Assume that the fluid is an ideal Fermi gas at zero temperature and calculate $S(\vec{q})$ as a function of q/k_F.

15. For a complex field $\psi(\vec{r})$ we have derived the momentum operator

$$\vec{p} = \int \psi^\dagger(r)\,\frac{\hbar}{i}\,\nabla\psi(r)\, d\vec{r}$$

(a) Show that $D(\vec{d}) = \exp(\frac{-i}{\hbar}\,\vec{d}\cdot\vec{p})$ is a displacement operator for a finite displacement $\vec{d}$, i.e. that

$$\exp\,(\tfrac{i}{\hbar}\,\vec{d}\cdot\vec{p})\;\psi(\vec{r})\;\exp(\tfrac{-i}{\hbar}\,\vec{d}\cdot\vec{p}) = \psi(\vec{r}-\vec{d})$$

for both types of statistics.

(b) A non-relativistic Galilean transformation is one which sets a system in motion with a fixed velocity, say $\vec{v}$, and thereby displaces momenta while giving the coordinates a time-dependent displacement. What is the appropriate transformation of the field variable ψ? Find a unitary operator which carries out such a transformation and express it in terms of ψ and $\psi^\dagger$. If the interaction Hamiltonian of the system is invariant under Galilean transformations (as is the case, for example, for interparticle forces dependent only on separations) what is the corresponding conserved generator? i.e. What is the corresponding constant of the motion?

$U = e^{iG(t)}$ $\quad [G,H] \neq 0$

$\frac{dG}{dt} = \frac{1}{i\hbar}[G,H] + \frac{\partial G}{\partial t} = 0$

Physics 253a - Assignment #1 Solutions

#1.) The Hamiltonian is:

$$\hat{H} = \int d^3r\, \psi^\dagger(\vec{r})\left[-\frac{\hbar^2\nabla^2}{2m}\right]\psi(\vec{r}) + \frac{1}{2}\int d^3r\, d^3r'\, \psi^\dagger(\vec{r})\psi^\dagger(\vec{r}\,')V(\vec{r}-\vec{r}\,')\psi(\vec{r}\,')\psi(\vec{r})$$

For a gas of spin ½ particles quantized with periodic boundary conditions in a box of volume V, interacting through a spin-independent potential, the field operator is:

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}\lambda} a_{\vec{k}\lambda}\, e^{i\vec{k}\cdot\vec{r}}\, u_\lambda$$

First look at kinetic energy term:

$$\hat{T} = \int d^3r \frac{1}{\sqrt{V}}\sum_{\vec{k}\lambda} a^\dagger_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{r}} u^\dagger_\lambda \frac{1}{\sqrt{V}}\sum_{\vec{k}'\lambda'} a_{\vec{k}'\lambda'} \frac{\hbar^2 k'^2}{2m} e^{i\vec{k}'\cdot\vec{r}} u_{\lambda'}$$

$$= \frac{1}{V}\sum_{\substack{\vec{k}\vec{k}'\\ \lambda\lambda'}} a^\dagger_{\vec{k}\lambda} a_{\vec{k}'\lambda'}\, \delta_{\lambda\lambda'} \frac{\hbar^2 k'^2}{2m} \underbrace{\int d^3r\, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}}_{\text{This is } V\delta_{\vec{k},\vec{k}'}}$$

Thus, $$\hat{T} = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m} a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda}\,.$$

Next, the potential energy term:

$$\hat{V} = \frac{1}{2}\int \frac{d^3r\, d^3r'}{V^2}\left(\sum_{\vec{k}'\sigma} a^\dagger_{\vec{k}'\sigma} e^{-i\vec{k}'\cdot\vec{r}} u^\dagger_\sigma\right)\left(\sum_{\vec{p}'\sigma'} a^\dagger_{\vec{p}'\sigma'} e^{-i\vec{p}'\cdot\vec{r}\,'} u^\dagger_{\sigma'}\right)V(\vec{r}-\vec{r}\,')$$

$$\times\left(\sum_{\vec{p}\lambda'} a_{\vec{p}\lambda'}\, e^{i\vec{p}\cdot\vec{r}\,'} u_{\lambda'}\right)\left(\sum_{\vec{k}\lambda} a_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{r}} u_\lambda\right)$$

$$= \frac{1}{2V^2}\sum_{\substack{\vec{k}\vec{k}'\vec{p}\vec{p}'\\ \lambda\lambda'\sigma\sigma'}} a^\dagger_{\vec{k}'\sigma} a^\dagger_{\vec{p}'\sigma'} a_{\vec{p}\lambda'} a_{\vec{k}\lambda}\, \delta_{\sigma\lambda}\delta_{\sigma'\lambda'} \int d^3r\, d^3r'\, e^{i(\vec{p}-\vec{p}\,')\cdot\vec{r}\,'}$$

$$\times\; V(\vec{r}-\vec{r}\,')\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}$$

Change variables to $\bar{x} \equiv \bar{r} - \bar{r}'$, $\bar{y} \equiv \bar{r}$, the integral becomes (the Jacobian is 1):

$$\int d^3x\, d^3y\; e^{i(\bar{p}-\bar{p}')\cdot(\bar{y}-\bar{x})}\, V(\bar{x})\, e^{i(\bar{k}-\bar{k}')\cdot\vec{y}}$$

$$= \int d^3y\; e^{i(\bar{p}-\bar{p}'+\bar{k}-\bar{k}')\cdot\bar{y}} \int d^3x\; e^{-i(\bar{p}-\bar{p}')\cdot\bar{x}}\, V(\bar{x})$$

Letting $\bar{q} \equiv \bar{k}' - \bar{k}$, we have

$$= V\, \delta_{\bar{p}-\bar{p}';\, \bar{k}'-\bar{k}}\, \tilde{V}(\bar{p}-\bar{p}') = V\, \delta_{\bar{p}-\bar{p}';\, \bar{k}'-k}\, \tilde{V}(\vec{q})$$

where $\tilde{V}(\bar{q}) = \int d^3x\, e^{-i\bar{q}\cdot\bar{x}}$. So:

$$\hat{V} = \frac{1}{2V} \sum_{\substack{\bar{p}\bar{k}\bar{q} \\ \lambda\lambda'}} \tilde{V}(\vec{q})\, a^{+}_{\bar{k}+\bar{q}\,\lambda}\, a^{+}_{\bar{p}-\bar{q}\,\lambda'}\, a_{\bar{p}\lambda'}\, a_{\bar{k}\lambda}$$

Thus:

$$H = \sum_{\bar{k}\lambda} \frac{\hbar^2 \bar{k}^2}{2m} a^{+}_{\bar{k}\lambda} a_{\bar{k}\lambda} + \frac{1}{2V} \sum_{\bar{q}} \tilde{V}(\bar{q}) \sum_{\substack{\bar{k}\bar{p} \\ \lambda\lambda'}} a^{+}_{\bar{k}+\bar{q}\,\lambda}\, a^{+}_{\bar{p}-\bar{q}\,\lambda'}\, a_{\bar{p}\lambda'}\, a_{\bar{k}\lambda}$$

#2.) The first order shift in energy is $\Delta E = \langle F | H_{int} | F \rangle$. Hence, we look at:

$$M \equiv \langle F | \sum_{\substack{\bar{k}\bar{p} \\ \lambda\lambda'}} a^{+}_{\bar{k}+\bar{p}\,\lambda}\, a^{+}_{\bar{p}-\bar{q}\,\lambda'}\, a_{\bar{p}\lambda'}\, a_{\bar{k}\lambda} | F \rangle$$

Case 1: $\bar{q} = 0$. Now we have $\sum_{\substack{\bar{k}\bar{p} \\ \lambda\lambda'}} a^{+}_{\bar{k}\lambda}\, \hat{N}_{\bar{p}\lambda'}\, a_{\bar{k}\lambda}$ to deal with.

But $\hat{N}_{\bar{p}\lambda'}\, a_{\bar{k}\lambda} = a_{\bar{k}\lambda}(\hat{N}_{\bar{p}\lambda'} - \delta_{\bar{p}\bar{k}}\delta_{\lambda\lambda'})$. Thus we have:

$$M = \langle F | \sum_{\substack{\bar{k}\bar{p} \\ \lambda\lambda'}} \hat{N}_{\bar{k}\lambda}(\hat{N}_{\bar{p}\lambda'} - \delta_{\bar{p}\bar{k}}\delta_{\lambda\lambda'}) | F \rangle$$

$$= \langle F | \hat{N}(\hat{N} - 1) | F \rangle = N(N-1).$$

This contributes an amount $\frac{N(N-1)}{2V}\tilde{V}(\bar{q}=0)$ to the energy shift.

Case 2: $\bar{q} \neq 0$. Now the $a^{\dagger}_{\bar{p}-\bar{q}\lambda'}$ must create back the electron destroyed by $a_{\bar{k}\lambda}$, so get

$$M = \sum_{\substack{\bar{k}\bar{p} \\ \lambda\lambda}} \delta_{\lambda\lambda'}\,\delta_{\bar{p},\bar{k}+\bar{q}} \langle F|\, a^{\dagger}_{\bar{k}+\bar{q}\lambda} a^{\dagger}_{\bar{k}\lambda} a_{\bar{k}+\bar{q}\lambda} a_{\bar{k}\lambda} \,|F\rangle$$

Since $\bar{q} \neq 0$, we can anticommute, and get:

$$M = -\sum_{\bar{k}\lambda} \langle F|\, \hat{N}_{\bar{k}+\bar{q}\lambda} \hat{N}_{\bar{k}\lambda} \,|F\rangle$$
$$= -\sum_{\bar{k}\lambda} \Theta(k_F - |\bar{k}+\bar{q}|)\Theta(k_F - k)$$

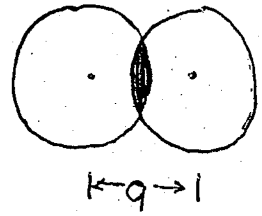
Thus, we get a contribution to the energy shift of

$$-\sum_{\bar{q}\neq 0} \tilde{V}(\bar{q})\, I(\bar{q})$$

where
$$I(\bar{q}) = +\frac{1}{2V}\sum_{\bar{k}\lambda} \Theta(k_F - |\bar{k}+\bar{q}|)\Theta(k_F - k)$$
$$= +\int \frac{d^3k}{(2\pi)^3}\, \Theta(k_F - |\bar{k}+\bar{q}|)\Theta(k_F - k)$$

Hence: $$\Delta E = \frac{N(N-1)}{2V}\tilde{V}(\bar{q}=0) - \sum_{\bar{q}\neq 0} \tilde{V}(\bar{q})\, I(\bar{q})$$

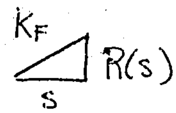
It is clear that $I(\bar{q})$ depends only on $|\bar{q}|$, and is the volume of overlap of two spheres of radius k_F whose centers are a distance q apart (divided by $(2\pi)^3$):



Clearly, $I = 0$ if $q > 2k_F$

Equivalently, $I(q) = \frac{2}{(2\pi)^3}$ (volume of $\odot$) $\; q/2$

$$= \pi \int_{q/2}^{k_F} R^2(s)\,ds$$



$$= \pi \int_{q/2}^{k_F} (k_F^2 - s^2)\,ds = \pi\left(k_F^3 - k_F^2 \tfrac{q}{2} - k_F^3/3 + (\tfrac{q}{2})^3/3\right)$$

$$= \frac{2\pi}{3} k_F^3 \left\{ 1 - \frac{3}{2}\frac{q}{2k_F} + \frac{1}{2}\left(\frac{q}{2k_F}\right)^3 \right\}$$

Thus $$I(q) = \frac{1}{(2\pi)^3} \frac{4\pi k_F^3}{3} \left\{ 1 - \frac{3}{2}\frac{q}{2k_F} + \frac{1}{2}\left(\frac{q}{2k_F}\right)^3 \right\}$$

#3.) $\tilde{V}(\vec{q}=0) \equiv 0$; $\tilde{V}(\vec{q}) = 4\pi e^2/q^2$

Hence $$\Delta E = -\sum_{\vec{q}} \frac{4\pi e^2}{q^2} I(q) = -\frac{(4\pi e)^2 V k_F^3}{(2\pi)^6\, 3} \int_0^{2k_F} 4\pi\, dq \left\{ 1 - \frac{3}{2}\frac{q}{2k_F} + \frac{1}{2}\left(\frac{q}{2k_F}\right)^3 \right\}$$

$$= -\frac{e^2 V k_F^3}{\pi^3\, 3}\, 2k_F \underbrace{\int_0^1 dx \left\{ 1 - \frac{3}{2}x + \frac{1}{2}x^3 \right\}}_{=\,3/8}$$

$$= -\frac{e^2 V k_F^3}{4\pi^3}\, k_F$$

Now $V k_F^3 = V \frac{3\pi^2 N}{V} = 3\pi^2 N$ and $k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$

Use $\frac{N}{V} = \frac{3}{4\pi}\frac{1}{r_s^3 a_0^3}$, $k_F = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s a_0}$

So

$$\Delta E = -\underbrace{\frac{3}{2\pi}\left(\frac{9\pi}{4}\right)^{1/3}}_{=\,0.916} N \frac{e^2}{2a_0}$$

The Kinetic Energy is $E_0 = \frac{3}{5} N \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N \frac{\hbar^2}{2m}\left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2 a_0^2}$

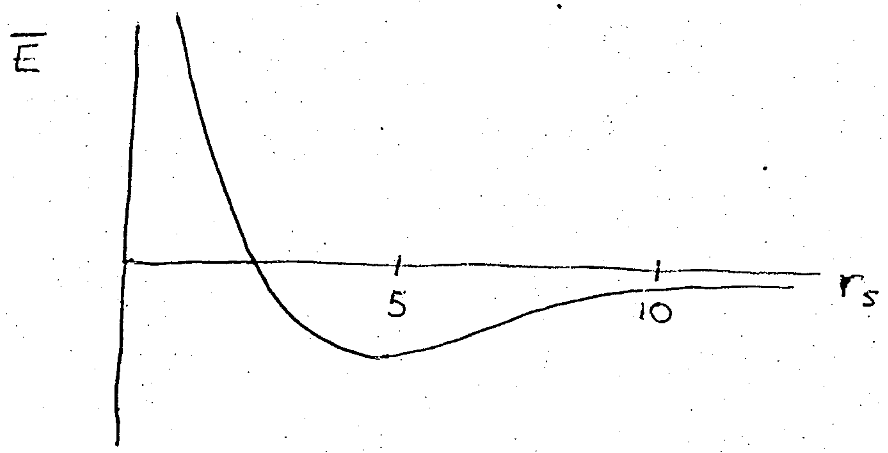
Use $a_0 \equiv \hbar^2/me^2$;

$$E_0 = \frac{3}{5}\,\frac{1}{2}\left(\frac{9\pi}{4}\right)^{2/3} \frac{N e^2}{r_s^2 a_0}$$

Hence $E_0 + \Delta E \equiv E = N \frac{e^2}{2a_0} \left\{ \frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2} - \frac{3}{2\pi}\left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \right\}$

Note that $\Delta E / E_0 \to 0$ as $r_s \to 0$ ($N/V \to \infty$, or high density limit). This means that for a very dense electron gas the coulomb interaction becomes a small perturbation! The total coulomb energy goes as (average separation)$^{-1}$, while the kinetic energy that is associated with confining an electron to a region of the order r_s^3 in volume goes as r_s^{-2}. Hence, for small r_s our answer exact! Putting in the numbers

$$\frac{E}{Ne^2/2a_0} = \frac{2.21}{r_s^2} - \frac{0.916}{r_s} \equiv \bar{E}$$



In equilibrium, the energy is a minimum, so $\frac{d\bar{E}}{dr_s} = 0$

$$\frac{d\bar{E}}{dr_s} = -\frac{2(2.21)}{r_s^3} + \frac{0.916}{r_s^2} = 0 \Rightarrow r_{s\,min} = \frac{2(2.21)}{0.916} = 4.83$$

$$\bar{E}(r_{s\,min}) = \frac{2.21}{(4.83)^2} - \frac{0.916}{(4.83)} = -.095$$

Thus $\frac{E_{min}}{N} = -.095 \frac{e^2}{2a_0} = -.095(13.6\,\text{eV}) = -1.29\,\text{eV}$

Since the jellium is pretty crude, these values are surprisingly close to those of a real Na crystal.

#4.) For an ideal Fermi gas at $T=0$ $|\Psi\rangle = |F\rangle$ and

$$S(\bar q) = \int d^3r\, e^{-i\bar q\cdot\bar r} \langle F|\, n(\bar r) n(0)\,|F\rangle$$

Look at $n(\bar r) = \frac{1}{V}\sum_{\bar p \bar q' \lambda'} a^\dagger_{\bar p+\bar q'\lambda'}\, a_{\bar p\lambda'}\, e^{-i\bar q'\cdot\bar r}$

so

$$\int d^3r\, e^{-i\bar q\cdot\bar r} n(\bar r) = \sum_{\bar p\lambda'} a^\dagger_{\bar p-\bar q\,\lambda'}\, a_{\bar p\lambda'}$$

Also, $n(\bar 0) = \frac{1}{V}\sum_{\bar k \bar q'' \lambda} a^\dagger_{\bar k+\bar q''\lambda}\, a_{\bar k\lambda}$

Hence, $S(\bar q) = \frac{1}{V}\sum_{\substack{\bar k\bar p\bar q'' \\ \lambda\lambda'}} \langle F|\, a^\dagger_{\bar p-\bar q\,\lambda'}\, a_{\bar p\lambda'}\, a^\dagger_{\bar k+\bar q''\lambda}\, a_{\bar k\lambda}\,|F\rangle$

By momentum considerations, only $\bar q'' = \bar q$ can contribute:

$$S(\bar q) = \frac{1}{V}\sum_{\bar k\bar p\lambda\lambda'} \langle F|\, a^\dagger_{\bar p-\bar q,\lambda'}\, a_{\bar p\lambda'}\, a^\dagger_{\bar k+\bar q\lambda}\, a_{\bar k\lambda}\,|F\rangle$$

Case 1: $\bar q = 0$

$$S(\bar q=0) = \frac{1}{V}\langle F|\,\hat N \hat N\,|F\rangle = \frac{N^2}{V}$$

Case 2: $\bar q \neq 0$, Must have $\bar p = \bar k + \bar q$, $\lambda = \lambda'$

$$S(\bar q) = \frac{1}{V}\sum_{\bar k\lambda} \langle F|\, a^\dagger_{\bar k\lambda}\, a_{\bar k+\bar q\lambda}\, a^\dagger_{\bar k+\bar q\lambda}\, a_{\bar k\lambda}\,|F\rangle$$

$$= \frac{1}{V}\sum_{\bar k\lambda} \langle F|\, \hat N_{\bar k\lambda}(1-\hat N_{\bar k+\bar q\lambda})\,|F\rangle$$

$$= \frac{N}{V} - \frac{1}{V}\underbrace{\sum_{\bar k\lambda}\langle F|\,\hat N_{\bar k\lambda} N_{\bar k+\bar q\lambda}\,|F\rangle}_{=\,2V\,I(q)}$$

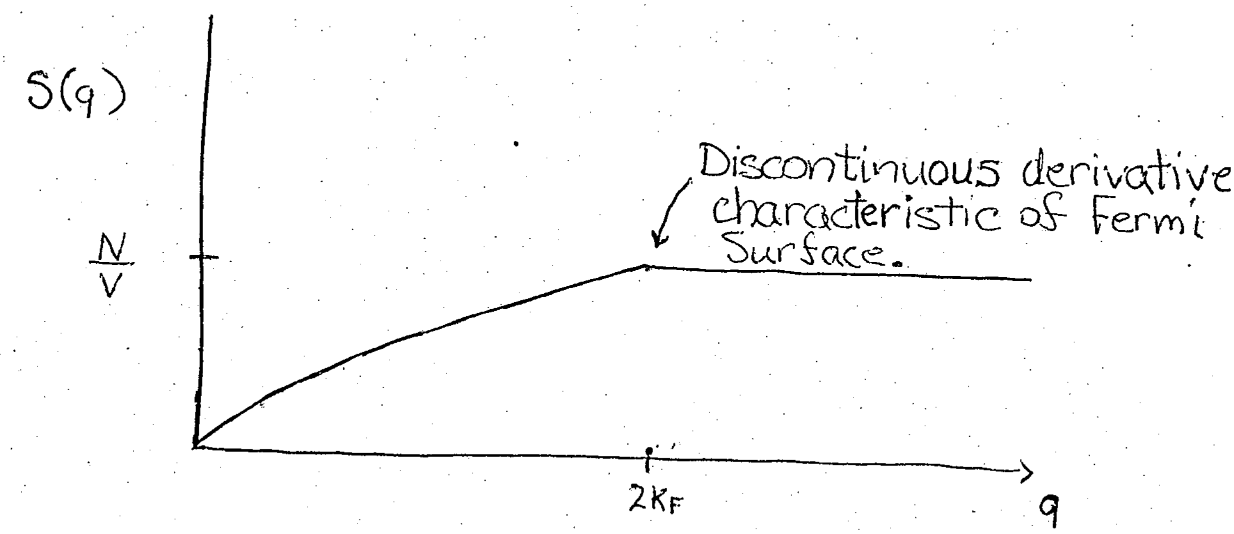
$$= \frac{N}{V} - 2I(q)$$

$$S(\bar{q}) = \frac{N}{V} - 2\,\frac{1}{2}\,\frac{N}{V}\left\{1 - \frac{3}{2}\frac{q}{2k_F} + \frac{1}{2}\left(\frac{q}{2k_F}\right)^3\right\}\theta\left(1-\frac{q}{2k_F}\right)$$

$$S(\bar{q}) = \frac{N}{V}\left[1 - \left\{1 - \frac{3}{2}\frac{q}{2k_F} + \frac{1}{2}\left(\frac{q}{2k_F}\right)^3\right\}\theta\left(1-\frac{q}{2k_F}\right)\right]$$

Let $x \equiv q/2k_F$:

$$S(x) = \begin{cases} N^2/V & x = 0 \\ \frac{3N}{2V}\left[x\left(1-\frac{1}{3}x^2\right)\right] & x < 1 \\ \frac{N}{V} & x > 1 \end{cases}$$



5.) (a) First look at $[\bar{P}, \psi(\bar{r})]$:

$$[\bar{P}, \psi(\bar{r})] = \frac{\hbar}{i}\int d^3r' d^3r'' \left\{\bar{\nabla}''\,[\psi^\dagger(\bar{r}')\psi(\bar{r}''), \psi(\bar{r})]\right\}\delta(\bar{r}'-\bar{r}'')$$

Now $[AB,C] = [A,C]B + A[B,C]$
$= A\{B,C\} - \{A,C\}B$

So $[\psi^\dagger(\bar{r}')\psi(\bar{r}''), \psi(\bar{r})] = [\psi^\dagger(\bar{r}'), \psi(\bar{r})]\psi(\bar{r}'')$ for BE
$= -\{\psi^\dagger(\bar{r}'), \psi(\bar{r})\}\psi(\bar{r}'')$ for FD

Both are equal to $-\delta(\bar{r}'-\bar{r})\psi(\bar{r}'')$
and

$$[\bar{P}, \psi(\bar{r})] = -\frac{\hbar}{i}\bar{\nabla}\psi(\bar{r})$$

Now $[P_i, \nabla_j \psi(\bar{r})] = \nabla_j [P_i, \psi(\bar{r})] = -\frac{\hbar}{i} \nabla_i \nabla_j \psi(\bar{r})$

and $e^{\lambda B} A e^{-\lambda B} = A + \lambda [B,A] + \frac{\lambda^2}{2!} [B,[B,A]] + \cdots$

Thus:

$$e^{\frac{i}{\hbar}\bar{p}\cdot\bar{d}} \psi(\bar{r}) e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{d}} = \psi(\bar{r}) + \frac{i}{\hbar}\left(-\frac{\hbar}{i}\right)\bar{d}\cdot\bar{\nabla}\psi(\bar{r}) + \frac{(-1)^2}{2!}(\bar{d}\cdot\bar{\nabla})^2\psi(\bar{r}) + \cdots$$
$$= \psi(\bar{r}) - \bar{d}\cdot\bar{\nabla}\psi(\bar{r}) + \frac{(-1)^2}{2!}(\bar{d}\cdot\bar{\nabla})^2\psi(\bar{r}) + \cdots$$
$$= \psi(\bar{r}-\bar{d})$$

(b) Under a Galilean Transformation, the wave function should transform as:

$$|\Psi(\bar{r}_1 \bar{r}_2 \cdots \bar{r}_N)|^2 \longrightarrow |\Psi(\bar{r}_1 - \bar{v}t, \bar{r}_2 - \bar{v}t, \cdots)|^2$$

and since $\Psi(\bar{r}_1 \cdots \bar{r}_N) = \frac{1}{\sqrt{N!}} \langle 0| \psi(\bar{r}_N) \cdots \psi(\bar{r}_1) |\Psi\rangle$, this tells us that the field operators must transform as

$$U^{+}(\bar{v})\psi(\bar{r})U(\bar{v}) = \psi(\bar{r}-\bar{v}t)e^{i\theta(\bar{r},\bar{v})} \qquad (\theta \text{ real})$$

Also, in momentum space,

$$|\Psi(\bar{p}_1 \bar{p}_2, \cdots \bar{p}_N)|^2 \longrightarrow |\Psi(\bar{p}_1 - m\bar{v}, \bar{p}_2 - m\bar{v}, \cdots)|^2$$

and since $\Psi(\bar{p}_1 \bar{p}_2 \cdots \bar{p}_N) = \frac{1}{\sqrt{N!}} \langle 0| a_{\bar{p}_N} a_{\bar{p}_{N-1}} \cdots a_{\bar{p}_1} |\Psi\rangle$ this tells us that

$$U^{+}(\bar{v}) a_{\bar{p}} U(\bar{v}) = a_{\bar{p}-\frac{m\bar{v}}{\hbar}} e^{i\varphi(\bar{p},\bar{v})} \qquad (\varphi \text{ real})$$

