Physics 253

Prof. Glauber Lyman 331A, afternoons. Take home final

Videotapes of Coleman's course in Cabot. ㉚

Course Assistant: David Ling: 4626

Texts: Bjorken + Drell (2nd vol. more relevant)

Final Exam: Jan 17-20

17 - 20

Exam: Monday at 1:00 → Thursday at 1:00

Final Exam May 20-23

I Second Quantization

Field theories are essentially many-particle theories
Pauli: 1925, the exclusion principle, prior to Uhlenbeck-Goodsmi
Explain Mendeleev by putting 2 electrons in each Bohr orbit
Dirac 1926: Wavefunctions are antisymmetric structures

1) $\Psi(\vec{r}_1, \cdots \vec{r}_n)$ antisymmetric → Pauli
2) If Hamiltonian is symmetric, Ψ preserves antisymmetry.
3) Symmetric Ψ explains photons.

<u>1924</u> S.N. Bose and Einstein '24-'25 Count particles only by their states, gives Bose-Einstein statistics for photons (bosons)
<u>1926</u>: Fermi statistics of a gas of free particles obeying Pauli principle: The Fermi-Dirac statistics for fermions.
Spin Statistics Theorem: Integer spin → bosons
half-odd integer spin → fermions.

<u>1926 Heisenberg</u> Helium spectrum: Ortho- and para-spectra
Extends Diracs idea to include spin states. Whether the spins are parallel or anti-parallel constrains the accessible orbits.

<u>Ferromagnetism</u> Electron spins tend to line up, because of the exclusion principle. In Ferromagnetic situation anomalously, the spins parallel actually lowers the energy.

State vector $|\Psi\rangle$ - N-particle system.
Spin zero: $\Psi(r_1, r_2 \cdots r_n) = \langle r_1, \cdots r_n | \Psi \rangle$
Spin j: 2j+1 comp. spinors: Ψ a $(2j+1)^N$ comp. spinor.

$\Psi_{s_1 \cdots s_N}(r_1 \cdots r_N) = \langle r_1 s_1, r_2 s_2 \cdots r_N s_N | \Psi \rangle$

Let x_i denote (r_i, s_i)

$\Psi(x_1 \cdots x_N) = \langle x_1 \cdots x_N | \Psi \rangle$

To integrate, must remember to add over the spin variables.

P_{ik} = Permutation of i, k, particles

$$P_{ik}\Psi(x_1 \cdots x_N) = \begin{matrix} \Psi(x_1 \cdots x_N) & \text{bosons} \\ -\Psi(x_1 \cdots x_N) & \text{fermions} \end{matrix}$$

General: $P\Psi(x_1 \cdots x_N) = (-1)^p \Psi(x_1 \cdots x_N)$ p = parity of the permutation

1927-28: Pascual Jordan, Oscar Klein, Eugene Wigner:

"Second Quantization": Independent of Particle Number

This H does not change particle number.

Typical Many-Particle-Hamiltonian: $H = \sum_{j=1}^{N} h_j(p_j, x_j)$ $+ \sum_{j,k} V(x_j x_k)$ + possible (unlikely?) many-particle forces.

Try to diagonalize Hamiltonian between two states describing 10^{21} electrons.

Take $\{\varphi_j(x)\}$ a complete set of space-spin wave functions. Set up Slater determinantal equation to get antisymmetry, then expand Ψ in these determinantal functions.

Occupation number representation:

$n_j = 0, 1, 2 \ldots$ Specify states by specifying a set of occupation numbers.

To specify the n_jth state: $| \rangle_j = C_0|0\rangle_j + C_1|1\rangle_j$ fermions

Bosons: $| \rangle_j = C_0|0\rangle_j + C_1|1\rangle_j + \cdots\cdots$

To normalize: $\sum_n |C_n|^2 = 1$

(Bose Statistics)

Here $\langle n|n'\rangle = \delta_{nn'}$, the spin-space state $|\ \rangle_j$ is a sum of orthogonal states.

All states together: $|\ \rangle = |\ \rangle_1 \otimes |\ \rangle_2 \otimes |\ \rangle_3 \otimes |\ \rangle \cdots$

1 state: inf-dim Hilbert space

All states: Direct of Hilbert spaces.

The Nothing state: $|0\rangle = \prod_{\text{all } j} |0\rangle_j$

Given state vectors $|0\rangle, |1\rangle, |2\rangle$ - —

Operators a, a^+ such that $[a, a^+] = 1$, $a^+ a|n\rangle = n|n\rangle$

Then $a|n\rangle = \sqrt{n}|n-1\rangle$, $\quad a|0\rangle = 0 \quad [a_j, a_k] = 0$

$a^+|n\rangle = \sqrt{n+1}|n+1\rangle \quad a a^+|n\rangle = n+1|n\rangle \quad [a_j^+, a_k^+] = 0, [a_j, a_k^+] = \delta$

These are harmonic oscillator operators, but this has nothin to do with the harmonic oscillator.

Define a_j, a_j^+ for each state j.

$\psi(x) = \sum_j a_j \varphi_j(x)$, reduces number of particles by one.