In order to determine the effect of $U(\bar{v})$ on the field variables, we must determine $\theta(\bar{r},\bar{v}), \varphi(\bar{p},\bar{v})$.

$$U^{+}(\bar{v})\psi(\bar{r})U(\bar{v}) = \frac{1}{\sqrt{V}} \sum_{\bar{p}\lambda} U^{+}(\bar{v}) a_{\bar{p}\lambda} U(\bar{v}) e^{i\bar{p}\cdot\bar{r}}$$

$$U^\dagger(\bar v)\psi(\bar r)U(\bar v) = \frac{1}{\sqrt{V}}\sum_{\bar p\lambda} a_{\bar p-\frac{m\bar v}{\hbar}\lambda}\, e^{i\varphi(\bar p,\bar v)}\, e^{i\bar p\cdot\bar r}$$
$$= \frac{1}{\sqrt{V}}\sum_{\bar p\lambda} a_{\bar p\lambda}\, e^{i\varphi(\bar p+\frac{m\bar v}{\hbar},\bar v)}\, e^{i(\bar p+\frac{m\bar v}{\hbar})\cdot\bar r}$$
$$= e^{+\frac{im\bar v\cdot\bar r}{\hbar}}\underbrace{\frac{1}{\sqrt{V}}\sum_{\bar p\lambda} a_{\bar p\lambda}\, e^{i\bar p\cdot\bar r+i\varphi(\bar p+\frac{m\bar v}{\hbar},\bar v)}}$$

In order for this to be $\psi(\bar r-\bar v t)$ times a phase, we must have $\varphi(\bar k,\bar v) = -\bar v t\cdot\bar k$, so:

$$\boxed{U^\dagger(\bar v)\psi(\bar r)U(\bar v) = e^{\frac{im\bar v}{\hbar}\cdot(\bar r-\bar v t)}\,\psi(\bar r-\bar v t)}$$

Thus,
$$\Theta(\bar r,\bar v) = -m\bar v\cdot(\bar r-\bar v t)/\hbar$$
$$\varphi(\bar p,\bar v) = -\bar p\cdot\bar v t$$
and we have:

$$\boxed{U^\dagger(\bar v)a_{\bar p}U(\bar v) = e^{-i\bar p\cdot\bar v t}\,a_{\bar p-\frac{m\bar v}{\hbar}}}$$

Aside: In single particle Q.M., we had
$$U^\dagger(\bar v)\bar r U(\bar v) = \bar r+\bar v t$$
$$U^\dagger(\bar v)\bar p U(\bar v) = \bar p+m\bar v$$

Writing $U(\bar v) = e^{i\bar v\cdot\bar F}$, we have, in infinitesimal case:

$$(1-i\bar v\cdot\bar F)\bar r(1+i\bar v\cdot\bar F) = \bar r - i[\bar v\cdot\bar F,\bar r] = \bar r+\bar v t$$
$$(1-i\bar v\cdot\bar F)\bar p(1+i\bar v\cdot\bar F) = \bar p - i[\bar v\cdot\bar F,\bar p] = \bar p+m\bar v$$

Try $\bar F = \alpha\bar r+\beta\bar p$:

$$-i[\bar v\cdot\bar F,\bar r] = -i[\bar v\cdot\bar p,\bar r]\beta = i\beta[\bar r,\bar v\cdot\bar p] = -\hbar\beta\bar v$$
$$\Rightarrow \beta = -t/\hbar$$

$$-i[\bar v\cdot\bar F,\bar p] = -i\alpha[\bar v\cdot\bar r,\bar p] = \alpha\hbar\bar v \quad\Rightarrow\quad \alpha = m/\hbar$$

$$\boxed{\bar F = \frac{1}{\hbar}(m\bar r-t\bar p) \quad\text{and}\quad U(\bar v) = e^{\frac{i}{\hbar}\bar v\cdot(m\bar r-t\bar p)}}$$

Thus, let us guess that $\bar{F} = \frac{1}{\hbar}(m\bar{R} - \bar{P}t)$

where $\bar{R} \equiv \int \psi^{\dagger}(\bar{r})\,\bar{r}\,\psi(\bar{r})\,d^3\bar{r}$, and see what it buys us. We need only show that $\psi(\bar{r})$ transforms correctly, since that implies that $a_{\bar{k}\lambda}$ transforms correctly.

Infinitesimal case:

$$U^{\dagger}(\bar{v})\,\psi(\bar{r})\,U(\bar{v}) = \psi(\bar{r}) - i[\bar{v}\cdot\bar{F}, \psi(\bar{r})]$$

Now, $e^{+\frac{im\bar{v}}{\hbar}\cdot(\bar{r}-\bar{v}t)}\,\psi(\bar{r}-\bar{v}t) = \psi(\bar{r}) + \frac{im\bar{v}\cdot\bar{r}}{\hbar}\psi(\bar{r}) + -t\bar{v}\cdot\bar{\nabla}\psi(\bar{r}) + O(\bar{v}^2)$

So, we want $[\bar{v}\cdot\bar{F}, \psi(\bar{r})] = -\frac{m\bar{v}\cdot\bar{r}}{\hbar}\psi(\bar{r}) - it\,\bar{v}\cdot\bar{\nabla}\psi(\bar{r})$

Consider:

$$\frac{m}{\hbar}[\bar{v}\cdot\bar{R}, \psi(\bar{r})] = \frac{m\bar{v}}{\hbar}\cdot\int[\psi^{\dagger}(\bar{r}')\psi(\bar{r}'), \psi(\bar{r})]\,\bar{r}'\,d^3r'$$

$$= -\frac{m\bar{v}}{\hbar}\cdot\int\delta(\bar{r}-\bar{r}')\,\bar{r}'\,d^3\bar{r}' = -\frac{m\bar{v}\cdot\bar{r}}{\hbar}$$

Next:

$$-\frac{t\bar{v}}{\hbar}\cdot[\bar{P}, \psi(\bar{r})] = -\frac{t\bar{v}}{\hbar}\cdot -\frac{\hbar}{i}\,\bar{\nabla}\psi(\bar{r}) = -it\,\bar{v}\cdot\bar{\nabla}\psi(\bar{r})$$

Thus our generator works! Note that $\frac{\partial\bar{F}}{\partial t} = -\frac{1}{\hbar}\bar{P}$

Examine $U^{\dagger}(\bar{v})\,H\,U(\bar{v})$. Clearly the interaction term is left alone (phases cancel, and variable change returns operator to original form), so look at Kinetic Energy:

$$U^{\dagger}(\bar{v})\,\hat{T}\,U(\bar{v}) = \int d^3r\,e^{-\frac{im\bar{v}}{\hbar}\cdot(\bar{r}-\bar{v}t)}\,\psi^{\dagger}(\bar{r}-\bar{v}t)\left(-\frac{\hbar^2\nabla^2}{2m}\right)e^{\frac{im\bar{v}}{\hbar}\cdot(\bar{r}-\bar{v}t)}\,\psi(\bar{r}-\bar{v}t)$$

$$= \int d^3r\,\psi^{\dagger}(\bar{r}-\bar{v}t)\left[-\frac{\hbar^2}{2m}\left(\bar{\nabla}+\frac{im\bar{v}}{\hbar}\right)^2\right]\psi(\bar{r}-\bar{v}t)$$

$$U^\dagger(\bar{v})\,\hat{T}\,U(\bar{v}) = \hat{T} + \bar{v}\cdot\vec{\mathbb{P}} + \tfrac{1}{2}mv^2\hat{N}$$

This is $\boxed{U^\dagger(\bar{v})HU(\bar{v}) = H + \bar{v}\cdot\vec{\mathbb{P}} + \tfrac{1}{2}mv^2\hat{N}}$

Or, to order $\bar{v}$:

$$-i[\bar{v}\cdot F, H] = \bar{v}\cdot\vec{\mathbb{P}} \implies \boxed{[\vec{F}, H] = i\vec{\mathbb{P}}}$$

Hence, the Heisenberg eqn. of motion for $\vec{F}$ is

$$i\hbar\frac{d\vec{F}}{dt} = [\vec{F}, H] + i\hbar\frac{\partial\vec{F}}{\partial t}$$

$$= i\vec{\mathbb{P}} + i\hbar\left(\frac{-1}{\hbar}\right)\vec{\mathbb{P}} = 0$$

Thus, $\vec{F} = \text{const.} = m\vec{R}_0/\hbar$

or

$$\vec{R} = \bar{R}_0 + \frac{\mathbb{P}t}{m}$$

What we have derived is the uniform motion of the center of mass $\vec{r} = \vec{R}/N$,

$$\bar{r} = \bar{r}_0 + \frac{\bar{\mathbb{P}}}{M}t$$

total mass $= Nm$.

Physics 253 a

Assignment 2

1. Let $f(\vec{r}\ t)$ be a solution to the Klein-Gordon equation. We assume that f and its time derivative are specified on the hyperplane corresponding to time t'. Show that at any other time t we have

$$\surd\ f(\vec{r}\ t) = -\int \Delta(\vec{r} - \vec{r}\,',\ t - t')\overleftrightarrow{\partial}'_0\ f(\vec{r}\,'\ t')\ d\vec{r}\,'.$$

Use this relation to evaluate the general (unequal-time) commutators of the free complex fields ϕ, $\phi^\dagger$ and their time derivatives.

$\surd$2. Show that for a complex scalar field $\phi(x)$ interacting with an external electromagnetic vector potential $A^\mu(x)$ the interaction Hamiltonian density is

$$\mathcal{H}_I = j^\mu A_\mu - \frac{e^2}{\hbar^2}\phi^\dagger\phi\,(\vec{A})^2 .$$

$\surd$3. The theory of the free complex scalar field which we have developed is symmetrical under the operation of charge reflection or charge "conjugation". This is the transformation $\phi \to \phi^\dagger$, $\phi^\dagger \to \phi$ or alternatively interchange of the operators b_k and c_k. Show that this is a canonical transformation by finding a unitary operator C which induces it, i.e. a C such that

$$C^{-1}\phi(x)\ C = \phi^\dagger(x) \quad \text{etc.}$$

(You may find it a help to refer to the Hermitian fields ϕ_1 and ϕ_2 in doing this.) How does charge conjugation transform the current vector j^μ? Is the Hamiltonian of problem 2 invariant under charge conjugation? If C has eigenstates what can you say about their charge? All 0

$C j^\mu C^{-1} = -j^\mu$ see pg 113
$C A^\mu C^{-1} = -A^\mu$

4. Consider two different neutral scalar fields $\phi_1(x)$ and $\phi_2(x)$ which have quanta of mass μ_1 and μ_2 respectively. Let the fields be coupled by an interaction of the form

$$\mathcal{H}_I = g\,\phi_1(x)\,\phi_2(x)\ .$$

Show that by means of a simple canonical transformation the total Hamiltonian for the coupled fields may be expressed as the sum of two free-field (i.e. uncoupled) Hamiltonians. Find the appropriate transformation of ϕ_1 and ϕ_2 and the unitary operator which induces it. What are the quantum masses for the new uncoupled fields? What are the eigenstates of the Hamiltonian? For $\mu_1 = \mu_2$ what bounds on the coupling constant g are necessary to preserve the ground state?

Physics 253a – Problem Set # 2
Solutions

1.) We take as the definition of $\Delta(\bar{r}t)$ the following:
$$\Delta(\bar{r}t) \equiv -\frac{1}{(2\pi)^3}\int d^3\bar{k}\, e^{i\bar{k}\cdot\bar{r}}\,\frac{\sin\omega_{\bar{k}}t}{\omega_{\bar{k}}}, \qquad \omega_{\bar{k}}^2 = \bar{k}^2+\mu^2$$
The following properties of Δ are evident:
(i) $(\Box^2+\mu^2)\Delta(x-y)=0$
(ii) $\Delta(\bar{r}-\bar{r}', 0) = 0$
(iii) $\partial_{t'}\Delta(\bar{r}-\bar{r}'; t-t')|_{t'=t} = \delta(\bar{r}-\bar{r}')$
Now, let $f(\bar{r}t)$ be a solution to the Klein Gordon eqn. Then
$$0 = f(\bar{r}'t')(\Box'^2+\mu^2)\Delta(\bar{r}-\bar{r}'; t-t') - \Delta(\bar{r}-\bar{r}'; t-t')(\Box'^2+\mu^2)f(\bar{r}'t')$$
Integrate over all $\bar{r}'$, and over t' from t_0 to t:
$$0 = \int_{t_0}^{t} dt'\,d^3\bar{r}'\left\{\partial_{t'}[f(\bar{r}'t')\overleftrightarrow{\partial}_{t'}\Delta(\bar{r}-\bar{r}'; t-t')] - \bar{\nabla}'\cdot[f(\bar{r}'t')\bar{\nabla}'\Delta(\bar{r}-\bar{r}'; t-t') - \Delta(\bar{r}-\bar{r}'; t-t')\bar{\nabla}'f(\bar{r}'t')]\right\}$$
The second term vanishes by Gauss' Theorem, and hence:
$$0 = \int d^3\bar{r}'\,[f(\bar{r}'t')\overleftrightarrow{\partial}_{t'}\Delta(\bar{r}-\bar{r}'; t-t')]\Big|_{t_0}^{t}.$$
Inserting (ii) & (iii) we have
$$f(\bar{r}t) = \int d^3\bar{r}'[f(\bar{r}'t')\overleftrightarrow{\partial}_{t'}\Delta(\bar{r}-\bar{r}'; t-t')]_{t'=t_0}$$

Now, we turn to charged scalar fields, $\varphi(x)$, $\varphi^\dagger(y)$. All commutation relations can be derived (by differentiation and conjugation) from $[\varphi(x),\varphi(y)]$ and $[\varphi(x),\varphi^\dagger(y)]$.
It is clear that $[\varphi(x),\varphi(y)]$ involves only $[\varphi,\dot{\varphi}]$ at equal times, and this commutator vanishes, so $[\varphi(x),\varphi(y)]=0$ for all x,y.
Next, we look at $[\varphi(x),\varphi^\dagger(y)]$:
$$[\varphi(x),\varphi^\dagger(y)] = \left[\int d^3\bar{r}'\,[\varphi(\bar{r}'t_y)\overleftrightarrow{\partial}_{t_y}\Delta(\bar{x}-\bar{r}'; t_x-t_y)],\ \varphi^\dagger(\bar{y},t_y)\right]$$

Now only nonvanishing equal time commutator is $[\varphi(\bar{r}t),\dot{\varphi}^\dagger(\bar{r}t)] = i\hbar\delta(\bar{r}-\bar{r}')$
so
$$[\varphi(x),\varphi^\dagger(y)] = -\int d^3\bar{r}'\,\Delta(\bar{x}-\bar{r}'; t_x-t_y)\,[\dot{\varphi}(\bar{r}'t_y),\varphi^\dagger(\bar{y}t_y)]$$
$$= i\hbar\int d^3\bar{r}'\,\Delta(\bar{x}-\bar{r}'; t_x-t_y)\,\delta(\bar{r}'-\bar{y}) = i\hbar\,\Delta(x-y)$$

This, of course, agrees with the results obtained by expressing the charged fields in terms of neutral fields.

2.) Coupling of charged field to external EM field: The Lagrangian density:

$$\mathcal{L} = (\partial_\mu - \tfrac{ie}{\hbar}A_\mu)\varphi^\dagger(\partial^\mu + \tfrac{ie}{\hbar}A^\mu)\varphi - \mu^2\varphi^\dagger\varphi$$

Canonical momenta $\pi_\mu = (\partial_\mu - \frac{ie}{\hbar}A_\mu)\varphi^\dagger$, $\mathcal{L} = \pi_\mu \pi^{\dagger\mu} - \mu^2\varphi^\dagger\varphi$

The Hamiltonian density is

$$\mathcal{H} = \pi\dot{\varphi} + \pi^\dagger\dot{\varphi}^\dagger - \mathcal{L} = \pi(\pi^\dagger - \tfrac{ie}{\hbar}A_0\varphi) + \pi^\dagger(\pi + \tfrac{ie}{\hbar}A_0\varphi^\dagger) - \mathcal{L}$$

Thus

$$\mathcal{H} = \pi(\pi^\dagger - \tfrac{ie}{\hbar}A_0\varphi) + \pi^\dagger(\pi + \tfrac{ie}{\hbar}A_0\varphi^\dagger) - \pi\pi^\dagger + \bar{\pi}\cdot\bar{\pi}^\dagger + \mu^2\varphi^\dagger\varphi$$

$$= \pi^\dagger\pi + \mu^2\varphi^\dagger\varphi + \frac{ie}{\hbar}A_0(\pi^\dagger\varphi^\dagger - \pi\varphi) + \bar{\pi}\cdot\bar{\pi}^\dagger$$

In terms of canonical variables, the free field Hamiltonian density $\mathcal{H}_0 = \pi^\dagger\pi + \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi + \mu^2\varphi^\dagger\varphi$, so the interaction Hamiltonian is

$$\mathcal{H}_I = \frac{ie}{\hbar}A_0(\pi^\dagger\varphi^\dagger - \pi\varphi) + \bar{\pi}\cdot\bar{\pi}^\dagger - \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi$$

The gauge current is $j_\mu = \frac{ie}{\hbar}(\pi^\dagger_\mu\varphi^\dagger - \pi_\mu\varphi)$, thus

$\mathcal{H}_I = j_0A_0 + \bar{\pi}\cdot\bar{\pi}^\dagger - \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi$, or $\mathcal{H}_I = j_\mu A^\mu + \{\bar{\pi}\cdot\bar{\pi}^\dagger - \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi + \bar{j}\cdot\bar{A}\}$

Now

$$\{\bar{\pi}\cdot\bar{\pi}^\dagger - \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi + \bar{j}\cdot\bar{A}\} = \{\bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi + \frac{ie}{\hbar}\bar{A}\cdot(\bar{\nabla}\varphi^\dagger\varphi - \varphi^\dagger\bar{\nabla}\varphi) + \frac{e^2}{\hbar^2}\bar{A}^2\varphi^\dagger\varphi - \bar{\nabla}\varphi^\dagger\cdot\bar{\nabla}\varphi + \bar{j}\cdot\bar{A}\}$$

$$= \{\frac{ie}{\hbar}\bar{A}\cdot(\bar{\nabla}\varphi^\dagger\varphi - \varphi^\dagger\bar{\nabla}\varphi) + \frac{e^2}{\hbar^2}\bar{A}^2\varphi^\dagger\varphi + \frac{ie}{\hbar}\bar{A}\cdot(\bar{\nabla}\varphi\varphi^\dagger - \bar{\nabla}\varphi^\dagger\varphi) - \frac{2e^2}{\hbar^2}\bar{A}^2\varphi^\dagger\varphi\}$$

$$= -\frac{e^2}{\hbar^2}\bar{A}^2\varphi^\dagger\varphi \qquad (\text{we have used } [\varphi^\dagger, \bar{\nabla}\varphi] = 0)$$

Thus $\mathcal{H}_I = j_\mu A^\mu - \frac{e^2}{\hbar^2}\bar{A}^2\varphi^\dagger\varphi$

3.) Looking for a unitary C which carries out charge conjugation $C^{-1}\varphi(x)C = \varphi^\dagger(x)$ & $C^{-1}\varphi^\dagger(x)C = \varphi(x)$.
In terms of the real fields φ_1, φ_2: $C^{-1}\varphi_1(x)C = \varphi_1(x)$ and $C^{-1}\varphi_2(x)C = -\varphi_2(x)$, so C can be constructed from φ_2 field variables alone.
Let $N_2 = \sum_k a_k^\dagger a_k$ be the number operator for the quanta of the φ_2 field. We construct C by noting that $N_2\varphi_2^{(\pm)} = \varphi_2^{(\pm)}(N_2 \mp 1)$. Then we look at $C = (-1)^{N_2}$:

$$(-1)^{N_2}\varphi_2(-1)^{N_2} = (-1)^{N_2}\{\varphi_2^{(+)} + \varphi_2^{(-)}\}(-1)^{N_2} = \varphi_2^{(+)}(-1)^{N_2-1}(-1)^{N_2} + \varphi_2^{(-)}(-1)^{N_2+1}(-1)^{N_2}$$

$$= -\varphi_2 \qquad \text{looks good!}$$

$$C^\dagger = [(-1)^{N_2}]^\dagger = C \qquad [(-1)^{N_2}]^2 = (-1)^{2N_2} = 1, \text{ so } C^{-1} = C.$$

Hence, C is unitary.

How does the gauge current transform under C?
Recall $$j_\mu = \frac{ie}{\hbar}(\partial_\mu\varphi\,\varphi^\dagger - \partial_\mu\varphi^\dagger\varphi) - \frac{2e^2}{\hbar^2}A_\mu\varphi^\dagger\varphi$$
$$= j_\mu^{(0)} - \frac{2e^2}{\hbar^2}A_\mu\varphi^\dagger\varphi$$

$$C^{-1}j_\mu^{(0)}C = \frac{ie}{\hbar}(\partial_\mu\varphi^\dagger\varphi - \partial_\mu\varphi\,\varphi^\dagger) = -j_\mu^{(0)}$$

$C^{-1}\varphi^\dagger\varphi C = \varphi\varphi^\dagger = \varphi^\dagger\varphi$. Hence, the full gauge current doesn't transform in a simple way under charge conjugation. The Hamiltonian of problem 2 is thus not invariant under charge conjugation. The problem is that A_μ represents an applied EM field, not a dynamical variable. There is no reason to expect positive and negative charges to behave the same way in a given external EM field. If, however, we elevate A_μ to the status of a quantum field, then C must reverse A_μ, because it changes the charge of the sources of A_μ. Thus, let us consider the transformation of j, and $\mathcal{H}$ with A_μ representing a quantum field:
$C^{-1}j_\mu C = -j_\mu$, so $C^{-1}\mathcal{H}C = \mathcal{H}$, and the coupling is C invariant. Also, for the free theory the charge is $Q = \int j_0^{(0)}(x)d^3x$ so that $C^{-1}QC = -Q$. Thus if $|cq\rangle$ is a C-eigenstate, and a charge eigenstate, then $cq = 0$, But $C^2 = 1 \Rightarrow c^2 = 1 \Rightarrow q = 0$.

4.) Consider $\mathcal{L} = \frac{1}{2}\{\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 - \mu_1^2\varphi_1^2 - \mu_2^2\varphi_2^2 - 2g\varphi_1\varphi_2\}$

In vector notation $\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, with $M \equiv \begin{pmatrix}\mu_1^2 & g\\ g & \mu_2^2\end{pmatrix}$ the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\{\partial_\mu\varphi^T \mathbb{1}\partial^\mu\varphi - \varphi^T M\varphi\}$$

M real symmetric $\Rightarrow$ M can be diagonalized by orthogonal transformation S, $D = S^+MS$ is diagonal, $D = \begin{pmatrix}\mu_a^2 & 0\\ 0 & \mu_b^2\end{pmatrix}$ and with $\psi = S^T\varphi$,

$$\mathcal{L} = \frac{1}{2}\{\partial_\mu\psi^T\partial^\mu\psi - \psi^T D\psi\}$$

The real masses are $\mu_{a,b}^2$, and are determined by the eigenvalue problem for M:

$\det|M - \lambda\mathbb{1}| = 0 \Rightarrow \lambda^2 - (\mu_1^2+\mu_2^2)\lambda + (\mu_1^2\mu_2^2 - g^2) = 0$

Solutions are $\mu_{a,b}^2 = \frac{1}{2}\{(\mu_1^2+\mu_2^2) \pm \sqrt{(\mu_1^2-\mu_2^2)^2 + 4g^2}\}$

Suppose $\mu_1 = \mu_2 = \mu$, $\mu_{a,b}^2 = \mu^2 \pm g$. For a ground state, the masses must be real, $\Rightarrow |g| \leq \mu^2$

Take $S = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. Then $S^T M S = \begin{pmatrix}\text{junk} & \frac{\mu_1^2-\mu_2^2}{2}\sin 2\theta + g\cos 2\theta\\ \frac{\mu_1^2-\mu_2^2}{2}\sin 2\theta + g\cos 2\theta & \text{junk}\end{pmatrix}$

Uncoupling $\Rightarrow$ $\tan 2\theta = \frac{2g}{\mu_2^2 - \mu_1^2}$ determines θ, hence S.

Let U be the unitary operator which transforms the fields:

$\psi = U^+\varphi U$, $U(\theta) = e^{i\theta G}$ with $G^+ = G$

Infinitesimal case: $\psi_1 = \varphi_1 - \theta\varphi_2 = \varphi_1 - i\theta[G,\varphi_1]$

$\psi_2 = \varphi_2 + \theta\varphi_1 = \varphi_2 - i\theta[G,\varphi_2]$

Thus $[G,\varphi_1] = -i\varphi_2$ & $[G,\varphi_2] = i\varphi_1$

The operator $G = \hbar^{-1}\int d^3x'\{\varphi_2(\bar{x}')\dot{\varphi}_1(\bar{x}') - \varphi_1(\bar{x}')\dot{\varphi}_2(\bar{x}')\}$ does the job. So,

$$U(\theta) = \exp\left\{\frac{i\theta}{\hbar}\int d^3x'\{\varphi_2(\bar{x}')\dot{\varphi}_1(\bar{x}') - \varphi_1(\bar{x}')\dot{\varphi}_2(\bar{x}')\}\right\}$$

The eigenstates of the Hamiltonian are generated by the correct creation operators $a_k^{(a)+}$, $a_k^{(b)+}$ acting on the vacuum:

$$\begin{pmatrix}a_k^{(a)+}\\ a_k^{(b)+}\end{pmatrix} = S^T\begin{pmatrix}a_k^{(1)+}\\ a_k^{(2)+}\end{pmatrix} \quad \text{or} \quad \begin{aligned}a_k^{(a)+} &= U^+(\theta)a_k^{(1)+}U(\theta)\\ a_k^{(b)+} &= U^+(\theta)a_k^{(2)+}U(\theta)\end{aligned}$$

Due Dec. 14 12/2/76

PHYSICS 253

Assignment 3

√1. A Hermitian field ϕ is coupled to an external source ρ via the interaction Hamiltonian density

$$\mathcal{H}_I = g\ \rho(x)\ \phi(x)$$

Show by explicit summation of the perturbation series that the time development operator is given by

$$U(t\ t_0) = :\exp\{-ig\int_{t_0}^{t} \rho(x)\ \phi(x)\ d^4x\}:$$
$$\cdot \exp\{-\frac{ig^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} \rho(x)\ \Delta_F(x-y)\ \rho(y)\ d^4x d^4y\}$$

√ 2. Prove the unitarity of the operator $U(t,t_0)$ derived in the previous problem

longer 3. A variant of the pair model we have discussed in class involves a neutral spin zero field ϕ coupled to a separable non-local external potential. The model is defined by the Hamiltonian

$$H = H_0 + \tfrac{1}{2}g:\left[\int u(\vec{r})\ \ \phi(\vec{r},t)\ d\vec{r}\ \right]^2:$$

in which H_0 is the free field Hamiltonian and $u(\vec{r})$ is a smooth function which vanishes rapidly for $r\to\infty$.

Show, by explicitly summing the perturbation series that for the scattering of a single quantum from the state $|\vec{k}>$ to the state $|\vec{k}'>$, U obeys a relation of the form

$$\langle \vec{k}' |\; U(\infty,-\infty)\; -1\; | \vec{k} \rangle = F(\vec{k},\vec{k}')\; \delta(\omega_k - \omega_{k'})$$

where F is a suitable (non-singular) function. Find an expression for F in terms of the Fourier transform of u,

$$v(\vec{k}) = \int e^{i\, k\cdot r} u(r)\; d\vec{r}.$$

4/ Let $\psi(x)$ and $\phi(x)$ be neutral spin zero fields. The ψ-quanta have mass μ while the ϕ-quanta have vanishing rest mass. We assume these fields to be coupled via the interaction Hamiltonian density

$$\mathcal{H}_I = g\; \psi(x)\; \phi^3(x),$$

so that a ψ-quantum decays spontaneously into three ϕ's. (a decay mode not wholly unlike one for the long-lived K^0.) Assuming g appropriately small, what is the relation between g and the lifetime of the ψ-quantum? If the ψ-quantum decays at rest, what is the momentum spectrum of one of the emerging ϕ-quanta?

Physics 253a - Problem Set # 3
Solutions

1.) For $\mathcal{H}_I = g\rho(x)\varphi(x)$, we want $U(t t_0)$. The Dyson expansion is

$$U(tt_0) = \sum_n \frac{(-ig)^n}{n!} \int d^4x_1 \cdots d^4x_n \, \rho(x_1)\cdots\rho(x_n)\, T\{\varphi(x_1)\cdots\varphi(x_n)\}$$

Now, use Wick's Theorem, and change integration variables so that in terms with j contractions, $\varphi(1)$ is paired with $\varphi(2)$, $\varphi(3)$ with $\varphi(4)$, etc. Thus, if we let $C(n,j)$ be the number of ways of choosing j pairs out of n objects, we have:

$$U = \sum_{nj} \frac{(-ig)^n}{n!} C(nj) \int d^4x_1 \cdots d^4x_n \, \rho(1)\cdots\rho(n)\, i\Delta_F(x_1-x_2)\cdots i\Delta_F(x_{2j-1}-x_{2j}) :\varphi(x_{2j+1})\cdots\varphi(n):$$

$$= \sum_{nj} \frac{(-ig)^n}{n!} C(nj) \left\{ \int d^4x\, d^4y\, \rho(x)\, i\Delta_F(x-y)\rho(y) \right\}^j : \left\{ \int d^4x\, \rho(x)\varphi(x) \right\}^{n-2j} :$$

We shall demonstrate that $C(nj) = \dfrac{n!}{2^j j!(n-2j)!}$, so that

$$U = :\sum_{n-2j} \frac{\left\{-ig\int \rho(x)\varphi(x)d^4x\right\}^{n-2j}}{(n-2j)!}: \quad \sum_j \frac{\left\{-\frac{ig^2}{2}\int \rho(x)\Delta_F(x-y)\rho(y)\, d^4x\, d^4y\right\}^j}{j!}$$

Thus, recognizing the exponentials, we find

$$U = \exp\left\{-\frac{ig^2}{2}\int \rho(x)\Delta_F(x-y)\rho(y)\, d^4x\, d^4y\right\} :\exp\left\{-ig\int \rho(x)\varphi(x)\, d^4x\right\}:$$

Proof that $C(nj) = \dfrac{n!}{2^j j!(n-2j)!}$: Let us denote the contraction $\varphi(1)\varphi(2)$ by 1/2, and a term like $\varphi(1)\varphi(2)\varphi(3)\varphi(4)\varphi(5)\varphi(6)\varphi(7)$ by 1/2 3 4 5/6 7. Thus the number of distinct j contraction terms in the Wick Expansion of $T(\varphi(1)\cdots\varphi(n))$ is:

$$C(n,j) = \frac{n!}{2^j\, j!\, (n-2j)!}$$

← # of permutations of fields.

(1/2 & 2/1 not distinct)

(contraction operations indistinguishable)

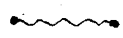
(Permutation of uncontracted fields doesn't give distinct term)

q.e.d.

A second way is to use the formula $U = \,:\exp\left(\sum_j \frac{\Theta(D_j)}{S(D_j)}\right):$ where $\Theta(D_j)$ is the operator associated with the connected diagram D_j, and $S(D_j)$ is the symmetry no..

Two connected diagrams:

(a) $\quad -ig\int \rho(x)\varphi(x)\,d^4x$

(b)  $\quad \int (-ig)\rho(x)(-ig)\rho(y)\,\overline{\varphi(x)\varphi(y)}\,d^4x\,d^4y$

$$= -ig^2\int \rho(x)\rho(y)\,\Delta_F(x-y)\,d^4x\,d^4y$$

$S(a) = 1$, $S(b) = 2$, so

$$U = \,:\exp\left(-ig\int \rho(x)\varphi(x)\,d^4x - \frac{ig^2}{2}\iint \rho(x)\rho(y)\,\Delta_F(x-y)\,d^4x\,d^4y\right):$$

q.e.d.

2.) Recall $U = :\exp\{-ig\int\rho(x)\varphi(x)d^4x\}: \exp\{-\frac{ig^2}{2}\int\rho(x)\Delta_F(x-y)\rho(y)d^4x d^4y\}$

First, let us define
$$A \equiv -ig\int\rho(x)\varphi^{(+)}(x)d^4x$$
$$B \equiv -ig\int\rho(x)\varphi^{(-)}(x)d^4x$$
$$\Theta = -\frac{g^2}{2}\int\rho(x)\Delta_F(x-y)\rho(y)d^4x\,d^4y$$

Thus, by the definition of normal order $U = e^{i\Theta}e^{B}e^{A}$.
Let us note, though, that
$$[B,A] = -g^2\int\rho(x)\rho(y)\,[\varphi^{(-)}(x),\varphi^{(+)}(y)]\,d^4x\,d^4y$$

Since $[\varphi^{(-)}(x),\varphi^{(+)}(y)]$ is a c-number, can replace it by its vacuum expectation value:
$$[\varphi^{(-)}(x),\varphi^{(+)}(y)] = -\langle 0|\varphi^{(+)}(y)\varphi^{(-)}(x)|0\rangle = -\langle 0|\varphi(y)\varphi(x)|0\rangle$$
$$= -i\Delta_+(y-x) \qquad (\hbar=1)$$

Thus,
$$[B,A] = ig^2\int\rho(x)\rho(y)\,\Delta_+(y-x)\,d^4x\,d^4y$$
$$= \frac{ig^2}{2}\int\rho(x)\rho(y)\,\{\Delta_+(x-y)+\Delta_+(y-x)\}\,d^4x\,d^4y$$
$$= \frac{g^2}{2}\int\rho(x)\rho(y)\,\Delta_1(x-y)\,d^4x\,d^4y$$

Since $[B,A]$ is a c-number, we can use the well known result $e^{B}e^{A} = e^{A+B}e^{-\frac{1}{2}[A,B]}$ to write
$$U = e^{A+B}\exp\left\{-\frac{ig^2}{2}\int\rho(x)\rho(y)d^4x\,d^4y\left[\Delta_F(x-y)+\frac{i}{2}\Delta_1(x-y)\right]\right\}$$

Now $(A+B)^\dagger = -(A+B)$, so U is unitary if $\Delta_F + \frac{i}{2}\Delta_1$ is purely real. But we saw in class that $\Delta_F = \frac{1}{2}\epsilon(x^0)\Delta - \frac{i}{2}\Delta_1$ and Δ is explicitly real, so U is unitary! q.e.d.!

3.) Hamiltonian $H_I = \frac{g}{2} : \left[\int v(\bar r)\varphi(\bar r t)\, d\bar r \right]^2 :$. Calculate amplitude for scattering from $|k\rangle$ to $|k'\rangle$ in lowest order, where $|k\rangle, |k'\rangle$ are relativistically normalized, $\langle k|k'\rangle = (2\pi)^3 2\omega_k \delta(\bar k - \bar k')$. This amplitude corresponds to the vertex $k \text{—×—} k'$:

$$k \text{—×—} k' = -i \int_{-\infty}^{\infty} dt\, d\bar r\, d\bar r'\, \frac{g}{2} v(\bar r) v(\bar r') \langle k' | :\varphi(\bar r t)\varphi(\bar r' t): | k\rangle$$

Now $\langle k' | :\varphi(rt)\varphi(r't): |k\rangle$ is easily shown to be $e^{-ik\cdot x'} e^{ik'\cdot x} + (x \leftrightarrow x')$ where $x = (\bar r t)$, $x' = (\bar r', t')$. The two terms give the same contribution, so the $\frac{1}{2}$ is cancelled. We find

$$k \text{—×—} k' = -ig\, 2\pi\, \delta(\omega_k - \omega_{k'})\, \tilde v^*(\bar k')\tilde v(\bar k),$$ where $\tilde v$ is the spatial Fourier transform of $v(\bar r)$. Thus we know the Feynman rules for the vertex. As always, an internal meson line is $\overset{}{\underset{q}{\text{———}}} = i(q^2 - \mu^2 + i\epsilon)^{-1}$.

Thus, the one particle part of the S-matrix is:

$$\langle k'|S-1|k\rangle = k \text{—×—} k' + k \text{—×——×—} k' + \text{etc.}$$

The n^{th} term is:

$$k \text{—×—}_{q_1}\text{×—}_{q_2}\text{×}\cdots_{q_{n-1}}\text{×—} k' = (-2\pi i g)^n i^{n-1} \int \frac{d^4 q's}{(2\pi)^{4(n-1)}} \prod_{j=1}^{n-1} \frac{|\tilde v(\bar q_j)|^2}{q_j^2 - \mu^2 + i\epsilon}\, \tilde v(\bar k)\tilde v^*(\bar k')$$

$$\times\, \delta(\omega_k - q_0^{(1)})\,\delta(q_0^{(1)} - q_0^{(2)}) \cdots\cdots \delta(q_0^{(n-1)} - \omega_{k'})$$

$$= -ig\, 2\pi\, \delta(\omega_k - \omega_{k'})\, \tilde v(\bar k)\tilde v^*(\bar k')\, g^{n-1} \left[\frac{d^3 q}{(2\pi)^3} \frac{|\tilde v(\bar q)|^2}{\omega_k^2 - \bar q^2 - \mu^2 + i\epsilon} \right]^{n-1}$$

$$= -2\pi i g\, \delta(\omega_k - \omega_{k'})\, \tilde v(k)\tilde v^*(\bar k')\, \alpha^{n-1}$$

where, of course $\alpha \equiv g \int \frac{d^3 q}{(2\pi)^3} \frac{|\tilde v(\bar q)|^2}{\omega_k^2 - \omega_q^2 + i\epsilon}$

Summing the series, we find

$$\langle k'|S-1|k\rangle = -2\pi i g\, \delta(\omega_k - \omega_{k'})\, \tilde v(\bar k)\tilde v^*(\bar k')\, \frac{1}{1-\alpha}$$

For non-relativistic states $|\bar k\rangle = (2\pi)^{-3/2}(2\omega_k)^{-1/2}|k\rangle$, we have

$$\langle \bar k'|S-1|\bar k\rangle = -\frac{ig\, \tilde v(\bar k)\tilde v^*(\bar k')}{2\omega_k (2\pi)^2} \left[1 - g\int \frac{d^3 q}{(2\pi)^3} \frac{|\tilde v(\bar q)|^2}{\omega_k^2 - \omega_q^2 + i\epsilon} \right]^{-1} \delta(\omega_k - \omega_{k'})$$

4.) From class, transition probability $W = \int \langle p|\mathcal{H}_I(x)\,\mathbb{P}\,\mathcal{H}_I(x')|p\rangle\, d^4x\, d^4x'$ where $\mathcal{H}_I(x) = g\,\psi(x)\varphi^3(x)$, $\mathbb{P}$ is a projection operator onto the no-ψ, three-φ states, and $|p\rangle$ is the covariantly normalized one-ψ state of four momentum p. Look at the matrix element $\langle p|\psi(x)\varphi^3(x)\,\mathbb{P}\,\psi(x')\varphi^3(x')|p\rangle$. The projection operator allows us to rewrite this matrix element as $\langle p|\psi^{(-)}(x)\psi^{(+)}(x')|p\rangle\langle 0|\varphi^{(+)3}(x)\varphi^{(-)3}(x')|0\rangle$. As in class $\langle p|\psi^{(-)}(x)\psi^{(+)}(x')|p\rangle = \dfrac{e^{ip(x-x')}}{2\omega_p V}$. Next, we must pair off the $\varphi^{(+)}$'s and $\varphi^{(-)}$'s, and there are 3! pairings, each one giving $i\Delta^{(+)}(x-x')$, so we have the result from class, multiplied by 3!:

$$w = \frac{W}{T} = 3!\,\frac{g^2}{2\omega_p(2\pi)^{-4}}\int\frac{d^3k_1\, d^3k_2\, d^3k_3}{(2\pi)^3 2\omega_{k_1}(2\pi)^3 2\omega_{k_2}(2\pi)^3 2\omega_{k_3}}\,\delta^{(4)}(p-k_1-k_2-k_3)$$

At this point, put in fact that φ mass is zero, and note that $w = \dfrac{\mu}{\omega_p} w_{rest}$. Evaluate w_{rest}:

$$w_{rest} = \frac{3!\,g^2}{2\mu(2\pi)^{-4}}\int\frac{d^3k_1\, d^3k_2}{(2\pi)^9\, 2^3}\,\frac{\delta(\mu-k_1-k_2-|\bar{k}_1+\bar{k}_2|)}{k_1\, k_2\, |\bar{k}_1+\bar{k}_2|}$$

Use the delta function to replace $|\bar{k}_1+\bar{k}_2|$ by $\mu-k_1-k_2$, and change to dimensionless variables $\bar{k} = \mu^{-1}\bar{k}_1$, $\bar{p} = \mu^{-1}\bar{k}_2$:

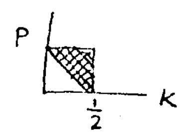
$$w_{rest} = \frac{3g^2\mu}{32\pi^2(2\pi)^3}\int d^3k\, d^3p\,\frac{\delta(1-k-p-|\bar{k}+\bar{p}|)}{kp(1-k-p)} \equiv \frac{3g^2\mu}{32\pi^2(2\pi)^3}\, I$$

Now, $$I = (4\pi)(2\pi)\int_0^\infty k\,dk\int_0^\infty p\,dp\int_{-1}^{1}dx\,\frac{\delta(1-k-p-\sqrt{k^2+p^2+2kpx})}{1-k-p}$$

Argument of δ-function is $f(x)$. $f(x)=0 \Rightarrow x = x_0 \equiv \dfrac{1-2(k+p)}{2kp}+1$
At x_0 $|f'(x_0)| = kp/(1-k-p)$, so:

$$I = 8\pi^2\int_0^\infty dk\int_0^\infty dp\int_{-1}^{1}\delta(x-x_0)$$

$|x_0|<1 \Rightarrow k+p>\frac{1}{2}$ and $k\le\frac{1}{2},\ p\le\frac{1}{2}$



$$I = 8\pi^2\int_0^{\frac{1}{2}}dk\left\{\tfrac{1}{2}-\left(\tfrac{1}{2}-k\right)\right\} = 8\pi^2\int_0^\infty dk\, k\,\theta\left(\tfrac{1}{2}-k\right) = \pi^2$$

Thus: $w_{rest} = \dfrac{3g^2\mu}{32(2\pi)^3} \Longrightarrow T_{rest} = \dfrac{32(2\pi)^3}{3g^2\mu}$

$$w = \int \mathcal{P}(\bar{k}_1)\,d^3\vec{k}_1 \Longrightarrow \mathcal{P}(\vec{k}_1) = \frac{3g^2}{8\mu(2\pi)^4}\,\frac{\theta(\frac{\mu}{2}-\bar{k}_1)}{|\bar{k}_1|}\ ;\ \frac{dw}{d|\vec{k}_1|} = \frac{3g^2 k_1\theta(\frac{\mu}{2}-k_1)}{4\mu(2\pi)^3}$$

253 s.c.t. check ~~[illegible]~~ $[a_+, a_+^\dagger] = 1$, $[a_-, a_-^\dagger] = 1$

Physics 253 a

Assignment 4 Due Thurs Jan 13.

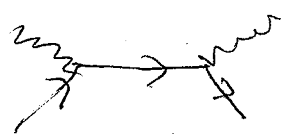
1. If charged, spin-zero "nucleons" interacted directly by means of a potential $V(\vec{r})$ their interaction Hamiltonian would presumably take the form

$$H_I = \tfrac{1}{2}\int \psi^\dagger(\vec{r}t)\, \psi(\vec{r}t)\, V(\vec{r}-\vec{r}\,')\, \psi^\dagger(\vec{r}\,'t)\, \psi(\vec{r}\,'t)\, d\vec{r}\, d\vec{r}\,'.$$

Show that there exists a function V which furnishes a description of "nucleon-nucleon" scattering equivalent to the lowest-order field theoretical treatment discussed in class (including the exchange term). Evaluate that V.

Yukawian

2. Find the differential cross section to order g^2 for the scattering of "mesons" by "nucleons" in the spin-zero theory we have discussed in class. Give its form both in the center of mass system and the rest system of the "nucleons".



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Physics 253a - Homework # 4 - Solutions

1.) Recall from class, for nucleon-nucleon scattering, we had to lowest order:

$$\langle p_1' p_2' | S-1 | p_1 p_2 \rangle = -i(2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2)\, g^2 \left\{ \frac{1}{(p_1'-p_1)^2-\mu^2+i\epsilon} + \frac{1}{(p_1-p_2')^2-\mu^2+i\epsilon} \right\}$$

This is elastic scattering, so $(p_1'-p_1)^2 = -\bar{q}^2$ in the center of mass, also $(p_1-p_2')^2 = -(\bar{q}+2\bar{p})^2$.

Now we look at the interaction

$$H_I(t) = \tfrac{1}{2} : \int d\bar{r}\, d\bar{r}'\, \psi^+(\bar{r}t)\psi(\bar{r}t) V(\bar{r}-\bar{r}') \psi^+(\bar{r}'t)\psi(\bar{r}'t) :$$

As in class, we use only the "nucleon" part of the field operators (forget about c and c^+). To lowest order:

$$\langle p_1' p_2' | S-1 | p_1 p_2 \rangle = -i \int dt\, d\bar{r}\, d\bar{r}'\, \tfrac{1}{2} V(\bar{r}-\bar{r}') \langle p_1' p_2' | : \psi^+(\bar{r}t)\psi(\bar{r}t)\psi^+(\bar{r}'t)\psi(\bar{r}'t) : | p_1 p_2 \rangle$$

Now, the above-quoted result was derived in class from the following:

$$\langle p_1' p_2' | S-1 | p_1 p_2 \rangle = -\frac{ig^2}{2} \int dx\, dx'\, \Delta_F(x-x') \langle p_1' p_2' | : \psi^+(x)\psi(x)\psi^+(x')\psi(x') : | p_1 p_2 \rangle$$

There is a great similarity between the two expressions. We exploit this by introducing $U(x-x') \equiv V(\bar{r}-\bar{r}')\delta(t-t')$. Then with $\tilde{U}(k) = \int d^4x\, e^{ikx} U(x)$, we have

$$\langle p_1' p_2' | S-1 | p_1 p_2 \rangle = -i(2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2) \left[\tilde{U}(p_1'-p_1) + \tilde{U}(p_1-p_2') \right]$$

Now

$$\tilde{U}(q) = \tilde{V}(\bar{q}), \text{ where } \tilde{V}(\bar{q}) = \int d\bar{r}\, e^{-i\bar{q}\cdot\bar{r}} V(\bar{r}), \text{ so}$$

we have in general:

$$\tilde{U}(p_1'-p_1) + \tilde{U}(p_1-p_2') = \tilde{V}(\bar{q}) + \tilde{V}(\bar{q}+2\bar{p})$$

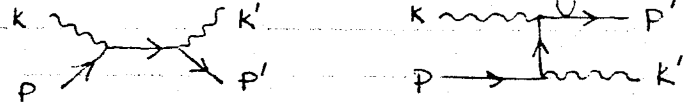
This must be equal to $-g^2 \left\{ \frac{1}{\bar{q}^2+\mu^2-i\epsilon} + \frac{1}{(\bar{q}+2\bar{p})^2+\mu^2-i\epsilon} \right\}$ for the center of mass amplitudes to agree. Thus

$$\tilde{V}(\bar{q}) = -\frac{g^2}{\bar{q}^2+\mu^2-i\epsilon}$$

or

$$V(r) = -\frac{g^2 e^{-\mu r}}{4\pi r}$$

2.) Meson-nucleon scattering to lowest order. Two graphs,



The S-matrix elements are:

$$\langle k'p'|S|kp\rangle = -g^2(2\pi)^4\delta^{(4)}(k'+p'-k-p)\left\{\frac{i}{(p+k)^2-m^2+i\epsilon}+\frac{i}{(k'-p)^2-m^2+i\epsilon}\right\}$$

The T-matrix is:

$$\langle k'p'|T|kp\rangle = -g^2\left\{\frac{1}{(p+k)^2-m^2+i\epsilon}+\frac{1}{(k'-p)^2-m^2+i\epsilon}\right\} = -g^2\left\{\frac{1}{s-m^2+i\epsilon}+\frac{1}{u-m^2+i\epsilon}\right\}$$

From class, $\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.o.m.,elastic}} = \frac{1}{64\pi^2 s}|T|^2$, so

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right)_{\text{c.o.m.}} = \frac{g^4}{64\pi^2 s}\left[\frac{1}{s-m^2}+\frac{1}{u-m^2}\right]^2 = \left.\frac{d\sigma(u,s)}{d\Omega}\right)_{\text{c.o.m.}}}$$

Now, we want $\frac{d\sigma}{d\Omega}$ in nucleon rest frame. Since σ is invariant,

$$\left.\frac{d\sigma}{d\Omega}\right)_{n\text{ at rest}} = \left.\frac{d\sigma}{d\Omega}\right)_{\text{c.o.m.}}\left(\frac{d\Omega_{\text{c.o.m.}}}{d\Omega_{n\text{ at rest}}}\right)$$

We must calculate this

let subscript c refer to c.o.m. frame, ℓ to lab frame. Define relativistic factors $\gamma_1, \gamma_2, \beta_1, \beta_2$ by (unprimed $\leftrightarrow$ before collision, primed $\leftrightarrow$ after collision),

$|\bar{k}_c| = \mu\beta_1\gamma_1 = |\bar{k}_c'|$ $\quad |k_c^0| = \mu\gamma_1 = k_c'^0$ $\quad \gamma = (1-\beta^2)^{-1/2}$

$|\bar{p}_c| = m\beta_2\gamma_2 = |\bar{p}_c'|$ $\quad p_c^0 = m\gamma_2 = p_c'^0$

The c.o.m. condition is $|\bar{p}_c| = |\bar{k}_c| \Longrightarrow \mu\beta_1\gamma_1 = m\beta_2\gamma_2$

Eliminate β's, $\boxed{\mu(\gamma_1^2-1)^{1/2} = m(\gamma_2^2-1)^{1/2}}$

Let γ be defined by $k_\ell^0 = \mu\gamma$. Lorentz transformation of momenta gives $(|\bar{k}_c| = \gamma_2(|\bar{k}_\ell| - \beta_2 k_\ell^0))$ (before collision)

$$\mu(\gamma_1^2-1)^{1/2} = \gamma_2\,\mu(\gamma^2-1)^{1/2} - (\gamma_2^2-1)^{1/2}\mu\gamma$$

or

$$\boxed{(\gamma_1^2-1)^{1/2} = \gamma_2(\gamma^2-1)^{1/2} - \gamma(\gamma_2^2-1)^{1/2}}$$

Let mass ratio be $\lambda \equiv \frac{\mu}{m}$. Can solve two boxed equations for γ_2, γ_1 as functions of γ, λ.