$\psi^+(x) = \sum_j a_j^+ \varphi_j^*(x)$

$[\psi(x), \psi(x')] = 0 \qquad [\psi^+(x), \psi^+(x')] = 0$

→ $[\psi(x), \psi^+(x')] = \sum [a_j, a_k^+] \varphi_j(x) \varphi_k^*(x') = \sum_j \varphi_j(x) \varphi_j(x') = \delta(x-x$

$\delta(x-x') = \delta_{s,s'} \delta(r-r')$

This is a Bose-Einstein Field

Review: Take complete orthonormal set $\{\varphi_j\}$ of "mode" functio

Can be solutions to some Schroedinger equation

$|\{n_j\}\rangle = |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots$ a product space

Bose-Einstein: Use operators from Harmonic Oscillator (Book Keep operators

$[a_i, a_j^+] = \delta_{ij} \quad [a_i, a_j] = [a_i^+, a_j^+] = 0$ (In other words, the numbers of things in each state are completely independent

Wave-function operator (Field operators)

$\psi(x) = \sum a_j \varphi_j(x)$

Canonical relations between a_j's for different complete $\{\varphi_j\}$

$\psi^\dagger(x) = \sum a_j^\dagger \varphi_j^\dagger(x)$ a_j's operators, φ_j's functions

$[\psi(x), \psi(x')] = [\psi^\dagger(x), \psi^\dagger(x')] = 0$

$[\psi(x), \psi^\dagger(x')] = \delta(x-x')$ (remembering that x may include spin coordinates

(analogy to $(\psi^*\psi|)$: $\psi^\dagger(x)\,\psi(x) = \overset{\text{operator for}}{\text{a density}}$:

$$\int_{\substack{\text{spins}\\ +\text{coordinates}}} \psi^\dagger_{(x)} \psi(x)\, dx = \sum_s \int \psi_s^\dagger(\vec r)\, \psi_s(\vec r)\, d\vec r = \sum a_j^\dagger a_j$$

= Total number of particles in system $= N = \sum a_j^\dagger a_j$

This is the $\sum n_i$, since $a_j^\dagger a_j |\{n_i\}\rangle = n_j |\{n_i\}\rangle$

Write $a_j^\dagger a_j = N_j$ a_j: annihilation, $a_j^\dagger$: creation

$[N, \psi(x)] = \sum_j [a_j^\dagger a_j, \psi(x)] = \sum_j [a_j^\dagger, \psi(x)] a_j = -\psi(x)$

$N\psi(x) = \psi(x)(N-1)$, so $\psi(x)$ is an annihilation operator on all states:

$$\int \psi^\dagger_{(x)} \underbrace{\psi^\dagger(x')\, \psi(x')}\, \psi(x))\, dx = \int \psi^\dagger(x)\, N\, \psi(x)\, dx$$

$$= \int \psi^\dagger(x)\, \psi(x)\, dx\, (N-1) = N(N-1)$$

Connection with configuration space wave functions.

One-particle wave function: $\Phi(x) = \varphi_j(x)$

Start with $|0\rangle = |\{0\}_i\rangle$

$|1\rangle = a_j^\dagger |0\rangle = \int \Phi(x)\, \psi^\dagger(x)\, dx\, |0\rangle$

2-particles:

a) $i \neq j$: $\Phi(x_1, x_2) = \frac{1}{\sqrt2}\left(\varphi_i(x_1)\varphi_j(x_2) + \varphi_j(x_1)\varphi_i(x_2)\right)$

$|2\rangle = a_i^\dagger a_j |0\rangle$ This is inherently symmetry, since we do not deal with coordinate labelling

$|2\rangle = \frac{1}{\sqrt2}\int \Phi(x_1, x_2)\, \psi^\dagger(x_1)\, \psi^\dagger(x_2)\, dx_1\, dx_2\, |0\rangle$

$= \frac{1}{\sqrt2}\frac{1}{\sqrt2}(a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger)|0\rangle = a_i^\dagger a_j^\dagger |0\rangle$

If $i=j$ $\Psi(x_1,x_2) = \varphi_i(x_1)\varphi_i(x_2)$

Using $|2\rangle \frac{1}{\sqrt{2}}\int\Psi(x_1,x_2)\,\psi^+(x_1)\,\psi^+(x_2)\,dx_1\,dx_2\,|0\rangle$

$|2\rangle = \frac{1}{\sqrt{2}}\int \varphi_i(x_1)\,\varphi_i(x_2)\,\psi^+(x_1)\,\psi^+(x_2)\,dx_1\,dx_2|0\rangle$

$|2\rangle = \frac{1}{\sqrt{2}}\,a_i^+a_i^+|0\rangle$ See Harmonic Oscillator: $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$

In general: given $\Psi(x_1,\dots x_N)$

write $\frac{1}{\sqrt{N!}}\int\Psi(x_1\cdots x_N)\,\psi^+(x_1)\cdots\psi^+(x_N)\,dx_1\cdots dx_N|0\rangle = |N\rangle$

(All this can be done through analogy to spin functions
spin-Statistics connection?