The results are
$$\gamma_1 = \frac{\gamma+\lambda}{(1+2\gamma\lambda+\lambda^2)^{1/2}}$$
$$\gamma_2 = \frac{1+\lambda\gamma}{(1+2\gamma\lambda+\lambda^2)^{1/2}}$$

Clearly λ is trivial to calculate, and γ is the ratio of meson energy to meson mass in lab frame, also trivial to calculate. γ_2 transforms four vectors from lab to c.o.m. frame.

Scattering angles: (Take scattering in x-y plane, relative motion along x).

c.o.m. Θ_c
$$\vec{K}_c' = |\vec{K}_c|(\cos\Theta_c\hat{x} + \sin\Theta_c\hat{y})$$
lab Θ_ℓ
$$\vec{K}_\ell' = |\vec{K}_\ell|(\cos\Theta_\ell\hat{x} + \sin\Theta_\ell\hat{y})$$

Lorentz transform
$$(\vec{K}_\ell')_x = \gamma_2(|\vec{K}_c|\cos\Theta_c + \beta_2 K_c^0) = |\vec{K}_\ell|\cos\Theta_\ell$$
$$(\vec{K}_\ell')_y = |\vec{K}_c|\sin\Theta_c = |\vec{K}_\ell|\sin\Theta_\ell$$

Take the ratio, and use $k_c^0 = \mu\gamma_1$, $|\vec{k}_c| = \mu\beta_1\gamma_1$

$$\tan\Theta_\ell = \frac{\sin\Theta_c}{\gamma_2(\cos\Theta_c + \beta_2\frac{k_c^0}{|\vec{k}_c|})} = \frac{\sin\Theta_c}{\gamma_2(\cos\Theta_c + \frac{\beta_2}{\beta_1})}$$

$$\boxed{\tan\Theta_\ell = \frac{\sin\Theta_c}{(\gamma_2\cos\Theta_c + \gamma_1\lambda)}}$$ where we have used $\frac{\beta_2}{\beta_1} = \frac{\gamma_1\lambda}{\gamma_2}$

By differentiating the above equation with respect to Θ_c, one obtains

$$\left(\frac{d\Omega_{c.o.m.}}{d\Omega_\ell}\right) = \frac{d(\cos\Theta_c)}{d(\cos\Theta_\ell)} = \frac{[\sin^2\Theta_c + (\gamma_2\cos\Theta_c + \gamma_1\lambda)^2]^{3/2}}{|\gamma_2 + \lambda\gamma_1\cos\Theta_c|}$$

and thus, the lab cross-section is

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{[\sin^2\Theta_c + (\gamma_2\cos\Theta_c + \gamma_1\lambda)^2]^{3/2}}{|\gamma_2 + \gamma_1\lambda\cos\Theta_c|}\frac{g^4}{64\pi^2 s}\left[\frac{1}{s-m^2} + \frac{1}{u-m^2}\right]^2}$$

Physics 253a

Final Examination

The completed examination paper is to be returned to the secretaries within 72 hours of the time at which the problems are picked up.

The only reference materials you are allowed to use are the lecture notes for the course, the homework problems and their solutions, the two volumes of Bjorken and Drell and standard table of integrals and of the properties of special functions. (Please cite references to the latter tables when or if you use them.)

If you have questions regarding the statement of the problems please phone me (Office: 495-2869 Home: 648-8546) or consult Mr. Ling who will be available in Jefferson 462 G from 4 to 5 P.M. and from 10 to 11 A.M. from Monday to Thursday January 17-20. Mr. Ling will also be available by phone during some periods at his home: 498-5023.

Please do all four problems. Their point credits are shown in parentheses.

1. (20) Find the amplitude (to lowest order in the coupling constant g) for the scattering of a "nucleon" by an "antinucleon" in the spin-zero theory which couples the meson field ϕ to the "nucleon" field ψ via the interaction Hamiltonian density

$$\mathcal{H}_I = g:\ \psi^\dagger(x) \quad \psi(x) \quad \phi(x):$$

Evaluate the corresponding differential cross section in the center-of-mass system.

2. (20) We have noted in class that if in the theory described in problem 1, the meson mass μ exceeds twice the "nucleon" mass m the meson decays into a nucleon-antinucleon pair. Find the decay rate in the laboratory system for a heavy meson of this type with momentum $\vec{k}$. The matrix $\langle\beta|\mathcal{T}|\alpha\rangle$ is defined via the relation

$$\langle\beta|\mathcal{S}-1|\alpha\rangle = (2\pi)^4\ i\delta(P_\beta - P_\alpha)\ \langle\beta|\mathcal{T}|\alpha\rangle.$$

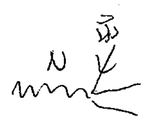
The diagonal element of $\mathcal{T}$ in a one-meson state, as we have noted in class, is real-valued for $\mu<2m$ and complex-valued for $\mu>2m$. Show explicitly (to second order in g) that these diagonal elements obey the identity which follows from the unitarity of $\mathcal{S}$.

3. (30) In this problem we discuss a spin-zero analogue of the decay of the μ-meson. Let us assume that we have the following complex fields: ψ_μ with quanta of mass μ, ψ_e with mass $m<\mu$ and $\psi_{\nu\mu}$ and $\psi_{\nu e}$ which are different "nuetrino" fields both with mass zero. Then the interaction Hamiltonian density

$$\mathcal{H}_I = G\{\psi^\dagger_\mu \, \psi_{\nu\mu} \, \psi^\dagger_{\nu e} \, \psi_e + \text{Herm. conj.}\}$$

describes for example, the decay

$$\mu^+ \to \bar{\nu}_\mu + \nu_e + e^+ .$$



a) Find the momentum spectrum of the electrons resulting from the decay of μ's at rest. Find the inverse life-time assuming $\mu \gg m$.

b) It has long been suspected that this decay process actually takes place in two stages, through the mediation of still another field ϕ_W with quanta of mass $M>\mu$. Let us assume that the actual interaction Hamiltonian density is

$$\mathcal{H}_I = g\{\psi^\dagger_\mu \, \psi_{\nu\mu} \, \phi_w + \psi^\dagger_{\nu e} \, \psi_e \, \phi^\dagger_w + \text{Herm. Conj.}\}$$

Find once again the momentum spectrum of the decay electrons and the inverse lifetime for $\mu \gg m$.

c) Compare in a thumbnail sketch the spectra which result from calculations a) and b). How could the spectrum shape be used to determine the mass M? When would the method become insensitive to M? What is the relation between the coupling constants g and G if the two calculations are to yield the same lifetime?

4. (30) The problem which follows involves a number of parallels with phenomena observed to take place in the interactions of neutral K mesons.

In problem 4 of the second problem set we discussed a situation in which two neutral spin-zero fields ϕ_1 and ϕ_2 are coupled by an interaction of the form

$$\mathcal{H}_I = g\ \phi_1(x)\ \phi_2(x).$$

We noted there that by defining two new fields ψ_1 and ψ_2 which are simply linear combinations of ϕ_1 and ϕ_2 the total Hamiltonian may be brought into a form in which ψ_1 and ψ_2 are free fields. The "physical" particles of the problem are thus the quanta of ψ_1 and ψ_2.

Let us assume for simplicity that the two fields ϕ_1 and ϕ_2 (when uncoupled) have quanta of equal mass μ. These fields although we have thus far given them similar descriptions, may in fact be quite distinguishable. It could happen for example that the quanta of the fields ϕ_1 and ϕ_2 are subject to different scattering interactions. Let us assume that the quanta of ϕ_1 are scattered by a static potential $\mathcal{V}(\vec{r})$, i.e. the interaction Hamiltonian density contains an additional term

$$\mathcal{H}_I = \mathcal{V}(\vec{r}) : \phi_1^{\,2}(\vec{r}\ t) : ,$$

while the quanta of ϕ_2 are not scattered at all.

a) Let us imagine a scattering experiment in which a beam of freely propagating ψ_1-quanta is incident upon the potential $\mathcal{V}(\vec{r})$. Show that the scattering process will accomplish the miracle of generating ψ_2-quanta. Find to lowest order in v the differential cross sections for production of ψ_2-quanta and for scattering of ψ_1-quanta.

b) When N scattering centers are present their potential may be written as $\sum_{j=1}^{n} v(\vec{r} - \vec{r}_j)$ where $\vec{r}_j$ are the positions of their centers. Let us assume that the points $\vec{r}_j$ all lie in a plane perpendicular to the momentum of the incident ψ_1 beam and that they are randomly (and uniformly) distributed over a disk of radius a within that plane. We take $N \gg 1$ and a to be of macroscopic magnitude. Find the angular distribution of ψ_2 quanta generated by the disk (assuming that the incident beam is collimated over the full area of the disk).

c) If the coupling constant g is exceedingly small the mass difference of the ψ_1 and ψ_2 quanta may be far too small to detect directly. Show that it may indeed be measured however by means of a suitable double collision experiment. Assume for definiteness that the first scatterer is centered at the origin and the second at the point $\vec{R}$ which is far enough from the origin that energy conservation may be assumed to hold precisely in propagation out to it. Assume that the range of the potentials v is negligibly small in comparison with the distance R. You may find the following identity helpful:

$$\lim_{\varepsilon\to 0} \frac{1}{x+i\varepsilon} = \lim_{\varepsilon\to 0} \frac{x}{x^2+\varepsilon^2} - \frac{i\varepsilon}{x^2+\varepsilon^2}$$

$$= \text{Principal value } \left(\frac{1}{x}\right) - i\pi\,\delta(x)$$

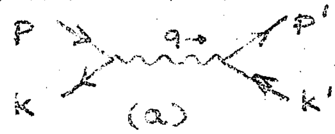
If you find the asymptotic forms required for $R\to\infty$ to be unmanageable you may (at some expense in credit) assume that each potential is a delta function. (Delta functions are too singular to represent actual potentials in three dimensions. They are usable in lowest-order Born approximations but generate nonsense in the higher orders.)

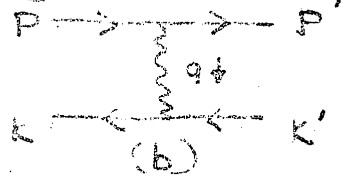
PHYSICS 253 a - FINAL EXAMINATION

— SOLUTIONS —

1.) $\mathcal{H}_I = g : \psi^\dagger(x)\psi(x)\varphi(x) :$

Nucleon-anti-nucleon scattering. There are two diagrams to lowest order:





(Time runs from left to right, i.e. primed momenta refer to values after the collision).

S-matrix elements are:

$$S_{(a)} = \int \frac{d^4q}{(2\pi)^4}(-ig)(2\pi)^4\delta^{(4)}(p+k-q)\frac{i}{q^2-\mu^2+i\epsilon}(-ig)(2\pi)^4\delta^{(4)}(p'+\cdots$$

$$= -ig^2(2\pi)^4\frac{1}{(p+k)^2-\mu^2+i\epsilon}\delta^{(4)}(p'+k'-p-k)$$

$$S_{(b)} = \int\frac{d^4q}{(2\pi)^4}(-ig)(2\pi)^4\delta^{(4)}(p'+q-p)\frac{i}{q^2-\mu^2+i\epsilon}(-ig)(2\pi)^4\delta^{(4)}(k+q-k'$$

$$= -ig^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)\frac{1}{(p'-p)^2-\mu^2+i\epsilon}$$

Thus with Mandelstam variables $s \equiv (p+k)^2$, $t \equiv (p'-p)^2$,

$$S = -ig^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)\left\{\frac{1}{s-\mu^2}+\frac{1}{t-\mu^2}\right\}$$

This gives us the T-matrix (or invariant amplitude)

$$T = -g^2\left\{\frac{1}{s-\mu^2}+\frac{1}{t-\mu^2}\right\}$$

The center of mass elastic cross-section is

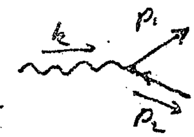
$$\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 s}\left|\frac{1}{s-\mu^2}+\frac{1}{t-\mu^2}\right|^2$$

2) Decay of a "meson" $\varphi \to N + \bar{N}$ $\mu > 2m$

$$\frac{W}{VT} = \text{Decay rate / unit volume} = \text{Beam density} \times \text{Decay rate / particle}$$

$$= 2\omega_k \cdot w$$

$$= \frac{g^2}{(2\pi)^2} \int \delta^4(k - p_1 - p_2)\, \delta(p_1^2 - m^2)\, \delta(p_2^2 - m^2)\, d^4p_1 d^4p_2$$



$$= \frac{g^2}{(2\pi)^2} \int \delta^3(\vec{k} - \vec{p}_1 - \vec{p}_2)\, \delta(\omega_k - \omega_{p_1} - \omega_{p_2}) \frac{d\vec{p}_1}{2\omega_{p_1}} \frac{d\vec{p}_2}{2\omega_{p_2}}$$

This expression is invariant. Nothing is lost by evaluating it in the "meson" rest frame $\vec{k} = 0$, $\omega_k = \mu$

$$\frac{W}{VT} = \frac{g^2}{(2\pi)^2} \int \delta(\mu - 2\sqrt{p_1^2 + m^2})\, \frac{4\pi p_1^2\, dp_1}{4\,\omega_{p_1}^2}$$

$$= \frac{g^2}{(2\pi)^2} \cdot \frac{\pi p_1^2}{\omega_{p_1}^2} \cdot \frac{\omega_{p_1}}{2p_1}$$

$$= \frac{g^2}{8\pi} \frac{p_1}{\omega_{p_1}} = \frac{g^2}{8\pi} \frac{\sqrt{\frac{\mu^2}{4} - m^2}}{\frac{\mu}{2}} = \frac{g^2}{8\pi} \sqrt{1 - \frac{4m^2}{\mu^2}}$$

By definition of the T matrix –

$$\langle \beta | S - 1 | \alpha \rangle = (2\pi)^4\, i\, \delta(P_\beta - P_\alpha) \langle \beta | T | \alpha \rangle$$

On the other hand for the diagram

we have written in class

$$\langle k' | S^{(2)} - 1 | k \rangle = \delta^4(k - k')\, g^2\, I$$

where I is an integral we discussed at length. In "regulated" form I is given by

$$I_R = i\pi^2 \left\{ \log \frac{M^2}{m^2} - \int_0^1 \log\left(1 - \frac{k^2}{m^2} x(1-x) - i\epsilon\right) dx \right\}$$

for $M \to \infty$.

The diagonal element of the T-matrix is evidently given (to second order) by

$$\langle k| T |k\rangle = \frac{g^2}{(2\pi)^4 i}\, I$$

The imaginary part of $\langle k| T |k\rangle$ is then given by

$$\mathrm{Im}\,\langle k| T |k\rangle = \frac{g^2}{16\pi^2}\,\mathrm{Im}\left\{\log\frac{M^2}{m^2} - \int_0^1 \log\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right)dx\right\}$$

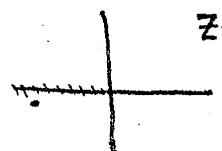
Note that our use of I_R here rather than the infinite integral I makes no difference. The imaginary part is finite and independent of the regulator mass for $M\to\infty$.

Now for $k^2<4m^2$ $\quad \mathrm{Re}\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right)>0$ for $0<x<1$.

For $k^2>4m^2$ however $1-\frac{k^2}{m^2}x(1-x)$ becomes <u>negative</u> for x between the bounds $\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{m^2}{k^2}}$

Hence the imaginary part of the logarithmic integral is given by

$$\mathrm{Im}\int_0^1 \log\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right)dx = -\pi\int_0^1 \theta\left(\frac{k^2}{m^2}x(1-x)-1\right)dx$$
$$= -2\pi\sqrt{\frac{1}{4}-\frac{m^2}{k^2}}$$
$$= -\pi\sqrt{1-\frac{4m^2}{k^2}}$$



$\log z = \log|z| + i\arg z$

$z = 1-\frac{k^2}{m}x(1-x)-i\epsilon$

$\arg z = -\pi$

for $1-\frac{k^2}{m}x(1-x)<0$

(The negative sign is due to $\epsilon>0$.)

The diagonal element of the unitarity identity is

$$2\,\mathrm{Im}\,T_{\alpha\alpha} = (2\pi)^4\sum_\beta \delta^4(P_\beta-P_\alpha)\,|T_{\beta\alpha}|^2$$
$$= \frac{W}{VT} = \text{Decay rate / unit volume}$$

From our calculations we have

$$2\,\mathrm{Im}\,\langle k| T |k\rangle = \frac{2g^2}{16\pi^2}\cdot\pi\sqrt{1-\frac{m^2}{4k^2}} = \frac{g^2}{8\pi}\sqrt{1-\frac{4m^2}{k^2}} = \frac{W}{VT}$$

i.e. The identity is satisfied.

3.) Muon decay $\mu^+ \longrightarrow \bar{\nu}_\mu + \nu_e + e^+$.

Part a) $\mathcal{H}_I = G\{\psi^\dagger_\mu \psi_{\nu_\mu} \psi^\dagger_{\nu_e} \psi_{e^+} + h.c.\}$

From class, we know

$$w = \frac{G^2}{2\omega_p}\frac{1}{(2\pi)^5}\int_{k_0,q_0,q_0'>0} d^4k\,d^4q\,d^4q'\ \delta(k^2-m^2)\,\delta(q^2)\,\delta(q'^2)\,\delta^{(4)}(p-k-q-q$$

Let $\mathcal{U}(p-k) \equiv \int_{q_0,q_0'>0} d^4q\,d^4q'\ \delta(q^2)\delta(q'^2)\,\delta^{(4)}(p-k-q-q')$.

Two cases to consider. First suppose $p-k$ is timelike, $(p-k)^2>$
Then there is a frame in which $\bar{p}=\bar{k}$, and since $\mathcal{U}$ is invariant, we evaluate it there:

$$\mathcal{U}(p-k) = \int \frac{d\bar{q}}{2|\bar{q}|}\frac{d\bar{q}'}{2|\bar{q}'|}\ \delta(p_0-k_0-|\bar{q}|-|\bar{q}'|)\,\delta^{(3)}(\bar{q}+\bar{q}') = \pi\int_0^\infty dq\ \delta(p_0-k_0-2$$

$$= \pi/2$$

Next, suppose $p-k$ is spacelike, $(p-k)^2<0$. Then there is a fram in which $p_0=k_0$, and thus

$$\mathcal{U}(p-k) = \int \frac{d\bar{q}}{2|\bar{q}|}\frac{d\bar{q}'}{2|\bar{q}'|}\ \delta(|\bar{q}|+|\bar{q}'|)\,\delta^{(3)}(\bar{p}-\bar{k}-\bar{q}-\bar{q}') = 0$$ because $|\bar{q}|+|\bar{q}'|>0$.

Hence, $\mathcal{U}(p-k) = \frac{\pi}{2}\theta((p-k)^2)$. The transition rate becom

$$w = \frac{G^2}{2\omega_p}\frac{1}{(2\pi)^5}\int_{k_0>0} d^4k\ \delta(k^2-m^2)\,\frac{\pi}{2}\theta((p-k)^2)$$

Take muon at rest, $p=(\mu,\bar{0})$, then,

$$w = \frac{G^2}{2\mu}\frac{(\pi/2)}{(2\pi)^5}\int\frac{d^3\bar{k}}{2\omega_{\bar{k}}}\ \theta((\mu-\omega_{\bar{k}})^2-\bar{k}^2)$$

Now $(\mu-\omega_{\bar{k}})^2 = \mu^2-2\mu\omega_{\bar{k}}+\bar{k}^2+m^2$. The theta function deman that $\omega_{\bar{k}} < (\mu^2+m^2)/2\mu$. Let us imagine that m is much smaller than μ. Ignoring m, we have

$$w = \frac{G^2}{2\mu}\frac{(\pi/2)}{(2\pi)^5}\int\frac{d^3k}{2|\bar{k}|}\ \theta\left(\frac{\mu}{2}-|\bar{k}|\right)$$

Thus the spectrum is $\frac{dw}{dk} = \frac{G^2}{8\mu(2\pi)^3}k\,\theta\left(\frac{\mu}{2}-k\right)$.
and the lifetime is

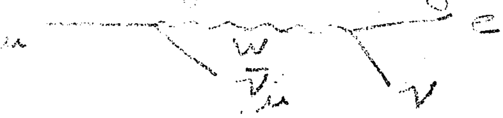
$$\frac{1}{\tau} = \frac{\mu G^2}{64(2\pi)^3}$$

Part b) Here we include the W-boson, via the interaction Hamiltonian,

$$\mathcal{H}_I = g\{\psi_\mu^\dagger \psi_{\nu_\mu} \phi_W + \psi_{\nu_e}^\dagger \psi_e \phi_W^\dagger + h.c.\}$$

To lowest order, there is only one diagram contribut

to $\mu^+ \to \bar{\nu}_\mu + \nu_e + e^+$,



The T-matrix is

$$\langle kqq'|T|p\rangle = -g^2\left\{\frac{1}{(p-q)^2 - M^2 + i\epsilon}\right\}$$

where again p = muon momentum $\quad q$ = μ-neutrino mom.

k = electron momentum $\quad q'$ = e-neutrino mo

From class, we have

$$w = \frac{1}{2\omega_p}\frac{1}{(2\pi)^6}\int_{k_0, q_0, q_0' > 0} d^4k\, d^4q\, d^4q'\, \delta(k^2 - m^2)\,\delta(q^2)\,\delta(q'^2)\,\delta(p-k-q-q')|\langle kqq'|$$

Let us define the invariant integral $I(p,k)$ by

$$I(p,k) = \int_{q_0, q_0' > 0} d^4q\, d^4q'\, \delta(q^2)\,\delta(q'^2)\,\delta(p-k-q-q')\left|\frac{1}{(p-q)^2 - M^2 + i\epsilon}\right|$$

As in part (a), for $p-k$ spacelike, I vanishes. Assu then that $p-k$ is timelike, and transform to frame in which $\bar{p} - \bar{k} = 0$.

$$I = \int \frac{d^3\bar{q}}{4|\bar{q}|^2}\,\delta(p_0 - k_0 - 2|\bar{q}|)\left|\frac{1}{(p_0 - |\bar{q}|)^2 - (\bar{p} - \bar{q})^2 - M^2}\right|^2$$

Do the $|\bar{q}|$ integral, which leaves an angular integrati

$|\bar{q}| \to \frac{1}{2}(p_0 - k_0)$,

$$I = \frac{2\pi}{8}\int_{-1}^{1} d(\cos\theta)\left|\frac{1}{\left(\frac{p_0+k_0}{2}\right)^2 - \bar{p}^2 - \left(\frac{p_0-k_0}{2}\right)^2 + 2p\left(\frac{p_0-k_0}{2}\right)\cos\theta - M^2}\right|^2$$

$$= \frac{\pi}{4}\int_{-1}^{1} d(\cos\theta)\left|\frac{1}{(p_0k_0 - \bar{p}^2 - M^2) + |\bar{p}|(p_0 - k_0)\cos\theta}\right|^2$$

Let $\quad A \equiv p_0k_0 - \bar{p}^2 - M^2, \quad B \equiv |\bar{p}|(p_0 - k_0) \qquad y \equiv A + B\cos\theta$

$$\mathcal{I} = \frac{\pi}{4B}\int_{A-B}^{A+B}\frac{dy}{y^2} = -\frac{\pi}{4B}\left[(A+B)^{-1} - (A-B)^{-1}\right] = \frac{\pi}{2}\,\frac{1}{A^2-B^2}$$

$\mathcal{I} = \frac{\pi}{2}\frac{1}{A^2-B^2}$. Now we need to rewrite A, B in terms of invariants. In this frame $\bar{p} = \bar{k}$,
so $p_0 k_0 - \bar{p}^2 = p_0 k_0 - \bar{p}\cdot\bar{k} = p\cdot k$, and thus $A = p\cdot k - M^2$
Next, look at $B^2 = |\bar{p}|^2 (p_0 - k_0)^2 = (p_0^2 - \mu^2)(p_0 - k_0)^2$
Since $\bar{p} - \bar{k} = 0$, $p_0(p_0 - k_0) = p\cdot(p-k)$, and $(p_0 - k_0)^2 = (p-k)^2$
so $B^2 = [p\cdot(p-k)]^2 - [p-k]^2\mu^2$

In order to evaluate the lifetime and the spectrum when the muon is at rest, we take $m = 0$, and $\bar{p} = 0$. Then

$$A = \mu k_0 - M^2 = \mu|\bar{k}| - M^2$$
$$B^2 = [\mu(\mu - |\bar{k}|)]^2 - \mu^2[\mu^2 - 2\mu|\bar{k}| + \cancel{m^2}] = \mu^2|\bar{k}|^2$$
$$A^2 - B^2 = M^4 - 2\mu|\bar{k}|M^2 = M^2(M^2 - 2\mu|\bar{k}|).$$

Finally, then

$$w = \frac{g^4}{2\mu}\frac{1}{(2\pi)^3}\int\frac{d^3\bar{k}}{2|\bar{k}|}\,\frac{\pi}{2}\,\frac{1}{M^2(M^2 - 2\mu|\bar{k}|)}\,\theta\left(\frac{\mu}{2} - |\bar{k}|\right)$$

Thus, the spectrum is $\dfrac{dw}{dk} = \dfrac{g^4\,|\bar{k}|}{8\mu(2\pi)^3 M^4}\,\dfrac{\theta\left(\frac{\mu}{2} - |\bar{k}|\right)}{\left(1 - \frac{2\mu|\bar{k}|}{M^2}\right)}$

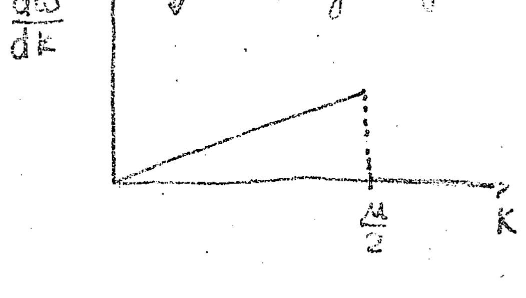
and the lifetime is:

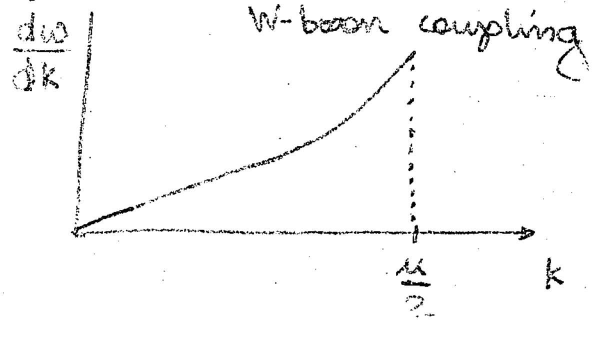
$$\frac{1}{\tau} = \frac{g^4}{8\mu M^4(2\pi)^3}\int_0^{\mu/2}\frac{k\,dk}{\left(1 - \frac{2\mu k}{M^2}\right)} = \frac{g^4}{8\mu M^4(2\pi)^3}\left(\frac{\mu}{2}\right)^2\int_0^1\frac{x\,dx}{\left[1 - \left(\frac{\mu}{M}\right)^2 x\right]}$$

or

$$\frac{1}{\tau} = \frac{g^4}{8\mu M^4(2\pi)^3}\,\frac{\mu^2}{4}\left[\left(\frac{M}{\mu}\right)^4 \ln\left\{\frac{1}{1 - \left(\frac{\mu}{M}\right)^2}\right\} - \left(\frac{M}{\mu}\right)^2\right]$$

c.) The spectra are shown below.





Thus, there is a deviation from linearity in the spectrum due to the intermediate W boson, and from an examination of experimental data one can infer a value of M/μ.

As $M \to \infty$, the spectrum becomes linear, and if

$$\frac{g^4}{M^4} = G^2$$

the lifetimes agree.

If M were much larger than μ, then the spectrum would depend only on g^4/M^4, and one couldn't learn anything about M alone.

4. With the interaction $H_I = g\,\varphi_1\varphi_2$ the Lagrangian density is

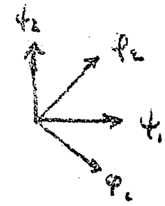
$$\mathcal{L} = \tfrac{1}{2}\left\{\partial_\mu\varphi_1\partial^\mu\varphi_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2 - \mu^2(\varphi_1^2+\varphi_2^2) - 2g\varphi_1\varphi_2\right\}$$

We set

$$\psi_1 = \tfrac{1}{\sqrt{2}}(\varphi_1+\varphi_2) \qquad\qquad \varphi_1 = \tfrac{1}{\sqrt{2}}(\psi_1-\psi_2)$$

$$\psi_2 = \tfrac{1}{\sqrt{2}}(\varphi_2-\varphi_1) \qquad\qquad \varphi_2 = \tfrac{1}{\sqrt{2}}(\psi_1+\psi_2)$$

(Note: Interchanged identifications of ψ_1 and ψ_2 are possible as is the insertion of phase factors. All these only alter unimportant overall phase factors below.)



Note
$$\psi_1^2+\psi_2^2 = \varphi_1^2+\varphi_2^2$$
$$\psi_1^2-\psi_2^2 = 2\varphi_1\varphi_2$$

Hence

$$\mathcal{L} = \tfrac{1}{2}\left\{\partial_\mu\psi_1\partial^\mu\psi_1 + \partial_\mu\psi_2\partial^\mu\psi_2 - (\mu^2+g)\psi_1^2 - (\mu^2-g)\psi_2^2\right\}$$

i.e. The fields ψ_1 and ψ_2 create freely propagating, uncoupled quanta with masses given by $\mu_1^2=\mu^2+g$, $\mu_2^2=\mu^2-g$ respectively.

We can introduce (but don't necessarily need) the Fourier resolutions

$$\varphi_i(x) = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_k}\left\{a_i(k)e^{-ikx} + a_i^\dagger(k)e^{ikx}\right\}d\vec{k}$$

$$\psi_i(x) = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_k}\left\{b_i(k)e^{-ikx} + b_i^\dagger(k)e^{ikx}\right\}d\vec{k}$$

which imply

$$b_1 = \tfrac{1}{\sqrt{2}}(a_1+a_2)$$
$$b_2 = \tfrac{1}{\sqrt{2}}(a_2-a_1)$$

The scattering interaction involves the φ_1 quanta only:

$$H_I = v(\vec{r}):\varphi_1^2:$$
$$= \tfrac{1}{2}v(\vec{r}):(\psi_1^2+\psi_2^2-2\psi_1\psi_2):$$

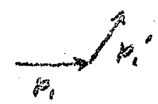
4(a) Scattering – production processes in lowest order:

The incident ψ_1's are a superposition containing φ_1's. The φ_1's are scattered, but these represent a superposition containing ψ_2's. Hence the scattering process generates ψ_2's as well as ψ_1's.

Initial one ψ_1 state $\quad |i\rangle = |\vec{p}_1\rangle = b_1^\dagger(\vec{p}_1)|0\rangle$

Final one ψ_1 state $\quad |f\rangle = |\vec{p}_1{}'\rangle = b_1^\dagger(\vec{p}_1{}')|0\rangle$, or alternatively

" one ψ_2 state $\quad |f\rangle = |\vec{p}_2{}'\rangle = b_2^\dagger(\vec{p}_2{}')|0\rangle$



Consider first the scattering process $\psi_1 \to \psi_1$. Only the ψ_1^2 term in H_I contributes.

$$\langle f|\, S^{(1)} - 1\,|i\rangle = -i\langle f|\int \mathcal{H}_I(x)\, d^4x\,|i\rangle$$

$$= -\frac{i}{2}\langle \vec{p}_1{}'|:\int \psi_1^2(x)\, v(\vec{r})\, d^4x :|\vec{p}_1\rangle$$

$$= -i\int e^{-i(p_1 - p_1')x}\, v(r)\, d^4x$$

$$= -i(2\pi)\,\delta(p_1^0 - p_1^{0\prime})\,\tilde{v}(\vec{p}_1 - \vec{p}_1{}')$$

where $\tilde{v}(\vec{p}) = \int e^{i\vec{p}\cdot\vec{r}}\, v(\vec{r})\, d\vec{r}$.

The total transition probability per unit time for scattering is therefore

$$\frac{W}{T} = 2\pi\int \delta(p_1^0 - p_1^{0\prime})\,|\tilde{v}(\vec{p}_1 - \vec{p}_1{}')|^2\,\frac{d^3\vec{p}_1{}'}{(2\pi)^3\, 2p_1^{0\prime}}$$

$$= \frac{1}{(2\pi)^2}\int |\tilde{v}(\vec{p}_1 - \vec{p}_1{}')|^2\,\frac{p_1^2\, d\Omega_{p_1'}}{2p_1^0\,\frac{dp_1^{0\prime}}{dp_1'}}\Bigg|_{p_1^{0\prime} = p_1^0}$$

$$= \sigma_{scatt} \times (\psi_1 \text{ flux})$$

Now $\frac{dp^0}{dp} = \frac{p}{p^0}$ and the ψ_1 flux is $2p_1$.

Hence

$$\frac{d\sigma}{d\Omega_{p_1'}}\Bigg|_{\psi_1\to\psi_1} = \frac{1}{4(2\pi)^2}\,|\tilde{v}(\vec{p}_1 - \vec{p}_1{}')|^2$$

4 a) Cont'd. For the ψ_2-generating process only the $\psi_1 \psi_2$ term of H_I contributes and we find

$$\frac{V}{T} = \frac{1}{(2\pi)^2} \int |\tilde{v}(\vec{p}_1 - \vec{p}_2')|^2 \frac{p_2'^2 \, d\Omega_{p_2'}}{2p_2^0 \; \frac{dp_2^0}{dp_2'}} \Bigg|_{p_2^0 = p_1^0}$$

$$\frac{d\sigma}{d\Omega_{p_2'}}\Big|_{\psi_1 \to \psi_2} = \frac{1}{4(2\pi)^2} \frac{p_2'}{p_1} |\tilde{v}(\vec{p}_1 - \vec{p}_2')|^2$$

The factor $\frac{p_2'}{p_1}$ is essentially kinematical in origin. The outgoing ψ_2's have a velocity different in general from the incident ψ_1's since their mass is different.

4 b) For a single scattering center at $\vec{r}_j$ the potential is $v(\vec{r}-\vec{r}_j)$. The corresponding ψ_2 production amplitude is proportional to

$$\int e^{i(\vec{p}_1 - \vec{p}_2')\cdot \vec{r}}\, v(r - r_j)\, d\vec{r} = e^{i(\vec{p}_1 - \vec{p}_2')\cdot \vec{r}_j}\, \tilde{v}(\vec{p}_1 - \vec{p}_2')$$

The intensity of the ψ_2's radiated by a set of N fixed scatterers is therefore

$$\frac{d\sigma}{d\Omega_{p_2'}}\Big|_{\psi_1 \to \psi_2} = \frac{1}{4(2\pi)^2} \frac{p_2'}{p_1} |\tilde{v}(\vec{p}_1 - \vec{p}_2')|^2 \left|\sum_{j=1}^{N} e^{i(p_1 - p_2')\cdot r_j}\right|^2$$

$$= \frac{d\sigma}{d\Omega_{p_2'}}\Big|_{\substack{\psi_1 \to \psi_2 \\ \text{for one} \\ \text{scatterer}}} \cdot F(\vec{p}_1 - \vec{p}_2')$$

where $F(q) = \left| \sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{r}_j} \right|^2$

is a species of form factor for the the collection of scattering centers.

4(b) Cont'd.

Note $$F(q) = \sum_{j\ell} e^{i\vec{q}\cdot(\vec{r}_j - \vec{r}_\ell)}$$

$$= N + \sum_{j\neq \ell} e^{i\vec{q}\cdot(\vec{r}_j - \vec{r}_\ell)}$$

The first term, N, yields the contribution of N independent scattering centers, i.e. the sum of the contributions we'd have if each scattering center were present alone. The corresponding cross section $\frac{d\sigma}{d\Omega_{\pi'}} = N \left.\frac{d\sigma}{d\Omega_{\pi'}}\right|_{\substack{\text{for one}\\ \text{scatterer}}}$

is often referred to as that for the <u>incoherent</u> production process.

The remaining term in F, i.e. $\sum_{j\neq \ell} e^{i\vec{q}\cdot(\vec{r}_j - \vec{r}_\ell)}$ sums the interference terms for the amplitudes for production by <u>different</u> scattering centers — and therefore describes what is usually called <u>coherent</u> production. The actual sum may be evaluated for a disk geometry by integrating.

We first note

$$\sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{r}_j} \cong N \int_{\text{disk}} e^{i\vec{q}\cdot\vec{r}} \frac{d^2\vec{r}}{\pi a^2}$$

$$= N \cdot \frac{2}{a^2} \int_0^a J_0(qr)\, r\, dr$$

$$= \frac{2N}{q^2 a^2} \int_0^{qa} \frac{d}{dx}\left(x J_1(x)\right) dx$$

$$= 2N \frac{J_1(qa)}{qa}$$

Hence $$\sum_{\ell\neq j} e^{i\vec{q}\cdot(\vec{r}_j - \vec{r}_\ell)} = 4N(N-1)\left[\frac{J_1(qa)}{qa}\right]^2$$

$$\approx 4N^2 \left[\frac{J_1(qa)}{qa}\right]^2$$

4 b) cont'd

Hence the coherent angular distribution is

$$\left.\frac{d\sigma}{d\Omega_{p_i'}}\right|_{coh} = \left.\frac{d\sigma}{d\Omega_{p_i'}}\right|_{\substack{one \\ scatterer}} \cdot 4N^2 \left[\frac{J_1(|p_i - p_i'|a)}{|p_i - p_i'|a}\right]^2$$

The Bessel function factor is essentially a delta function. The disk generates via interference a forward-collimated beam of ϕ_2's. But it also generates an incoherent distribution of ϕ_2's distributed over all angles

$$\left.\frac{d\sigma}{d\Omega_{p_i'}}\right|_{inc} = N \left.\frac{d\sigma}{d\Omega_{p_i'}}\right|_{\substack{one \\ scatterer}}$$

The same thing happens of course in scattering as well as production. For sunlight going through the atmosphere the coherent beam produces the image of the sun. The incoherent distribution is "why the sky is blue".

4 c) Consider a double collision process in which a beam of ϕ_1's is incident on a first scattering center $v(\vec{r})$ and we detect particles which are subsequently scattered by another center $v(\vec{r}-\vec{R})$. (That can be arranged experimentally by setting up appropriate shielding barriers.) Let us look for example for ϕ_1's emerging from the second collision. These may be ϕ_1's from the original beam that have simply been scattered twice — or they may have been converted to ϕ_2's in the first collision and then back again in the second.

The two alternative reaction paths furnish interfering amplitudes. Since the ϕ_1 and ϕ_2 have different masses their amplitudes will oscillate at slightly different rates and the probability for the second scattering process will depend thereby on the distance R.

To be more specific what emerges from the first scattering process is a $\varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2)$. After the φ_1 has travelled a certain distance its ϕ_1 and ϕ_2 components fall into the same phase and the φ_1 has changed thereby into a φ_2. But φ_2's are not scattered and so for this value of R the cross section falls to zero. As R increases the incident particles become φ_1's again and are scattered. The double collision probability will show a periodic behavior as a function of R. The smaller the mass difference $(\mu_1 - \mu_2)$ the larger (and more measurable) will be the [illegible] distance.

4 c) cont'd

The general second-order amplitude for the ϕ_1 to ϕ_1 transition is

$$\langle f | S^{(2)} - 1 | i \rangle = \frac{(-i)^2}{2} \langle \vec{p}_1' | \iint \{ v(r') + v(r'-R) \} \{ v(r) + v(r-R) \} \cdot \frac{1}{4} T \{ :(\phi_1^2(x') + \phi_2^2(x') - 2\phi_1(x')\phi_2(x')): \times :(\phi_1^2(x) + \phi_2^2(x) - 2\phi_1(x)\phi_2(x)): \} d^4x \, d^4x' | \vec{p}_1 \rangle$$

We drop from this expression the terms due to double collisions with a <u>single</u> potential. These are just corrections to lowest order scattering by one center and are eliminated experimentally by the shielding geometry. The remaining ϕ_1 to ϕ_1 double collision amplitude is then

$$A = - \langle p_1' | \iint v(r'-R) v(r) \frac{1}{4} T \{ :(\phi_1^2(x') + \phi_2^2(x') - 2\phi_1(x')\phi_2(x')): :(\phi_1^2(x) + \phi_2^2(x) - 2\phi_1(x)\phi_2(x)): \} d^4x \, d^4x' | p_1 \rangle$$

The only terms which actually contribute are:

$$A = - \langle p_1' | \iint v(r'-R) v(r) \{ :\phi_1(x')\phi_1(x): \} \{ i\Delta_F(x'-x, \mu_1) + i\Delta_F(x'-x, \mu_2) \} d^4x \, d^4x' | p_1 \rangle$$

$$= -2i \iint v(r'-R) v(r) \, e^{-i(p_1 x - p_1' x')} \{ \Delta_F(x'-x, \mu_1) + \Delta_F(x'-x, \mu_2) \} d^4x \, d^4x'$$

$$= \frac{-2i}{(2\pi)^4} \int e^{i p_1' x'} v(r'-R) \, e^{ik(x'-x)} \left\{ \frac{1}{k^2 - \mu_1^2 + i\epsilon} + \frac{1}{k^2 - \mu_2^2 + i\epsilon} \right\} \cdot v(r) \, e^{-i p_1 x} \, d^4k \, d^4x \, d^4x'$$

The time integrations yield δ-functions and then the k^0 integration becomes trivial:

$$A = -\frac{2i}{(2\pi)^2} \delta(p_1^0 - p_1^{0\prime}) \int e^{-i p_1' r} v(r'-R) \, e^{ik\cdot(r'-r)} \left\{ \frac{1}{p_1^{0\,2} - k^2 - \mu_1^2 + i\epsilon} + \frac{1}{p_1^{0\,2} - k^2 - \mu_2^2 + i\epsilon} \right\} \cdot v(r) \, e^{i p_1 \cdot r} \, d\vec{k} \, d\vec{r} \, d\vec{r}'$$

9 c) Cont'd

The k-integrals which remain represent simply the Green's function for the Helmholtz equation

$$\frac{1}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot\vec{r}}}{k_0^2-k^2-i\epsilon}\,d\vec{k} = \frac{e^{ik_0|\vec{r}|}}{4\pi|r|}$$

If we let $\quad K_1 = \sqrt{p_1^{0\,2}-\mu_1^2} = |\vec{p}_1|$

$\qquad\qquad K_2 = \sqrt{p_1^{0\,2}-\mu_2^2}$

be the magnitudes of the propagation vectors for the ψ_1 and ψ_2 respectively then we have

$$A = 4\pi i\,\delta(p_1^0-p_1^{0\prime})\int e^{-i\vec{p}_1'\cdot\vec{r}'}\,v(r'-R)\left\{\frac{e^{iK_1|r'-r|}+e^{iK_2|r'-r|}}{4\pi|r'-r|}\right\}v(r)\cdot e^{i\vec{p}_1\cdot\vec{r}}\,d\vec{r}\,d\vec{r}'$$

Suppose first that $v(r)$ is a delta-function $\quad v(r) = V_0\,\delta(\vec{r})$

Then

$$A = i\,\delta(p_1^0-p_1^{0\prime})\,V_0^2\,e^{-i\vec{p}_1'\cdot R}\left(\frac{e^{iK_1R}+e^{iK_2R}}{R}\right)$$

The probability for double collisions leading to ψ_1 is proportional to

$$V_0^4\left|\frac{e^{iK_1R}+e^{iK_2R}}{R}\right|^2 = \frac{V_0^4}{R^2}\left(2+2\,\mathrm{Re}\,e^{i(K_1-K_2)R}\right)$$

$$= \frac{2V_0^4}{R^2}\left\{1+\cos(K_1-K_2)R\right\}$$

$$= \frac{4V_0^4}{R^2}\cos^2\tfrac{1}{2}(K_1-K_2)R$$

If the mass difference $\mu_1-\mu_2$ is very small, we will have

$K_1 \approx K_2 \approx K \qquad$ and $\qquad K_2-K_1 = \frac{\delta}{K}$

The cross section for the production of ψ_1's will oscillate with increasing R. The oscillations determine δ/K.

4 c) cont'd

But the potential $v(r)$ is not a delta function in general. Hence we return to the expression for A and expand the Green's functions.

Let $\vec{r}' = \vec{R} + \vec{r}''$. Then we can write

$$\frac{e^{ik|r'-r|}}{|r'-r|} = \frac{e^{ik|R+r''-r|}}{|R+r''-r|} \approx \frac{e^{ik|\vec{R}|}}{|\vec{R}|} e^{i\vec{K}\cdot(\vec{r}''-\vec{r})}$$

where we have set $\vec{K} = k\frac{\vec{R}}{|\vec{R}|}$

The general amplitude for the double collision process is then

$$A = i\,\delta(p_1^0 - p_1^{0\prime})\frac{e^{-i\vec{p}_0'\cdot\vec{R}}}{|R|}\Big\{\int e^{-i\vec{p}_1'\cdot\vec{r}''} v(r'')\, e^{i\vec{K}_1\cdot\vec{r}''} dr''\, e^{iK_1R}\int e^{-i\vec{K}_1\cdot\vec{r}} v(r) e^{i\vec{p}_1\cdot\vec{r}} d\vec{r}$$
$$+ \int e^{-i\vec{p}_1'\cdot\vec{r}''} v(r'') e^{i\vec{K}_2\cdot\vec{r}''} dr''\, e^{iK_2R}\int e^{-i\vec{K}_2\cdot\vec{r}} v(r) e^{i\vec{p}_1\cdot\vec{r}} d\vec{r}\Big\}$$

$$= i\,\delta(p_1^0 - p_1^{0\prime})\frac{e^{-i\vec{p}_1'\cdot\vec{R}}}{|R|}\Big\{\tilde{v}(\vec{K}_1 - \vec{p}_1')\,\tilde{v}(\vec{p}_1 - \vec{K}_1)\, e^{iK_1R}$$
$$+ \tilde{v}(\vec{K}_2 - \vec{p}_1')\,\tilde{v}(\vec{p}_1 - \vec{K}_2)\, e^{iK_2R}\Big\}$$

Since $\vec{K}_1$ and $\vec{K}_2$ differ only very slightly in magnitude and $\tilde{v}$ is a smoothly varying function we have

$$A \approx i\,\delta(p_1^0 - p_1^{0\prime})\frac{e^{-i\vec{p}_1'\cdot\vec{R}}}{|R|}\tilde{v}(\vec{K}_1 - \vec{p}_1')\,\tilde{v}(\vec{p}_1 - \vec{K}_1)\left\{e^{iK_1R} + e^{iK_2R}\right\}$$

This amplitude differs from the earlier one with $v = V_0\delta(r)$ only through the inclusion of the Fourier Transforms of v i.e. by representing correctly (to lowest order) the angular distributions of the scattering processes.

March 10, 1977

PHYSICS 253b

Problem set 1

1. Find the invariant differential cross section $d\sigma/d|t|$ in lowest order for the "Coulomb" scattering of two charged scalar particles of equal mass using (a) the Lorentz gauge, and (b) the radiation gauge, exhibiting separately for the latter the amplitudes contributed by the instantaneous Coulomb potential and the transverse field. Show by combining these contributions that the two answers are the same.

Scalar electrodynamics: (linear part)

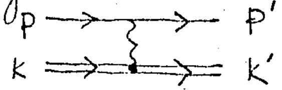
2. Calculate the invariant differential cross section $d\sigma/d|t|$ for the Compton scattering of photons [of initial momentum $\vec{k}$ and polarization $\hat{e}(k)$ and final momentum $\vec{k}'$ and polarization $\hat{e}'(k')$] by charged spin-zero particles of mass m and initial momentum $\vec{p}$. Satisfy yourself that there is no possibility of the emergence of photons with nonsensical longitudinal or "scalar" polarizations if the incident ones are polarized sensibly. Find the differential cross section for ~~[illegible]~~ unpolarized incident photons when the final polarization $\hat{e}'(k')$ is fixed and then sum over these polarizations. Find the total cross section averaged thus and summed over polarizations.

This involves the quadratic terms A^2 2-photons

Physics 253b   Problem Set #1 Solns

1.) Coulomb Scattering of Distinguishable charged scalar particles of equal mass: Particle a $\longrightarrow$, Particle b $\Longrightarrow$

Lorentz Gauge - Use the Feynman Rules from class, to calculate the T-matrix. There is one diagram:



$$T = \frac{1}{i}\left\{(-ie)(p+p')_\mu\left[\frac{-ig^{\mu\nu}}{(p'-p)^2+i\epsilon}\right](-ie)(k+k')_\nu\right\} = e^2\,\frac{(p+p')\cdot(k+k')}{(p'-p)^2+i\epsilon} = e^2\left(\frac{s-u}{t}\right.$$

Radiation Gauge - We saw in class that to second order in e, the Radiation Gauge S-matrix could be written as

$$S^{(2)} = \frac{i}{2}\int j(x)\cdot j(y)\,D_F(x-y)\,d^4x\,d^4y$$

Now, $j(x) = j^{(a)}(x) + j^{(b)}(x)$, so, for transitions from an a-b state to another a-b state (different) only the cross terms contribute (equal amounts), so

$$\langle p'k'|S^{(2)}|pk\rangle = i\int d^4x\,\langle p'|j^{(a)}(x)|p\rangle\cdot\langle k'|j^{(b)}(x)|k\rangle\,d^4y\,D_F(x-y)$$

Thus, we need
$$\begin{aligned}\langle p'|j_\mu(x)|p\rangle &= ie\langle p'|\varphi^\dagger(x)\partial_\mu\varphi(x)|p\rangle - ie\langle p'|\partial_\mu\varphi^\dagger(x)\,\varphi(x)|p\rangle\\ &= ie\left\{-ip_\mu e^{-i(p-p')\cdot x} - ip'_\mu e^{-i(p-p')\cdot x}\right\}\\ &= e(p+p')_\mu e^{-i(p-p')\cdot x}\end{aligned}$$

So,
$$\begin{aligned}\langle p'k'|S^{(2)}|pk\rangle &= ie^2(p+p')\cdot(k+k')\int d^4x\,d^4y\,e^{-i(p-p')\cdot x}\,e^{-i(k-k')\cdot y}\,\frac{d^4q}{(2\pi)^4}\,\frac{e^{-iq\cdot(x-y)}}{q^2+i\epsilon}\\ &= ie^2\frac{(p+p')\cdot(k+k')}{(p'-p)^2+i\epsilon}(2\pi)^4\,\delta^{(4)}(p+k-p'-k')\end{aligned}$$

Hence
$$T = e^2\frac{(p+p')\cdot(k+k')}{(p'-p)^2+i\epsilon}\text{, the same as in Lorentz Gauge!}$$

Now that we have demonstrated the equivalence, let us explicitly exhibit the coulomb and transverse terms. Recall from class

$$S = -\frac{1}{2}\sum_{\ell m}\int j^\ell(x)\langle 0|T\{A_\ell(x)A_m(y)\}|0\rangle j^m(y)\,d^4x\,d^4y + \frac{-i}{2}\int\frac{j_0(\vec r t)\,j_0(\vec r' t)}{4\pi|\vec r-\vec r'|}d\vec r\,d\vec r'$$

As before, only cross terms contribute to our scattering process, so the S-matrix becomes

$$\langle \cancel{p}k|S-1|pk\rangle = -i\sum_{lm}\int \langle p'|j_l^{(a)}(x)|p\rangle D^{F(T)}_{lm}(x-y)\langle k'|j_m^{(b)}(y)|k\rangle d^4x\, d^4y$$
$$-i\int \frac{\langle p'|j_0^{(a)}(\bar r t)|p\rangle\langle k'|j_0^{(b)}(\bar r' t)|k\rangle}{4\pi|\bar r-\bar r'|}d\bar r d\bar r' dt$$

The first term is the transverse term, $S^{(T)}$:

$$S^{(T)} = -i\sum_{lm}\int d^4x d^4y \frac{d^4q}{(2\pi)^4}\, e(p+p')_l e^{-i(p-p')\cdot x}\, e(k+k')_m e^{-i(k-k')\cdot y}\left(\delta_{lm}-\frac{q_l q_m}{\bar q^2}\right)\frac{e^{-iq\cdot(x-y)}}{q^2+i\epsilon}$$

Do x-y integrals, get δ-functions, then do q integral:

$$S^{(T)} = -ie^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)\frac{1}{(p-p')^2+i\epsilon}\left\{(\bar p+\bar p')\cdot(\bar k+\bar k') - \frac{(\bar p+\bar p')\cdot(\vec p-\vec p')(\bar k+\bar k')\cdot(\bar k-\bar k')}{(\bar p-\bar p')^2}\right\}$$

$$= ie^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)\frac{(p+p')\cdot(k+k')}{(p-p')^2+i\epsilon} + S^{(T)}_{Gauge}$$

where $S^{(T)}_{Gauge} = \dfrac{ie^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)}{(p-p')^2+i\epsilon}\left\{\dfrac{(\bar p'^2-\bar p^2)(\bar k^2-\bar k'^2)}{(\bar p-\bar p')^2} - (p+p')_0(k+k')_0\right\}$

The second term is the coulomb term, $S^{(c)}$:

$$S^{(c)} = -i\int \frac{e(p+p')_0 e^{-i(p-p')\cdot x}\, e(k+k')_0 e^{-i(k-k')\cdot y}}{4\pi|\bar r-\bar r'|}d\bar r d\bar r' dt \qquad \begin{array}{l} x=(\bar r t)\\ y=(\bar r' t)\end{array}$$

Doing the integrations:

$$S^{(c)} = -ie^2(2\pi)^4\delta^{(4)}(p'+k'-p-k)\frac{(p+p')_0(k+k')_0}{(\bar p-\bar p')^2}$$

Now, look at $T^{(T)}_{Gauge}+T^{(c)} = e^2\left\{\dfrac{(\bar p'^2-\bar p^2)(\bar k^2-\bar k'^2) - (p+p')_0(k+k')_0(\bar p-\bar p')^2 - (p_0+p_0')(k+k')_0(p-p')^2}{(\bar p-\bar p')^2[(p-p')^2+i\epsilon]}\right\}$

Now $(p-p')^2+(\bar p-\bar p')^2 = (p_0-p_0')^2 = (p'-p)_0(k-k')_0$, so

$$T^{(T)}_{Gauge}+T^{(c)} = e^2\left\{\frac{(\bar p'^2-\bar p^2)(\bar k^2-\bar k'^2) - (p_0'^2-p_0^2)(k_0^2-k_0'^2)}{(\bar p-\bar p')^2\,(p-p')^2}\right\}$$

Note, that $p_0^2=\bar p^2+m_a^2$ $\quad p_0'^2=\bar p'^2+m_a^2 \implies p_0'^2-p_0^2=\bar p'^2-\bar p^2$, ditto for k, k

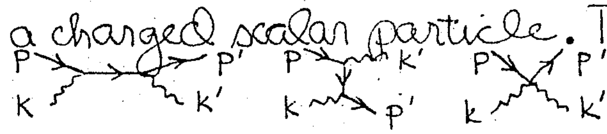
so

$T^{(T)}_{Gauge}+T^{(c)} = 0$! Now $(p+p')\cdot(k+k') = (p+p')\cdot(2k+p-p') = 2k\cdot p + 2k\cdot p'$
$= (s-m_a^2-m_b^2)+(-u+m_a^2+m_b^2)$

$$\frac{d\sigma}{d|t|} = \frac{1}{64\pi p^{*2}s}|T|^2 = \frac{1}{64\pi p^{*2}s}\frac{e^4}{t^2}(s-u)^2 = \frac{e^4}{64\pi p^{*2}s}\left(\frac{2s+t-4m^2}{t}\right)^2$$

as $s+t+u=4m^2$ for $m_a=m_b=m$.

2.) Consider Compton scattering by a charged scalar particle. To lowest order, there are three graphs:



From the Feynman rules

$$T = -e^2\left\{\frac{e(k)\cdot(2p+k)(2p'+k')\cdot e'(k')}{s-m^2} + \frac{e(k)\cdot(2p'-k)(2p-k')\cdot e'(k')}{u-m^2} - 2e(k)\cdot e'(k')\right\}$$

Gupta-Bleuler says there are three allowable photon polarizations - the two transverse polarizations (physical), and a linear superposition of scalar and longitudinal polarizations (unphysical) proportional to k. Thus, all allowable photon polarizations have $e(k)\cdot k = 0$, and we may write the T-matrix as:
$$T = 2e^2\left\{e(k)\cdot e'(k') + \frac{e(k)\cdot p'\, e'(k')\cdot p}{p'\cdot k} - \frac{e(k)\cdot p\, e'(k')\cdot p'}{p\cdot k}\right\}$$

which may be written as $T = 2e^2 e(k)\cdot a$ or $T = 2e^2\, b\cdot e'(k')$, where
$$a \equiv e'(k') + p'\frac{e'(k')\cdot p}{p'\cdot k} - p\frac{e'(k')\cdot p'}{p\cdot k}\ ; \quad b \equiv e(k) + p\frac{e(k)\cdot p'}{p\cdot k'} - p'\frac{e(k)\cdot p}{p'\cdot k'}$$

Let us note that $a\cdot k = k\cdot e'(k') + e'(k')\cdot p - e'(k)\cdot p' = e'(k')\cdot k' = 0$

$b\cdot k' = k'\cdot e(k) + e(k)\cdot p' - e(k)\cdot p = e(k)\cdot k = 0$

Thus, T vanishes if either the initial or final polarization is unphysical!

Cross-sections: $\boxed{\dfrac{d\sigma}{d|t|} = \dfrac{1}{64\pi p^{*2} s}|T|^2 = \dfrac{e^4}{16\pi p^{*2} s}|e(k)\cdot a|^2}$

We want to average over $e(k)$. Now $a\cdot k = 0 \Rightarrow \sum_{pol} e(k)\cdot a\, e(k)\cdot a = -a^2$

Proof: $a\cdot k = 0 \Rightarrow k = (k,0,0,k)$; $a = (\alpha,\beta,\gamma,\alpha)$.

Thus $\sum_{pol}|e(k)\cdot a|^2 = (-\beta)^2 + (-\gamma)^2 = \alpha^2+\beta^2+\gamma^2-\alpha^2 = -a^2$.

Averaging over initial spin of photon gives $\left.\dfrac{d\sigma}{d|t|}\right|_{av.} = -\dfrac{e^4}{32\pi p^{*2} s}a^2$

Calculate $a^2 = e'^2 + m^2\left(\dfrac{e'\cdot p}{p'\cdot k}\right)^2 + m^2\left(\dfrac{e'\cdot p'}{p\cdot k}\right)^2 + \dfrac{2e'\cdot p' e'\cdot p}{p'\cdot k} - \dfrac{2e'\cdot p\, e'\cdot p'}{p\cdot k} - 2p\cdot p'\dfrac{e'\cdot p\, e'\cdot p'}{p\cdot k\, p'\cdot k}$

Now $e'^2 = -1$, so $a^2 = -1 + m^2(x^2+y^2) + 2xy(p\cdot k - p'\cdot k - p\cdot p')$, where $x \equiv \dfrac{e'\cdot p}{p'\cdot k}$, $y \equiv \dfrac{e'\cdot p'}{p\cdot k}$

Also $p\cdot k - p'\cdot k - p\cdot p' = p\cdot k - p\cdot k' - p\cdot p' = p(k-k'-p') = -p^2 = -m^2$, so

$a^2 = -1 + m^2(x-y)^2$. We can write $x - y = e'\cdot g$, where $g = \dfrac{p}{p'\cdot k} - \dfrac{p'}{p\cdot k}$

or $g = \dfrac{p}{p\cdot k'} - \dfrac{p'}{p'\cdot k'}$. Note that $g\cdot k' = 0$!

The averaged cross-section is $\boxed{\left.\dfrac{d\sigma}{d|t|}\right|_{av} = \dfrac{e^4}{32\pi p^{*2} s}\left[1 - m^2\left|e'(k')\cdot\left\{\dfrac{p}{p\cdot k'} - \dfrac{p'}{p'\cdot k'}\right\}\right|^2\right]}$

Next, we want to sum over final polarization:

$$\left.\frac{d\sigma}{d|t|}\right|_{\substack{av\\ sum}} = \frac{e^4}{16\pi p^{*2} s} - \frac{e^4 m^2}{32\pi p^{*2} s}\sum_{pol}|e'(k')\cdot g|^2 = \frac{e^4}{16\pi p^{*2} s}\left\{1 + \tfrac{1}{2}m^2 g^2\right\}$$

Thus, we need $g^2 = \frac{m^2}{(p\cdot k')^2} + \frac{m^2}{(p'\cdot k')^2} - \frac{2p\cdot p'}{(p\cdot k')(p'\cdot k')} = m^2\left\{\frac{1}{p\cdot k'} - \frac{1}{p'\cdot k'}\right\}^2 + \frac{2m^2 - 2p\cdot p'}{(p\cdot k')(p\cdot k)}$

$$= m^2\left\{\frac{1}{p\cdot k'} - \frac{1}{p'\cdot k'}\right\}^2 + \frac{t}{(p'\cdot k)(p\cdot k)}$$

Now $p\cdot k' = p'\cdot k = \frac{1}{2}(s+t-m^2)$, and $p\cdot k = p'\cdot k' = \frac{1}{2}(s-m^2)$, so

$$g^2 = 4m^2\left\{\frac{1}{s+t-m^2} - \frac{1}{s-m^2}\right\}^2 + \frac{4t}{(s+t-m^2)(s-m^2)} = 4m^2\left(\frac{t}{(s+t-m^2)(s-m^2)}\right)^2 + 4\left(\frac{t}{(s+t-m^2)(s-m^2)}\right)$$

and hence:

$$\boxed{\left.\frac{d\sigma}{d|t|}\right|_{\substack{sum\\av}} = \frac{e^4}{16\pi p^{*2} s}\left\{1 + 2\left[\frac{m^2 t}{(s+t-m^2)(s-m^2)}\right] + 2\left[\frac{m^2 t}{(s+t-m^2)(s-m^2)}\right]^2\right\}}$$

The total cross-section is $\sigma = \int_{-4p^{*2}}^{0} \frac{d\sigma}{d|t|}\,dt = \frac{e^4}{16\pi p^{*2} s}\left\{4p^{*2} + 2I_1 + 2I_2\right\}$

where $I_j \equiv \int_{-4p^{*2}}^{0} \left[\frac{m^2 t}{(s+t-m^2)(s-m^2)}\right]^j dt$. These integrals are trivial:

$$I_1 = \frac{m^2}{s-m^2}4p^{*2} - m^2 \ln\left(\frac{s-m^2}{s-m^2-4p^{*2}}\right)$$

$$I_2 = \left(\frac{m^2}{s-m^2}\right)^2 4p^{*2} + \frac{m^2}{s-m^2}\,\frac{m^2}{s-m^2-4p^{*2}}\,4p^{*2} - 2\left(\frac{m^2}{s-m^2}\right)^2 \ln\left(\frac{s-m^2}{s-m^2-4p^{*2}}\right)$$

Putting it all together, using $s = (\sqrt{p^{*2}+m^2} + p^*)^2 \Rightarrow 4p^{*2} = (s-m^2)^2/s$, and doing some algebra:

$$\boxed{\sigma = \frac{e^4}{4\pi s}\left(\frac{s+m^2}{s-m^2}\right)^2 - \frac{e^4}{2\pi}\,\frac{m^2(s+m^2)}{(s-m^2)^3}\ln\left(\frac{s}{m^2}\right)}$$

If we let $s = m^2(1+\epsilon)$, then $\sigma = \frac{e^4}{4\pi m^2}F(\epsilon) = 4\pi r_0^2 F(\epsilon)$, where $r_0 = \frac{e^2}{4\pi m}$ (classical electron radius)

and $$F(\epsilon) = \frac{1}{1+\epsilon}\left(\frac{2+\epsilon}{\epsilon}\right)^2 - \frac{2(2+\epsilon)}{\epsilon^3}\ln(1+\epsilon)$$

As $\epsilon \to 0$ (low energy limit) $F(\epsilon) \to \frac{2}{3}$, so $\sigma \to \frac{8\pi r_0^2}{3}$ (Thompson cross section)

April 26, 1977

Physics 253b
Problem set 2

1. The Lagrangian density of a free neutral pseudoscalar meson field is

$$\mathcal{L}_{0m} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2),$$

while that for a free spin $\frac{1}{2}$ neutral "nucleon" field is

$$\mathcal{L}_{0N} = \overline{\psi}(i\not\partial - m)\psi.$$

The most frequently studied interaction of these fields is the pseudoscalar one which adds the interaction term

$$\mathcal{L}' = i\, g\, \overline{\psi}\, \gamma_5\, \psi\, \phi$$

to the total Lagrangian density. An alternative coupling, the pseudovector one can be written as

$$\mathcal{L}' = \frac{f}{\mu}\, \overline{\psi}\, \gamma_5\, \gamma^\mu\, \psi\, \partial_\mu\, \phi$$

(where the factor μ^{-1} has been inserted to give f the same dimensions as g.)

√₁

a) Construct the interaction Hamiltonians for these two coupling schemes.

b) Find the invariant matrix element (in momentum space) for nucleon-nucleon scattering in lowest (i.e. second) order for the pseudoscalar coupling.

c) Do the same for the pseudovector coupling and show that the result is equivalent to the one for pseudoscalar coupling provided f and g are in a certain relation. What relation?

Physics 253b – Problem Set #2 Solutions

1.) $\mathcal{L}_{0M} = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2) \Rightarrow \pi_\varphi = \dot{\varphi}$ and $\mathcal{H}_{0M} = \frac{1}{2}(\pi_\varphi^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \mu^2\varphi^2)$

$\mathcal{L}_{0N} = \bar{\Psi}(i\not{\partial} - m)\Psi \Rightarrow \pi_\psi = i\bar{\Psi}\gamma^0,\ \pi_{\bar{\psi}} = 0$, and $\mathcal{H}_{0N} = \bar{\Psi}(-i\gamma^j\partial_j + m)\Psi = \Psi^\dagger(\vec{\alpha}\cdot\frac{1}{i}\vec{\nabla} + \beta m)\Psi$

a.) First, take $\mathcal{L}' = ig\bar{\Psi}\gamma_5\Psi\varphi$. Since there are no derivatives in the perturbation, $\mathcal{H}' = -ig\bar{\Psi}\gamma_5\Psi\varphi$, and in the interaction picture

$$\boxed{\mathcal{H}_I = -ig\bar{\Psi}_I\gamma_5\Psi_I\varphi_I}$$

Next, take $\mathcal{L}' = f\mu^{-1}\bar{\Psi}\gamma_5\gamma^\mu\Psi\partial_\mu\varphi$. The canonical momenta π_φ now becomes $\pi_\varphi = \dot{\varphi} + f\mu^{-1}\bar{\Psi}\gamma_5\gamma^0\Psi$, and thus the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \pi_\varphi\dot{\varphi} - \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2) - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\mu\Psi\partial_\mu\varphi + \mathcal{H}_{0N} \\
&= \pi_\varphi(\pi_\varphi - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi) - \tfrac{1}{2}(\pi_\varphi - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi)^2 + \tfrac{1}{2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \tfrac{1}{2}\mu^2\varphi^2 - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\mu\Psi\partial_\mu\varphi + \mathcal{H}_{0N} \\
&= \underbrace{\tfrac{1}{2}(\pi_\varphi^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \mu^2\varphi^2) + \mathcal{H}_{0N}}_{\equiv\ \mathcal{H}_0} \underbrace{- \tfrac{1}{2}(f\mu^{-1}\bar{\Psi}\gamma^5\gamma^0\Psi)^2 - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\mu\Psi\partial_\mu\varphi}_{\equiv\ \mathcal{H}_1}
\end{aligned}$$

Thus
$$\begin{aligned}
\mathcal{H}_1 &= -f\mu^{-1}\bar{\Psi}\gamma^5\gamma^\mu\Psi\partial_\mu\varphi - \tfrac{1}{2}(f\mu^{-1}\bar{\Psi}\gamma^5\gamma^0\Psi)^2 \\
&= -f\mu^{-1}\bar{\Psi}\gamma^5\gamma^j\Psi\partial_j\varphi - f\mu^{-1}\bar{\Psi}\gamma^5\gamma^0\Psi(\pi_\varphi - \tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi) - \tfrac{1}{2}(f\mu^{-1}\bar{\Psi}\gamma^5\gamma^0\Psi)^2
\end{aligned}$$

or

$$\boxed{\mathcal{H}_1 = -\frac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi\pi_\varphi - \frac{f}{\mu}\bar{\Psi}\gamma^5\gamma^j\Psi\partial_j\varphi + \frac{1}{2}\left(\frac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi\right)^2}$$

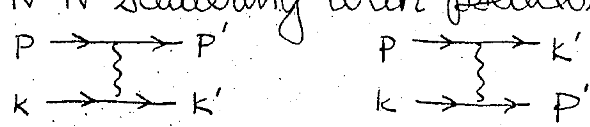
We saw in class that in the interaction picture $\pi_{\varphi_I} = \partial_0\varphi_I$, and thus

$$\boxed{\mathcal{H}_I = -\frac{f}{\mu}\bar{\Psi}_I\gamma^5\gamma^\mu\Psi_I\partial_\mu\varphi_I + \frac{1}{2}\left(\frac{f}{\mu}\bar{\Psi}_I\gamma^5\gamma^0\Psi_I\right)^2}$$

Note that $\mathcal{H}_I$ is not a Lorentz scalar! However, this "problem" is solved as for QED by noting that $T(\partial_\mu\varphi(x)\,\partial_\nu\varphi(y)) = \partial_\mu^x\partial_\nu^y T(\varphi(x)\varphi(y)) - i\delta_{\mu 0}\delta_{\nu 0}\delta^{(4)}(x-y)$. Thus, the relevant part of the S-matrix for N-N scattering to $O(f^2)$ is

$$\begin{aligned}
S^{(2)} &= -i\int d^4x\,\tfrac{1}{2}\left(\tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^0\Psi\right)^2 + \frac{(-i)^2}{2!}\int d^4x\,d^4x' :\left(-\tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\mu\Psi\right)_x\left(-\tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\nu\Psi\right)_{x'}: \left[\partial_\mu^x\partial_\nu^{x'}\varphi(x)\varphi(y) - i\delta_{\mu 0}\delta_{\nu 0}\delta^{(4)}(x-x')\right] \\
&= \frac{(-i)^2}{2!}\int d^4x\,d^4x' :\left(-\tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\mu\Psi\right)_x\left(-\tfrac{f}{\mu}\bar{\Psi}\gamma^5\gamma^\nu\Psi\right)_{x'}: \partial_\mu^x\partial_\nu^{x'}\varphi(x)\varphi(y)
\end{aligned}$$

b.) N-N scattering with pseudoscalar coupling; there are two diagrams:



Using the Feynman rules, we have

$$T = (-i)^2 \left\{ \frac{\bar{u}(p')(-ig\gamma_5)u(p)\,\bar{u}(k')(-ig\gamma_5)u(k)}{t-\mu^2} - \frac{\bar{u}(p')(-ig\gamma_5)u(k)\,\bar{u}(k')(-ig\gamma_5)u(p)}{u-\mu^2} \right\}$$

or

$$\boxed{T = g^2 \left\{ \frac{\bar{u}(p')\gamma_5 u(p)\,\bar{u}(k')\gamma_5 u(k)}{t-\mu^2} - \frac{\bar{u}(p')\gamma_5 u(k)\,\bar{u}(k')\gamma_5 u(p)}{u-\mu^2} \right\}}$$

c.) N-N scattering with pseudovector coupling. Because of the occurrence of $\partial_\mu^x \partial_\nu^{x'} \varphi(x)\varphi(x')$ the vertex $p \rightarrow p'$ has a factor of $i(\not{p}'-\not{p})$ associated with it. Thus

$$T = (-i)^2 \left\{ \frac{\bar{u}(p')(-\frac{if}{\mu}\gamma_5)(\not{p}'-\not{p})u(p)\,u(k')(-\frac{if}{\mu}\gamma_5)(\not{k}'-\not{k})u(k)}{t-\mu^2} - \frac{\bar{u}(p')(-\frac{if}{\mu}\gamma_5)(\not{p}'-\not{k})u(k)\,\bar{u}(k')(-\frac{if}{\mu}\gamma_5)(\not{k}'-\not{p})u(p)}{u-\mu^2} \right\}$$

or

$$\boxed{T = \left(\frac{f}{\mu}\right)^2 \left\{ \frac{\bar{u}(p')\gamma_5(\not{p}'-\not{p})u(p)\,u(k')\gamma_5(\not{k}'-\not{k})u(k)}{t-\mu^2} - \frac{\bar{u}(p')\gamma_5(\not{p}'-\not{k})u(k)\,\bar{u}(k')\gamma_5(\not{k}'-\not{p})u(p)}{u-\mu^2} \right\}}$$

Now, we use the plane-wave Dirac equation $\not{p}u(p) = mu(p)$, its Pauli adjoint $\bar{u}(p)\not{p} = m\bar{u}(p)$, and $\{\gamma_5, \not{a}\} = 0$ to write
$\bar{u}(p')\gamma_5(\not{p}'-\not{p})u(p) = -\bar{u}(p')\{\not{p}'\gamma_5 + \gamma_5\not{p}\}u(p) = -2m\,\bar{u}(p')\gamma_5 u(p)$, and hence

$$\boxed{T = \left(\frac{2mf}{\mu}\right)^2 \left\{ \frac{\bar{u}(p')\gamma_5 u(p)\,\bar{u}(k')\gamma_5 u(k)}{t-\mu^2} - \frac{\bar{u}(p')\gamma_5 u(k)\,\bar{u}(k')\gamma_5 u(p)}{u-\mu^2} \right\}}$$

The two T-matrices agree if $\left(\frac{2mf}{\mu}\right)^2 = g^2$, or $\boxed{f = \pm\frac{\mu}{2m}g}$

PHYSICS 253b

Problem set 3

1. The particles which participate in the muon decay, that is the muon (mass μ), the electron (mass m), and two different varieties of neutrino ν_μ and ν_e (presumed massless) all have spin $\frac{1}{2}$. The decay scheme is

$$\mu^+ \to \bar{\nu}_\mu + \nu_e + e^+.$$

The interaction Hamiltonian density responsible for the decay is a well established superposition of vector and axialvector couplings which is usually written as

$$H = \frac{G}{\sqrt{2}} [\bar{\psi}_{\nu\mu} \gamma^\mu (1-\gamma^5) \psi_\mu] [\bar{\psi}_e \gamma^\mu (1-\gamma^5) \psi_{\nu e}] + \text{Herm. Conj.}$$

(All four fields are represented by four-component spinor operators

a) By using the techniques developed last term for the scalar analogue of this problem and their obvious spinor generalizations find the momentum spectrum of the electrons emitted from the decay of a μ^+ at rest. (You may find certain integrations simplified by using a technique explained in the solutions to last term's final examination).

b) Find the inverse μ^+ lifetime in terms of G, assuming $\mu >> m$.

Physics 253b - Problem Set # 3 Solutions

1.) Let $\mathcal{H}_1 = \frac{G}{\sqrt{2}}[\bar{\Psi}_{\nu\mu} M^\mu \Psi_\mu][\bar{\Psi}_e M_\mu \Psi_{\nu e}]$, and $\mathcal{H}_2 = \mathcal{H}_1^\dagger$, where $M^\mu \equiv \gamma^\mu(1-\gamma^5)$. We shall take Ψ to be a lepton number lowering operator, and thus it is $\mathcal{H}_2$ which has a non vanishing matrix element for going from $|1_{\mu^+}\rangle$ to $|1_{e^+}, 1_{\nu_e}, 1_{\bar{\nu}_\mu}\rangle$. This gives us a transition rate

$$W = \int d^4x\, d^4x' \langle p|\mathcal{H}_1(x)\mathbb{P}\mathcal{H}_2(x')|p\rangle$$

where $|p\rangle$ is a relativistically normalized one μ^+ state, and $\mathbb{P}$ is a projection operator onto the "$0\mu^+$, $1e^+$, $1\nu_e$, $1\bar{\nu}_\mu$, 0 everything else" subspace. We may write

$$W = \frac{G^2}{2}\int d^4x\, d^4x' \langle p;0|\bar{\Psi}_{\nu\mu}(x)M^\mu\Psi_\mu(x)\mathbb{P}\bar{\Psi}_\mu(x')M^\nu\Psi_{\nu\mu}(x')|p;0\rangle \times$$

$$\times \langle 0;0|\bar{\Psi}_e(x)M_\mu\Psi_{\nu e}(x)\mathbb{P}\bar{\Psi}_{\nu e}(x')M_\nu\Psi_e(x')|0;0\rangle$$

Note that $\langle 0;0|\bar{\Psi}_e(x)M_\mu\Psi_{\nu e}(x)\mathbb{P}\bar{\Psi}_{\nu e}(x')M_\nu\Psi_e(x')|0;0\rangle = \langle 0|\bar{\Psi}^\alpha_e(x)\Psi^{\beta'}_e(x')|0\rangle\langle 0|\Psi^\beta_{\nu_e}(x)\bar{\Psi}^{\alpha'}_{\nu_e}(x')|0\rangle (M_\mu)^{\alpha\beta}(M_\nu)^{\alpha'\beta'}$

$$= \int \frac{d^3p'}{(2\pi)^3 2E_{p'}}(\not{p}'-m)_{\beta'\alpha}(M_\mu)^{\alpha\beta}\frac{d^3k'}{(2\pi)^3 2|\vec{k}'|}(\not{k}')^{\beta\alpha'}(M_\nu)^{\alpha'\beta'}e^{-i(k'+p')(x-x')}$$

$$= \int \frac{d^3p'}{(2\pi)^3 2E_{p'}}\frac{d^3k'}{(2\pi)^3 2|\vec{k}'|}e^{-i(k'+p')\cdot(x-x')}\mathrm{Tr}[(\not{p}'-m)M_\mu\not{k}'M_\nu]$$

Next, note that $\langle p;0|\bar{\Psi}_{\nu\mu}(x)M^\mu\Psi_\mu(x)\mathbb{P}\bar{\Psi}_\mu(x')M^\nu\Psi_{\nu\mu}(x')|p;0\rangle = \langle 0|\bar{\Psi}^\alpha_{\nu\mu}(x)\Psi^{\beta'}_{\nu\mu}(x')|0\rangle\langle p|\Psi^\beta_\mu(x)\bar{\Psi}^{\alpha'}_\mu(x')|p\rangle (M^\mu)^{\alpha\beta}(M^\nu)^{\alpha'\beta'}$

$$= \bar{V}_\beta(p)V_{\alpha'}(p)e^{ip(x-x')}\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}e^{-ik\cdot(x-x')}(\not{k})_{\beta'\alpha}(M^\mu)^{\alpha\beta}(M^\nu)^{\alpha'\beta'}$$

If we average over μ^+ spin, this becomes

$$= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}e^{i(p-k)\cdot(x-x')}\left[(\not{p}-\mu)_{\beta\alpha'}(M^\nu)^{\alpha'\beta'}(\not{k})^{\beta'\alpha}(M^\mu)^{\alpha\beta}\right]$$

$$= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}e^{i(p-k)\cdot(x-x')}\mathrm{Tr}[(\not{p}-\mu)M^\nu\not{k}M^\mu]$$

Hence;

$$\frac{W}{VT} = \frac{1}{2}\frac{G^2}{2}\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\frac{d^3k'}{(2\pi)^3 2|\vec{k}'|}\frac{d^3p'}{(2\pi)^3 2E_{p'}}(2\pi)^4\delta(k+k'+p'-p)\,\mathrm{Tr}[(\not{p}-\mu)M^\nu\not{k}M^\mu]\,\mathrm{Tr}[(\not{p}'-m)M_\mu\not{k}'$$

or $\dfrac{W}{VT} = \dfrac{G^2}{4(2\pi)^5}\int \dfrac{d\bar{k}'}{2k'}\dfrac{d\bar{k}}{2k}\dfrac{d\bar{p}'}{2E_{p'}}\,\delta^{(4)}(k+k'+p'-p)\,\mathrm{Tr}[(\not{p}-\mu)M^\nu \not{k} M^\mu]\,\mathrm{Tr}[(\not{p}'-m)M_\mu \not{k}' M_\nu]$

Consider $\mathrm{Tr}[(\not{p}-\mu)M^\nu\not{k}M^\mu] = \mathrm{Tr}[(\not{p}-\mu)\gamma^\nu(1-\gamma^5)\not{k}\gamma^\mu(1-\gamma^5)]$

$$= \mathrm{Tr}[(\not{p}-\mu)\gamma^\nu\not{k}\gamma^\mu(1-\gamma^5)^2] \qquad (\gamma^{5\,2}=1)$$
$$= 2\,\mathrm{Tr}[(\not{p}-\mu)\gamma^\nu\not{k}\gamma^\mu(1-\gamma^5)]$$
$$= 2\,\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu) - 2\,\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu\gamma^5) - \underbrace{2\mu\,\mathrm{Tr}(\gamma^\nu\not{k}\gamma^\mu)}_{=0} + \underbrace{2\mu\,\mathrm{Tr}(\gamma^\nu\not{k}\gamma^\mu\gamma^5)}_{=0}$$

Let us define $A \equiv \mathrm{Tr}[(\not{p}-\mu)M^\nu\not{k}M^\mu]\,\mathrm{Tr}[(\not{p}'-m)M_\mu\not{k}'M_\nu]$

$$= 4[\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu) - \mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu\gamma^5)][\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu) - \mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\gamma_5)$$

Now $\mathrm{Tr}(\not{a}\gamma^\mu\not{b}\gamma^\nu)$ is symmetric in (μ,ν), while $\mathrm{Tr}(\not{a}\gamma^\mu\not{b}\gamma^\nu\gamma^5)$ is antisymmet in (μ,ν), and so:

$$A = 4[\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu)\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu) + \mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu\gamma_5)\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\gamma_5)]$$

Now

$$\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu)\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu) = 16(p^\nu k^\mu + k^\nu p^\mu - k\cdot p\, g^{\mu\nu})(p'_\mu k'_\nu + p'_\nu k'_\mu - p'\cdot k' g_{\mu\nu})$$
$$= 16(2p\cdot k'\,p'\cdot k + 2\,p\cdot p'\,k\cdot k' + 0\,p\cdot k\,p'\cdot k')$$
$$= 32(p\cdot k'\,p'\cdot k + p\cdot p'\,k\cdot k')$$

~~next~~, note that $\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu\gamma^5)\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\gamma^5)$ is a Lorentz scalar, antisymmetric in (p,k), and in (p',k'), and is linear in the momenta p, p', k, k', thus

$$\mathrm{Tr}(\not{p}\gamma^\nu\not{k}\gamma^\mu\gamma^5)\mathrm{Tr}(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\gamma^5) = a(p\cdot p'\,k\cdot k' - p\cdot k'\,p'\cdot k)$$

Let us take $p = p' = (1,\bar{0})$, and $k = k' = (0,1,0,0)$. Then the right hand side is $-a$. The left hand side is:

$$\mathrm{Tr}(\gamma^0\gamma^\nu\gamma^1\gamma^\mu\gamma^5)\mathrm{Tr}(\gamma_0\gamma_\mu\gamma_1\gamma_\nu\gamma^5) = \mathrm{Tr}(\gamma^0\gamma^2\gamma^1\gamma^3\gamma^5)\mathrm{Tr}(\gamma_0\gamma_3\gamma_1\gamma_2\gamma^5) + \mathrm{Tr}(\gamma^0\gamma^3\gamma^1\gamma^2\gamma^5)\mathrm{Tr}(\gamma_0\gamma_2\gamma_1\gamma_3\gamma^5)$$
$$= 2\,\mathrm{Tr}(\gamma^0\gamma^2\gamma^1\gamma^3\gamma^5)\mathrm{Tr}(\gamma_0\gamma_3\gamma_1\gamma_2\gamma_5)$$
$$= -2\,\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5)\mathrm{Tr}(\gamma_0\gamma_1\gamma_2\gamma_3\gamma_5)$$
$$= -2\,\mathrm{Tr}(-i\gamma^5\gamma^5)\,\mathrm{Tr}(-i\gamma^5\gamma^5) = 2[\mathrm{Tr}(\gamma_5^2)]^2 = 2[\mathrm{Tr}(1)]^2 = 2(4)^2$$

Thus $a = -32$, and

$$A = 4[32(p\cdot k'\,p'\cdot k + p\cdot p'\,k\cdot k' - p\cdot p'\,k\cdot k' + p\cdot k'\,p'\cdot k)]$$
$$= 256\,p\cdot k'\,p'\cdot k$$

and,

$$\frac{W}{VT} = \frac{64G^2}{(2\pi)^5}\int \frac{d\bar{k}}{2|\bar{k}|}\frac{d\bar{k}'}{2|\bar{k}'|}\frac{d\bar{p}'}{2E_{p'}}\,\delta^{(4)}(k+k'+p'-p)\,p\cdot k'\,p'\cdot k$$

Let us define $I_{\mu\nu}(p-p') \equiv \displaystyle\int \frac{d\bar{k}}{2|\bar{k}|}\frac{d\bar{k}'}{2|\bar{k}'|}\,\delta^{(4)}(k+k'+p'-p)\,k'_\mu k_\nu$

Then $\frac{W}{VT} = \frac{64G^2}{(2\pi)^5}\int \frac{d\bar{p}'}{2E_{\bar{p}'}} p^\mu p'^\nu I_{\mu\nu}(p-p')$. Next, we shall evaluate $I_{\mu\nu}$. It is a second rank Lorentz tensor. First note that if $q^2<0$ there is a frame in which $q_0=0$, and hence $I_{\mu\nu}(q)$ contains a factor $\delta(|\bar{k}|+|\bar{k}'|)$ which always vanishes. Thus, $I_{\mu\nu}(q) \propto \theta(q^2)$. Take $q^2>0$, and go to the frame in which $q=(q_0,\bar{0})$. There

$$I_{\mu\nu}(q) = \int \frac{d\bar{k}}{2|\bar{k}|}\frac{d\bar{k}'}{2|\bar{k}'|}\delta(|\bar{k}|+|\bar{k}'|-q_0)\delta(\bar{k}+\bar{k}')(q-k)_\mu k_\nu$$

$$= \frac{1}{2}\int \frac{d\bar{k}}{4|\bar{k}|^2}\delta(|\bar{k}|-\tfrac{q_0}{2})(q-k)_\mu k_\nu = \frac{1}{8}\int d\Omega_{\bar{k}}(q-k)_\mu k_\nu\,\theta(q_0)$$

where $k=(\frac{q_0}{2},\frac{q_0}{2}\hat{n})$. Clearly, for $\mu\neq\nu$, we get zero. For $\mu=\nu=0$ we have

$$I_{00}(q) = \frac{\theta(q_0)}{8}4\pi\left(\frac{q_0}{2}\right)^2 = \frac{\pi\theta(q_0)}{8}q_0^2$$

For $\mu=\nu=j$, we have $I_{jj}(q) = \frac{\theta(q_0)}{8}\left(-\left(\frac{q_0}{2}\right)^2\right)\int d\Omega(n_j)^2 = -\frac{4\pi}{3}\frac{\theta(q_0)q_0^2}{32}$

Thus

$$I_{\mu\nu}(q) = \frac{\pi\theta(q_0)\theta(q^2)q_0^2}{24}\begin{Bmatrix}3&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{Bmatrix} = \frac{\pi\theta(q_0)\theta(q^2)q_0^2}{24}(2g_{\mu0}g_{\nu0}+g_{\mu\nu})$$

or, in any frame $I_{\mu\nu}(q) = \frac{\pi\theta(q_0)\theta(q^2)}{24}(2q_\mu q_\nu + g_{\mu\nu}q^2)$

We may write

$$\frac{W}{VT} = \frac{8\pi G^2}{3(2\pi)^5}\int\frac{d\bar{p}'}{2E_{p'}}\theta((p-p')^2)\theta(p_0-p_0')\left[2p\cdot(p-p')p'\cdot(p-p')+(p-p')^2p\cdot p'\right]$$

Now $2p\cdot(p-p')p'\cdot(p-p')+(p-p')^2p\cdot p' = 2[\mu^2-p\cdot p'][p\cdot p'-m^2]+(\mu^2+m^2-2p\cdot p')p\cdot p'$
$= -4(p\cdot p')^2+3(\mu^2+m^2)p\cdot p'-2\mu^2m^2$

We need the density of μ^+'s in the initial state:

$$\frac{1}{V}\int d^3x\langle j_0(x)\rangle = \int\frac{d^3x}{V}\langle p|\bar{\psi}\gamma^0\psi|p\rangle = \int\frac{d^3x}{V}\bar{v}(p)\gamma^0v(p) = \bar{v}(p)\gamma^0v(p)$$

Averaging over μ^+ spins, we find $\frac{1}{2}\sum\bar{v}(p)\gamma^0v(p) = \frac{1}{2}\sum\mathrm{Tr}(\gamma^0v(p)\bar{v}(p))$
$= \frac{1}{2}\mathrm{Tr}[\gamma^0(\not{p}-\mu)]$
$= 2p^0 = 2E_p$

Thus

$$\omega = \frac{W}{VT}\frac{1}{2E_p} = \frac{8\pi G^2}{3(2\pi)^5}\frac{1}{2E_p}\int\frac{d\bar{p}'}{2E_{p'}}\theta((p-p')^2)\theta(p_0-p_0')\left\{-4(p\cdot p')^2+3(\mu^2+m^2)p\cdot p'-2\mu^2m^2\right\}$$

a.) Take μ^+ at rest:

$$\omega = \frac{8\pi G^2}{3(2\pi)^5}\frac{1}{2\mu}\int \frac{d^3\bar{p}'}{2E_{\bar{p}'}}\,\Theta(\mu - E_{p'})\Theta(\mu^2 - 2\mu E_{p'} + m^2)\left\{-4\mu^2 E_{p'}^2 + 3(\mu^2+m^2)\mu E_{p'} - 2\mu\right.$$

Thus, the momentum spectrum is given by:

$$\frac{d\omega}{dp'} = \frac{G^2 p'^2}{12\pi^3 E_{p'}}\Theta(\mu - E_{p'})\Theta(\mu^2 - 2\mu E_{p'} + m^2)\left\{-4\mu^2 E_{p'}^2 + 3(\mu^2+m^2)\mu E_{p'} - 2\mu^2 m^2\right.$$

b.) Now take $m \to 0$ $\qquad \Theta(\mu - E_{p'}) \to \Theta(\mu - p')$

$\qquad \Theta(\mu^2 - 2\mu E_{p'} + m^2) \to \Theta(\mu - 2p')$

Thus:

$$\frac{d\omega}{dp'} = \frac{G^2 p'^2}{12\pi^3 p'}\Theta(\mu - 2p')\left\{-4\mu^2 p'^2 + 3\mu^3 p'\right\}$$

or

$$\frac{d\omega}{dp'} = \frac{G^2\mu^2 p'^2}{12\pi^3}\Theta(\mu - 2p')\left(3 - \frac{4p'}{\mu}\right)$$

so

$$\frac{1}{\tau} = \int_0^\infty \frac{d\omega}{dp'}dp' = \frac{G^2\mu^2}{12\pi^3}\int_0^{\frac{\mu}{2}}\left(3 - \frac{4p'}{\mu}\right)p'^2 dp'$$

$$= \frac{G^2\mu^2}{12\pi^3}\left[\left(\frac{\mu}{2}\right)^3 - \frac{1}{\mu}\left(\frac{\mu}{2}\right)^4\right] = \frac{G^2\mu^5}{12\pi^3}\left(\frac{1}{8} - \frac{1}{16}\right) = \frac{G^2\mu^5}{12\pi^3}\frac{1}{16}$$

or

$$\boxed{\frac{1}{\tau} = \frac{G^2\mu^5}{192\pi^3}}$$

May 1977

PHYSICS 253b

FINAL EXAMINATION

The completed examination paper is to be returned within 72 hours of the time at which the problem sets are picked up.

The only reference materials you are allowed to use are the lecture notes for the course (both semesters) the homework problems and their solutions, the two volumes of Bjorken and Drell and standard tables of integrals and of the properties of special functions. (In the unlikely event you need the latter references please cite them appropriately).

If you have questions regarding the statement of the problems please phone me (Office: 495-2869, Home: 648-8546) or consult Mr. Ling who will be available in Jefferson 462 G from 11:00 - 12:00 a.m. and 4:00 - 5:00 p.m. (Phone 495-4863). Mr. Ling will also be available by phone during some periods at his home: 498-5023.

Please do all three problems. Their credits are as shown. The final grade will be a suitably weighted average of the homework and examination scores.

1.(30) We have discussed extensively in class the radiation by an excited two-level system coupled in a simple way to the electromagnetic field. Let us assume now that the two-level system is initially unexcited~~, and that the vacuum is free of photons~~. A ~~steady coherent light beam of~~ photon plane wave~~s~~ with propagation vector $\vec{k}$ is ~~switched on~~ present at time $t = 0$ and the two-level system begins to scatter it. (Use the analogous coupled (optional) oscillator model~~s~~ to) Find the time-dependent state of the field at later times. ~~Derive the inelastic differential cross section for the two-level system, i.e. the angular and frequency distributions of the scattered quanta.~~ You are free to make all of the simplifying assumptions we made in class (including neglect of photon polarizations). The level of accuracy desired is only that of the Weisskopf-Wigner approximation.

2.(40) Electrons and positrons circulating in opposite directions around storage rings can collide and annihilate, producing oppositely charged pairs of more massive particles. Assume that only electromagnetic couplings are operative and that the electrons and positrons are unpolarized. Calculate to lowest order in the charge the invariant differential cross sections for production of $\mu^{\pm}$ pairs (which you may assume simply to be a pair of heavy electrons) and $\pi^{\pm}$ pairs. Notation: Let $p_{\pm}$ and $p'_{\pm}$ be the initial and final four-momenta, respectively, for the particles with the indicated sign of the charge. Express your results insofar as possible in terms of the Mandelstam variables

$$s = (p_{+} + p_{-})^2$$

$$t = (p_{+} - p'_{+})^2 .$$

Specialize your result to the center-of-mass system and find the distribution of positively charged final particles in momentum and angle.

3.(30) The weak interaction which describes muon decay,

$$H = \frac{G}{\sqrt{2}} [\bar{\psi}_{\nu_\mu}\gamma^\mu(1-\gamma^5)\psi_\mu][\bar{\psi}_e\gamma_\mu(1-\gamma^5)\psi_{\nu_e}] + \text{Herm. Conj.}$$

also leads to the possibility of certain inelastic collision processes. A beam of μ-neutrinos, for example, (produced by pion decay) which collides with electrons can produce negative muons together with electron-neutrinos,

$$\nu_\mu + e^- \rightarrow \mu^- + \nu_e . \qquad t = (\nu - e)^2$$

A few presumed examples of this reaction have in fact been found in a large bubble chamber at CERN. Use the above interaction (which omits a possible contribution from the newly-discovered "neutral current") to find the invariant differential and total cross sections for the production of μ^- (with the final neutrinos of course remaining unobserved). You may assume the neutrinos are massless and the electrons initially at rest and unpolarized. Note however that the incident μ-neutrinos are fully polarized with their spins directed oppositely to their momenta (helicity = -1). Show that if they had the opposite helicity the cross section would vanish. The coupling constant is given by $G = (1.29/\mu^2) \times 10^{-7}$ where μ, the muon mass is 105 MeV. Evaluate the total cross section for incident μ-neutrinos of 20 GeV. (Note a useful constant is $\hbar c = 1.97 \times 10^{-11}$ MeV cm). $\sim 8\times 10^{-44}\,\text{cm}^2$

Physics 253 b

Final Examination Solutions

May 1977

1. A) ~~Formulation~~ Formulation of interaction of two-level system with the radiation field: Let ground state energy $= \varepsilon_0 = 0$

" excited " " $= \varepsilon_1 = \hbar\omega$

Hamiltonian:

$$H = \tfrac{1}{2}\hbar\omega(1+\sigma_z) + M\{\sigma_+ \mathcal{E}^{(+)} + \sigma_- \mathcal{E}^{(-)}\} + \sum_{k'} \hbar\omega_{k'} a^\dagger_{k'} a_{k'}$$

where M = Electric dipole matrix element.

The quantity $\;\mathcal{N} \equiv \tfrac{1}{2}(1+\sigma_z) + \sum a^\dagger_{k'} a_{k'}$

is conserved, i.e. $[\mathcal{N}; H] = 0$

Hence for a state which begins with a single photon and no other excitation, i.e. $\mathcal{N}=1$, we can write

$$|t\rangle = \alpha(t)\,|1,0\rangle + \sum_{k'} \beta_{k'}(t)\,|0,\vec{k}'\rangle$$

where $|1,0\rangle$ has the two-level system excited and no photons and $|0,\vec{k}'\rangle$ has no excitation, but a photon of propagation vector $\vec{k}'$ present.

By writing $\;M\sqrt{\frac{\omega_{k'}}{2\hbar V}} = i\lambda_{k'}$, the Schrödinger equation for $|t\rangle$ can be reduced to the set of coupled equations

$$\dot{\alpha} = -i\omega\alpha - i\sum \lambda_{k'}\beta_{k'}$$

$$\dot{\beta}_k = -i\omega_k\beta_k - i\lambda^*_k\alpha$$

The initial conditions are $\quad \alpha(0)=0 \qquad \beta_{k'}(0) = \delta_{k'k}$

B) Coupled oscillator formulation:

Let the Hamiltonian for the coupled oscillator system be

$$H = \hbar\omega a^\dagger a + \hbar\sum_{k'}(\lambda_{k'} a^\dagger b_{k'} + \lambda^*_{k'} b^\dagger_{k'} a) + \sum \hbar\omega_k b^\dagger_k b_k$$

1 Cont'd We have shown in class that if the initial state of the oscillator system is the coherent state $|\alpha(0), \{\beta_k(0)\}\rangle$ then the state at time t – apart from a phase factor – is the coherent state $|\alpha(t)\ \{\beta_k(t)\}\rangle$ in which the time-dependent amplitudes satisfy precisely the differential equations noted earlier for the two-level system.

The initial conditions are $\alpha(0)=0$, $\beta_{k'}(0) = \delta_{k'k}\beta_k$ (which are not very different from those for the two-level system)

Solution by Laplace Transform

Let $\quad \bar{\alpha}(s) = \int_0^\infty \alpha(t)\, e^{-st}\, dt$.

Then $\quad \int \frac{d\alpha}{dt} e^{-st} dt = -\alpha(0) + s\,\bar{\alpha}(s)$

Likewise for $\bar{\beta}_k(s)$, etc.

The coupled differential equations then transform to

$$s\,\bar{\alpha}(s) - \alpha(0) = -i\omega\,\bar{\alpha}(s) - i\sum \lambda_k \bar{\beta}_k(s)$$

$$s\,\bar{\beta}_{k'}(s) - \beta_{k'}(0) = -i\omega_k \bar{\beta}_k(s) - i\lambda_k^* \,\bar{\alpha}(s)$$

Hence

$$\bar{\beta}_{k'}(s) = \frac{\beta_{k'}(0) - i\lambda_k^* \,\bar{\alpha}(s)}{s + i\omega_k}$$

If we let

$$F(s) \equiv \sum_{k'} \frac{|\lambda_{k'}|^2}{s + i\omega_{k'}}$$

we can write

$$\bar{\alpha}(s) = \frac{1}{s + i\omega + F}\left\{ \alpha(0) - i\sum \frac{\lambda_{k'}\,\beta_{k'}(0)}{s + i\omega_{k'}} \right\}$$

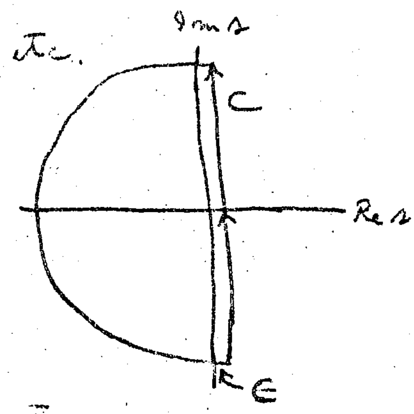
1. Cont'd   With $\alpha(0) = 0$ $\beta_{k'}(0) = \delta_{k'k}\beta_k$

$$\bar{\alpha}(s) = \frac{-i\lambda_k\beta_k}{(s+i\omega_k)(s+i\omega+F)}$$

Then $$\bar{\beta}_{k'}(s) = \frac{\delta_{k'k}\beta_k}{s+i\omega_k} - \frac{\lambda^*_{k'}\lambda_k\beta_k}{(s+i\omega_{k'})(s+i\omega_k)(s+i\omega+F)}$$

The time-dependent solutions are

$$\alpha(t) = \frac{1}{2\pi i}\int_c \bar{\alpha}(s)e^{st}\,ds$$ etc.



The Weisskopf-Wigner approximation consists in replacing $F(s)$ on the contour c by $F(\epsilon - i\omega) \equiv i\delta\omega + \kappa$, where $\delta\omega$ is a frequency shift and κ a damping constant.

Then

$$\alpha(t) = \frac{1}{2\pi i}\int_c \frac{-i\lambda_k\beta_k\, e^{st}\,ds}{(s+i\omega_k)(s+i\bar{\omega})} \qquad \text{where } \bar{\omega} \equiv \omega+\delta\omega-i\kappa$$

$$= \frac{-i\lambda_k\beta_k}{i(\bar{\omega}-\omega_k)}\left\{e^{-i\omega_k t} - e^{-i\bar{\omega}t}\right\}$$

$$= -\frac{\lambda_k\beta_k}{\omega+\delta\omega-\omega_k-i\kappa}\left\{e^{-i\omega_k t} - e^{-i(\omega+\delta\omega)t-\kappa t}\right\}$$

Note that for $t \to \infty$ there is a steady-state excitation amplitude for the two-level system - or for the central oscillator. The amplitude is resonant at the shifted frequency $\omega_k = \omega + \delta\omega$. The resonance width is κ.

1. Cont'd. Solution for $\beta_{k'}(t)$:

$$\beta_{k'}(t) = \frac{1}{2\pi i}\int_c \left\{ \frac{\beta_k \delta_{k'k}}{s+i\omega_k} - \frac{\lambda_{k'}^* \lambda_k \beta_k}{(s+i\omega_{k'})(s+i\omega_k)(s+i\bar{\omega})} \right\} e^{st}\, ds$$

$$= \delta_{k'k}\beta_k e^{-i\omega_k t} + \frac{i\lambda_{k'}^*\lambda_k\beta_k}{(\omega_{k'}-\omega_k)(\omega_k-\bar{\omega})(\bar{\omega}-\omega_{k'})}\left\{(\omega_k-\bar{\omega})e^{-i\omega_{k'}t} + (\bar{\omega}-\omega_{k'})e^{-i\omega_k t} + (\omega_{k'}-\omega_k)e^{-i\bar{\omega}t}\right\}$$

For modes k' degenerate with the k-th mode, $\omega_{k'}=\omega_k$, there is a second order pole in the integrand for which the residue requires a bit more calculation. Alternatively we can take the limit $\omega_{k'}\to\omega_k$ in the general result we have constructed. For modes degenerate with the initial one there is found to be a linear as well as oscillating and damped dependence on time.

What is the time dependent state of the field? The answer is somewhat different in the two formulations of the problem. For the coupled oscillator model the coherent state vector $|\alpha(t)\{\beta_k(t)\}\rangle$ factorizes. Hence the field is in the pure coherent state $|\{\beta_k(t)\}\rangle$. For the model of the two level system however there is no such factorization. The density operator for the entire system is given by

$$|t\rangle\langle t| = \left\{\alpha(t)|1\,0\rangle + \sum \beta_{k'}(t)|0\,k'\rangle\right\}\left\{\langle 1\,0|\alpha^*(t) + \sum\langle 0\,k''|\beta_{k''}^*(t)\right\}$$

Cont'd The state of the field alone is described by a reduced density operator which is reached by taking the partial trace over the states of the two-level system

$$\rho_{FIELD} = \langle 1 | t \rangle \langle t | 1 \rangle + \langle 0 | t \rangle \langle t | 0 \rangle$$

Two level system excited

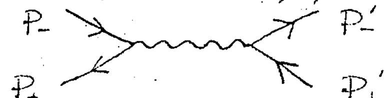
Two level system in ground state

$$= |\alpha(t)|^2 \, |0\rangle\langle 0| + \sum_{k'k''} \beta_{k'}(t) \, \beta^*_{k''}(t) \, |k'\rangle\langle k''|$$

vac. state

one-photon states

2.) First consider $e^+e^- \to \mu^+\mu^-$. To lowest order, the relevant diagram is

(electrons) 

(muons)

The T-matrix is given by $\quad T = e^2 \dfrac{\bar{v}_e(p_+)\gamma_\mu u_e(p_-)\,\bar{u}_\mu(p_-')\gamma^\mu v_\mu(p_+')}{s+i\epsilon}$.

Summing over final spins, and averaging over initial spins, we have

$$\left(\tfrac{1}{2}\right)^2 \sum |T|^2 = \frac{e^4}{4s^2} \sum |\bar{v}_e(p_+)\gamma_\mu u_e(p_-)|^2 \sum |\bar{u}_\mu(p_-')\gamma^\mu v_\mu(p_+')|^2$$

$$= \frac{e^4}{4s^2} \mathrm{Tr}(\gamma_\mu(\not{p}_- + m_e)\gamma_\nu(\not{p}_+ - m_e))\,\mathrm{Tr}(\gamma^\mu(\not{p}_+' - m_\mu)\gamma^\nu(\not{p}_-' + m_\mu))$$

The traces are easily evaluated:

$$\mathrm{Tr}(\gamma_\mu(\not{p}_- + m_e)\gamma_\nu(\not{p}_+ - m_e)) = 4[p_{-\mu}p_{+\nu} + p_{-\nu}p_{+\mu} - p_-\cdot p_+ g_{\mu\nu} - m_e^2 g_{\mu\nu}]$$
$$\mathrm{Tr}(\gamma^\mu(p_+' - m_\mu)\gamma^\nu(\not{p}_-' + m_\mu)) = 4[p_+'^\mu p_-'^\nu + p_+'^\nu p_-'^\mu - p_+'\cdot p_-' g^{\mu\nu} - m_\mu^2 g^{\mu\nu}]$$

Using $\quad s = (p_+ + p_-)^2 = 2m_e^2 + 2p_+\cdot p_-$

and $\quad s = (p_+' + p_-')^2 = 2m_\mu^2 + 2p_+'\cdot p_-'$, we have

$$\mathrm{Tr}(\gamma_\mu(\not{p}_- + m_e)\gamma_\nu(\not{p}_+ - m_e)) = 4[p_{-\mu}p_{+\nu} + p_{-\nu}p_{+\mu} - \tfrac{1}{2}s g_{\mu\nu}]$$
$$\mathrm{Tr}(\gamma^\mu(\not{p}_+' - m_\mu)\gamma^\nu(\not{p}_-' + m_\mu)) = 4[p_+'^\mu p_-'^\nu + p_+'^\nu p_-'^\mu - \tfrac{1}{2}s g^{\mu\nu}]$$

The product is

$$\mathrm{Tr}(\gamma_\mu(\not{p}_- + m_e)\gamma_\nu(\not{p}_+ - m_e))\mathrm{Tr}(\gamma^\mu(\not{p}_+' - m_\mu)\gamma^\nu(\not{p}_-' + m_\mu)) = 16[2(p_-\cdot p_+'\, p_-'\cdot p_+ + p_-\cdot p_-'\, p_+\cdot p_+') - s(p_-\cdot p_+ + p_-'\cdot p_+') + \tfrac{1}{4}s^2\, 4]$$

$$= 16[s^2 - s(p_-\cdot p_+ + p_-'\cdot p_+') + 2(p_-\cdot p_+'\, p_-'\cdot p_+ + p_-\cdot p_-'\, p_+\cdot p_+')]$$

Next, note that $\quad p_-\cdot p_+ + p_-'\cdot p_+' = s - m_e^2 - m_\mu^2$

so that $\quad s(s - p_-\cdot p_+ - p_-'\cdot p_+') = s(m_e^2 + m_\mu^2)$. Also, note that $p_-\cdot p_+' = p_-'\cdot p_+$, and that $\quad p_-\cdot p_-' = p_+\cdot p_+'$. Thus $\quad p_-\cdot p_+'\, p_-'\cdot p_+ + p_-\cdot p_-'\, p_+\cdot p_+' = (p_-\cdot p_+')^2 + (p_-\cdot p_-')^2$

Finally, we note that $\quad p_-\cdot p_+' = \tfrac{1}{2}(m_e^2 + m_\mu^2 - u) \quad p_-\cdot p_-' = \tfrac{1}{2}(m_e^2 + m_\mu^2 - t)$

and that $\quad s + t + u = 2(m_e^2 + m_\mu^2)$, so $\quad m_e^2 + m_\mu^2 - u = s + t - m_e^2 - m_\mu^2$

and

$$\left(\tfrac{1}{2}\right)^2 \sum |T|^2 = \frac{e^4}{4s^2} 16[s(m_e^2 + m_\mu^2) + 2[\tfrac{1}{2}(s + t - m_e^2 - m_\mu^2)]^2 + 2[\tfrac{1}{2}(m_e^2 + m_\mu^2 - t)]^2]$$

$$= \frac{2e^4}{s^2}[s^2 + 2st + 2(m_e^2 + m_\mu^2 - t)^2] \equiv \bar{T}^2(e\to\mu)$$

Recall from last semester that the c.o.m. cross-section is

$$\frac{d\sigma}{d\Omega_{p_+'}} = \frac{|\vec{p}_+'|}{64\pi^2 |\vec{p}_+| s} |\langle p_+' \cdot p_+' | T | p_+ \cdot p_+ \rangle|^2$$

We need to see if $\dfrac{dt}{d\Omega_{p_+'}}$ changes for an inelastic process like $e^+e^- \to \mu^+\mu^-$.

$$t = (p_+' - p_-)^2 = m_\mu^2 + m_e^2 - 2p_+'\cdot p_+ = m_\mu^2 + m_e^2 - 2\left(\sqrt{m_\mu^2 + \bar{p}_+'^2}\sqrt{m_e^2 + \bar{p}_+^2} - \bar{p}_+'\cdot\bar{p}_+\right)$$
$$= F(|\bar{p}_+|, |\bar{p}_+'|) + 2|\bar{p}_+'||\bar{p}_+|\cos\theta$$

Thus $d\Omega_{p_+'} = -2\pi\, d(\cos\theta) = -\dfrac{\pi\, dt}{|\bar{p}_+'||\bar{p}_+|}$, and hence $\dfrac{d\sigma}{dt} = -\dfrac{\pi}{|\bar{p}_+'||\bar{p}_+|}\dfrac{|\bar{p}_+'|\,|\langle|T|\rangle|^2}{64\pi^2|\bar{p}_+|s}$

or $\dfrac{d\sigma}{d|t|} = \dfrac{|\langle|T|\rangle|^2}{64\pi|\bar{p}_+|^2 s}$ as before! Also $s = 4E_{p_+}^2 \Rightarrow \bar{p}_+^2 = \dfrac{s - 4m_e^2}{4}$

so $\dfrac{d\sigma}{d|t|} = \dfrac{|\langle|T|\rangle|^2}{16\pi s(s - 4m_e^2)}$

Thus, we have for the muon production cross-section:

$$\frac{d\sigma}{d|t|}(e^+e^- \to \mu^+\mu^-) = \frac{e^4}{8\pi s^3(s - 4m_e^2)}\left\{s^2 + 2st + 2(m_e^2 + m_\mu^2 - t)^2\right\}$$

The distribution of final μ^+ momenta is trivial - there is one allowed μ^+ momenta in the center of mass $|\bar{p}_+'| = \frac{1}{2}(s - 4m_\mu^2)^{1/2}$

N for the angular distribution of the produced muons. We will need t (in the center of mass):

$$t = (p_+' - p_+)^2 = m_\mu^2 + m_e^2 - 2p_+'\cdot p_+ = m_\mu^2 + m_e^2 - 2(E_{p_+}E_{p_+'} - p_+ p_+'\cos\theta)$$

and $s = (2E_{p_+})^2 = (2E_{p_+'})^2$, so

$$t = m_\mu^2 + m_e^2 - \tfrac{1}{2}s + 2p_+ p_+'\cos\theta$$

This gives the invariant cross-section:

$$\frac{d\sigma}{d|t|} = \frac{e^4}{8\pi s^3(s - 4m_e^2)}\left\{s^2 + 2s(m_\mu^2 + m_e^2 - \tfrac{1}{2}s + 2p_+p_+'\cos\theta) + 2(\tfrac{1}{2}s - 2p_+p_+'\cos\theta)^2\right\}$$

$$= \frac{e^4}{8\pi s^3(s - 4m_e^2)}\left\{2s(m_\mu^2 + m_e^2) + \cancel{2s\,2p_+p_+'\cos\theta} + \tfrac{1}{2}s^2 + 8p_+^2p_+'^2\cos^2\theta - \cancel{2s\,2p_+p_+'\cos\theta}\right\}$$

$$= \frac{e^4}{8\pi s^3(s - 4m_e^2)}\left\{2s(m_\mu^2 + m_e^2) + \tfrac{1}{2}s^2 + \tfrac{1}{2}(s - 4m_e^2)(s - 4m_\mu^2)\cos^2\theta\right\}$$

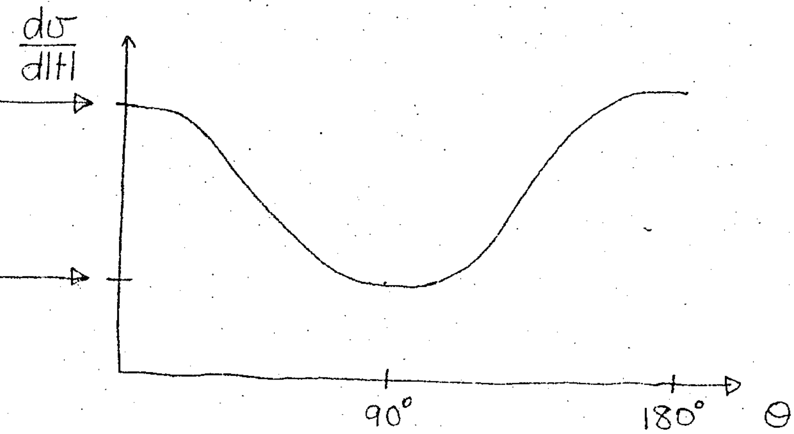
or

$$\boxed{\frac{d\sigma}{d|t|} = \frac{e^4}{16\pi s^3(s - 4m_e^2)}\left\{s^2 + 4s(m_\mu^2 + m_e^2) + (s - 4m_e^2)(s - 4m_\mu^2)\cos^2\theta\right\}}$$

Thus:

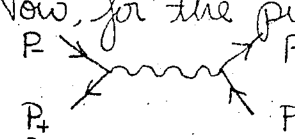
$$\left.\frac{d\sigma}{d|t|}\right|_{max} = \frac{e^4(s^2+8m_e^2m_\mu^2)}{8\pi s^3(s-4m_e^2)}$$

$$\left.\frac{d\sigma}{d|t|}\right|_{min} = \frac{e^4(s+4m_e^2+4m_\mu^2)}{16\pi s^2(s-4m_e^2)}$$



For $s=4m_\mu^2$ (threshold) the distribution is spherically symmetric and develops a forward (and backward) peak as s is increased.

Now, for the pion production problem. The relevant diagram is



The T-matrix is given by

$$T = \frac{e^2\,\bar{v}_e(p_+)\gamma^\mu u_e(p_-)(p_-'-p_+')_\mu}{s+i\epsilon} = \frac{e^2}{s}\bar{v}_e(p_+)(\not{p}_-'-\not{p}_+')u_e(p_-)$$

(i.e. the minus sign in front of p_+' in the term $p_-'-p_+$ is due to the fact that the momentum p_+ progresses against the arrow!)

Now, $p_+' = p_++p_--p_-'$, and $\bar{v}_e(p_+)(\not{p}_++\not{p}_-)u_e(p_-)=0$, so: $T = \frac{2e^2}{s}\bar{v}_e(p_+)\not{p}_-'u_e(p_-)$

Averaging over the e^+e^- spins, we have

$$\overline{|T|^2} = 4\left(\tfrac{1}{2}\right)^2\sum_{spins}|T|^2 = \frac{e^4}{s^2}Tr\{\not{p}_-'(\not{p}_-+m_e)\not{p}_-'(\not{p}_+-m_e)\} = \frac{e^4}{s^2}\left[Tr(\not{p}_-'\not{p}_-\not{p}_-'\not{p}_+)-m_e^2Tr(\not{p}_-'\not{p}_-')\right]$$

$$Tr(\not{p}_-'\not{p}_-') = 4m_\pi^2$$

$$\tfrac{1}{4}Tr(\not{p}_-'\not{p}_-\not{p}_-'\not{p}_+) = 2p_-\cdot p_-'\;p_+\cdot p_-' - m_\pi^2\,p_-\cdot p_+$$

Now $2p_-\cdot p_-' = -(t-m_\pi^2-m_e^2)$ and $p_-\cdot p_+ = \frac{1}{2}s-m_e^2$

Also $p_+\cdot p_-' = p_-'\cdot(p_+'+p_-'-p_-) = \frac{1}{2}s-m_\pi^2+m_\pi^2+\frac{1}{2}(t-m_\pi^2-m_e^2) = \frac{1}{2}(s+t-m_\pi^2-m_e^2)$

Thus:

$$\overline{|T|^2} = \frac{4e^4}{s^2}\left[-(t-m_\pi^2-m_e^2)\tfrac{1}{2}(s+t-m_\pi^2-m_e^2)-m_\pi^2(\tfrac{1}{2}s-m_e^2)-m_e^2m_\pi^2\right]$$

$$= \frac{4e^4}{s^2}\left[-\tfrac{1}{2}(t-m_\pi^2-m_e^2)^2-\tfrac{1}{2}s(t-m_\pi^2-m_e^2)-m_\pi^2(\tfrac{1}{2}s-m_e^2)-m_e^2m_\pi^2\right]$$

$$= \frac{4e^4}{s^2}\left[-\tfrac{1}{2}(t-m_\pi^2-m_e^2)^2-\tfrac{1}{2}s(t-m_\pi^2-m_e^2)-\tfrac{1}{2}sm_\pi^2\right]$$

$$= \frac{2e^4}{s^2}\left[s(m_\pi^2+m_e^2-t)-(m_\pi^2+m_e^2-t)^2-sm_\pi^2\right]$$

Thus,

$$\boxed{\frac{d\sigma}{d|t|} = \frac{e^4}{8\pi s^3(s-4m_e^2)}\{s(m_\pi^2+m_e^2-t)-(m_\pi^2+m_e^2-t)^2-sm_\pi^2\}}$$

Again, there is only one allowed momentum for the outgoing π^+, given by $|\vec{p}_+'| = \frac{1}{2}(s-4m_\pi^2)^{1/2}$ (in the center of mass).
Now for the angular distribution of the produced pions:

$$\frac{d\sigma}{d|t|} = \frac{e^4}{8\pi s^3(s-4m_e^2)}\left\{ s\left(\tfrac{1}{2}s - 2p_+p_+'\cos\theta\right) - \left(\tfrac{1}{2}s - 2p_+p_+'\cos\theta\right)^2 - sm_\pi^2 \right\}$$

$$= \frac{e^4}{8\pi s^3(s-4m_e^2)}\left\{ \tfrac{1}{4}s^2 - \cancel{2sp_+p_+'\cos\theta} + \cancel{2sp_+p_+'\cos\theta} - 4p_+^2p_+'^2\cos^2\theta - sm_\pi^2 \right\}$$

$$= \frac{e^4}{32\pi s^3(s-4m_e^2)}\left\{ s^2 - s4m_\pi^2 - 16p_+^2p_+'^2\cos^2\theta \right\}$$

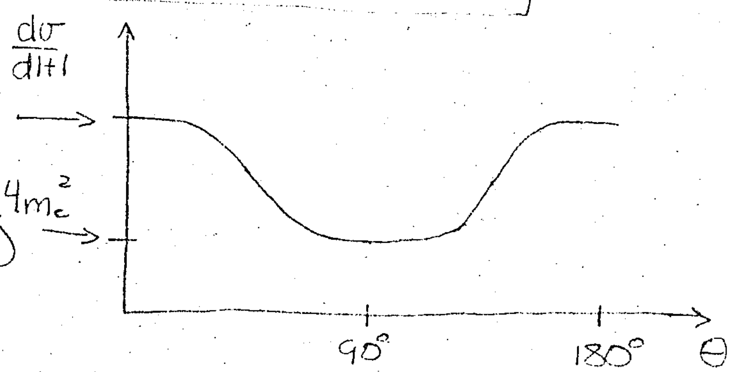
or

$$\boxed{\frac{d\sigma}{d|t|} = \frac{e^4(s-4m_\pi^2)}{32\pi s^3(s-4m_e^2)}\left\{ s - (s-4m_e^2)\cos^2\theta \right\}}$$

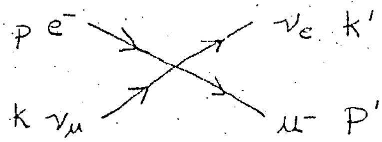
Thus:

$$\left.\frac{d\sigma}{d|t|}\right|_{max} = \frac{e^4(s-4m_\pi^2)}{32\pi s^2(s-4m_e^2)}$$

$$\left.\frac{d\sigma}{d|t|}\right|_{min} = \frac{e^4(s-4m_\pi^2)4m_e^2}{32\pi s^3(s-4m_e^2)}$$



Note: Even at threshold, there is anisotropy in $\frac{d\sigma}{d|t|}$!

3.) We are looking at the process 

With the coupling $\mathcal{H} = \frac{G}{\sqrt{2}}[\bar\Psi_{\nu_\mu} M^\mu \Psi_\mu][\bar\Psi_e M_\mu \Psi_{\nu_e}]$ + Herm. Conj. $(M_\mu \equiv \gamma_\mu(1-\gamma_5)$

we can easily calculate the T-matrix element associated with the above graph: $T = -\frac{G}{\sqrt{2}}[\bar u_{\nu_e}(k')M^\mu u_e(p)][\bar u_\mu(P')M_\mu u_{\nu_\mu}(k)]$

Let us square T, sum over final neutrino spins, sum over final muon spins, average over initial electron spins, and put in a helicity projection operator for the initial mu neutrino:

$$\frac{1}{2}\sum|T|^2 = \frac{1}{2}\frac{G^2}{2}\,\mathrm{Tr}[(\not p+m_e)M^\nu \not k' M^\mu]\,\mathrm{Tr}[(\not p'+m_\mu)M_\mu \mathcal{P}\not k M_\nu]$$

where $\mathcal{P} = \frac{1}{2}(1\pm\gamma_5)$ ($\pm$ for helicity $+1$ / for helicity -1). Suppose we take helicity $+1$, then there is a product $M_\mu \mathcal{P}_+ = \gamma_\mu(1-\gamma_5)\frac{1}{2}(1+\gamma_5) = \frac{1}{2}\gamma_\mu(1-\gamma_5^2)$

But $\gamma_5^2 = 1$, so $M_\mu \mathcal{P}_+ = 0$, and the reaction can't go. Now, since we get zero for $\mathcal{P}_+$, we might as well set $\mathcal{P}=1$, hence:

$$\frac{1}{2}\sum|T|^2 = \frac{G^2}{4}\mathrm{Tr}[(\not p+m_e)M^\nu \not k' M^\mu]\,\mathrm{Tr}[(\not p'+m_\mu)M_\mu \not k M_\nu]$$

$$= \frac{G^2}{4}\mathrm{Tr}(\not p M^\nu \not k' M^\mu)\,\mathrm{Tr}(\not p' M_\mu \not k M_\nu)$$ (The mass terms give vanishing traces)

We evaluated the product of these two traces in the solutions to problem set # 3, where we found that

$$\mathrm{Tr}(\not p M^\nu \not k' M^\mu)\,\mathrm{Tr}(\not p' M_\mu \not k M_\nu) = 256\, p\cdot k\; p'\cdot k'$$

Now $p\cdot k = \frac{1}{2}[2p\cdot k] = \frac{1}{2}[(p+k)^2 - p^2 - k^2] = \frac{1}{2}[s - m_e^2]$

and $p'\cdot k' = \frac{1}{2}[s - m_\mu^2]$, so

$$\frac{1}{2}\sum|T|^2 = \frac{G^2}{4}\,256\,\frac{1}{2}(s-m_e^2)\frac{1}{2}(s+m_\mu^2) = 16G^2(s-m_e^2)(s-m_\mu^2)$$

Even for inelastic processes, $\frac{d\sigma}{d|t|} = \frac{|T|^2}{64\pi p^{*2} s}$

Now, in the c.o.m. frame $s = (p^* + \sqrt{p^{*2}+m_e^2})^2 \Rightarrow 4p^{*2}s = (s-m_e^2)^2$

so

$$\frac{d\sigma}{d|t|} = \frac{|T|^2}{16\pi(s-m_e^2)^2}, \text{ or } \boxed{\frac{d\sigma}{d|t|} = \frac{G^2}{\pi}\left(\frac{s-m_\mu^2}{s-m_e^2}\right)}$$

Spherical distribution !!

Next, let us evaluate the total production cross-section σ:

$t = (p'-p)^2 = m_\mu^2 + m_e^2 - 2p\cdot p' = m_\mu^2 + m_e^2 - 2(E_p E_{p'} - \bar{p}\cdot\bar{p}')$

Largest value of $t = m_\mu^2 + m_e^2 - 2(E_p E_{p'} - |\bar{p}||\bar{p}'|)$

Smallest value of $t = m_\mu^2 + m_e^2 - 2(E_p E_{p'} + |\bar{p}||\bar{p}'|)$

$$t_{max} - t_{min} = 4|\bar{p}||\bar{p}'| = 2|\bar{p}|\ 2|\bar{p}'| = \sqrt{\frac{(s-m_e^2)^2}{s}}\sqrt{\frac{(s-m_\mu^2)^2}{s}} = \frac{(s-m_e^2)(s-m_\mu^2)}{s}$$

Thus $$\sigma = \int_{t_{min}}^{t_{max}} \frac{d\sigma}{d|t|} dt = \int_{t_{min}}^{t_{max}} \frac{G^2}{\pi}\left(\frac{s-m_\mu^2}{s-m_e^2}\right) dt = \frac{G^2}{\pi}\frac{(s-m_\mu^2)}{(s-m_e^2)}\ \frac{(s-m_e^2)(s-m_\mu^2)}{s}$$

or

$$\boxed{\sigma = \frac{G^2}{\pi}\frac{(s-m_\mu^2)^2}{s}}$$

The 20GeV neutrino refers to the frame in which the electron is at rest, thus if $E = 20\text{GeV}$, then $s = (E+m_e, \bar{E})^2 = 2m_e E + m_e^2 = m_e(m_e + 2E)$

Now, we can neglect m_e compared to $2E$ when $E = 20\text{ GeV}$, so $s \simeq 2m_e E$, and

$$\sigma = \frac{G^2}{\pi}\frac{(2m_e E - m_\mu^2)^2}{2m_e E}$$

Let us write $2m_e E = x\, m_\mu^2$, so $x = \dfrac{2(.511\text{MeV})(20\text{GeV})}{(105\text{MeV})^2} = 1.854$

Now $G m_\mu^2 = 1.29\times10^{-7}$

and thus $$\sigma = \frac{(1.29\times10^{-7})^2}{\pi}\frac{(1.854-1)^2}{1.854}\frac{1}{m_\mu^2} = \frac{2.084\times10^{-15}}{m_\mu^2}$$

Now, the muon compton wavelength is $\dfrac{1}{m_\mu} = \dfrac{m_e}{m_\mu}\dfrac{1}{m_e} = \left(\dfrac{.511}{105}\right)(3.86\times10^{-11}\text{cm})$

and hence we find that $= 1.88\times10^{-13}\text{cm}$

$$\sigma = 7.35\times10^{-41}\text{cm}^2 = 7.35\times10^{-17}\text{ barns}$$