Fermi-Dirac Particles

Given "mode" functions: $\{\varphi_j(x)\}$

$|\ \rangle_j = C_0|0\rangle_j + C_1|1\rangle_j$ for Fermions.

Invent operator U_j s.t: $U_j|1\rangle_j = |0\rangle_j$, $U_j|0\rangle_j = 0$

jth mode has 2 basis states:

U_j specifies all the matrix elements

$$\langle n|U_j|m\rangle = \begin{pmatrix}0&1\\0&0\end{pmatrix}$$: matrix representation of U

$$U^+ : \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

so $U_j^+|0\rangle_j = |1\rangle_j$, $U_j^+|1\rangle_j = 0$

real "number"

Let $a_j = \Lambda_j U_j$ $\quad a_j^+ = U_j^+\Lambda_j^* = \Lambda_j U_j^+$

$U_j^+U_j : \begin{pmatrix}0&0\\0&1\end{pmatrix}$, $U_j^+U_j = N_j$

$U_jU_j^+ : \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\quad U_jU_j^+ = 1-N_j$

We want $a_j^+a_j = N_j \Rightarrow |\Lambda_j|^2 = 1$

Want $\{a_i,a_j\} = 0$ (trivial for $i=j$)

$\{a_i^+,a_j^+\} = 0$, $\{a_i,a_j^+\} = \delta_{ij}$

Construction: adopt an ordering for φ_j

Let $\Lambda_j = \prod_{k=1}^{j-1}(1-2N_k)$ ← these commute with U_j

$N_k = U_k^+U_k$

Λ_j is thus not really a number, but an operator that commutes with U_j"

For $l<m$ $\quad a_l a_m = \Lambda_l U_l \Lambda_m U_m$ $\qquad a_m a_l = \Lambda_m U_m \Lambda_l U_l$

involves N_l before U_l $\qquad$ involves N_l after U_l

thus: this applied to $|0\rangle_l$ gives 0 for both, applied to $|1\rangle_l$, one is the negative of the other

Thus we have the anticommutation

$a_\ell a_m \overset{-}{=} a_m a_\ell$, $a_\ell^+ a_m^+ = -a_m^+ a_\ell^+$

For $\ell < m$ $\quad a_\ell^+ a_m = -a_m a_\ell^+$

For $l = m$ $\quad a_\ell^+ a_\ell = N_\ell \quad a_\ell a_\ell^+ = 1 - N_\ell$

so $\{a_\ell, a_\ell^+\} = 1$

1928 Pascual Jordan and Eugene Wigner

Define: $\Psi(x) = \Sigma\, a_j\, \varphi_j(x)$, $\Psi^+(x) = \Sigma\, a_j^+\, \varphi_j^+(x)$

$\{\Psi(x), \Psi(x')\} = \{\Psi^+(x), \Psi^+(x')\} = 0$

$\{\Psi(x), \Psi^+(x')\} = \delta(x - x')$

$$\int \Psi^+(x)\Psi(x)\,dx = \sum_j a_j^+ a_j = \Sigma N_{j\cancel{\neq}} = N$$

$$[N, \Psi(x)] = \Sigma[a_j^+ \cancel{\neq} a_j, \cancel{\Psi(x)}] = \Sigma(a_j^+ a_j \Psi(x) - \Psi(x) a_j^+ a_j)$$

$$= -\Sigma(a_j^+ \Psi(x) + \Psi a_j^+) a_j = -\Sigma a_j \varphi_j(x) = \underline{-\Psi(x)}$$

Thus $N\Psi(x) = \Psi(x)(N-1)$, same relation as for Bose ~~Einstein~~ field $\Psi(x)$ is an annihilation operator for Fermions.

$$\int \Psi^+(x)\Psi^+(x')\Psi(x')\Psi(x)\,dx\,dx'$$

Here ordering counts, since interchange gives − sign.

$$= \int \Psi^+(x)\, N\, \Psi(x)\,dx$$

$$= \int \Psi^+(x)\,\Psi(x)\,dx\,(N-1) = N(N-1)$$

Same as for Bosons,

Configuration space **States:** 1 particle $\Phi(x) = \varphi_j(x)$

$$|1\rangle = \int \Phi(x)\,\Psi^+(x)\,dx\,|0\rangle = a_j^+|0\rangle$$

2 particles; cannot have $i = j$

$$\Phi(x_1, x_2) = \tfrac{1}{\sqrt{2}}(\varphi_i(x_1)\varphi_j(x_2) - \varphi_j(x_1)\varphi_i(x_2))$$

Let $|2\rangle = \frac{1}{\sqrt{2}} \int \Phi(x_1, x_2)\,\Psi^+(x_2)\,\Psi^+(x_1)\,dx_1\,dx_2\,|0\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(a_i^+ a_j^+ - a_j^+ a_i^+)|0\rangle$

Because of anti commutation: $|2\rangle = a_i^+ a_j^+ |0\rangle$

Antisymmetry built in through anticommutation of the a_i,

Fermions:

$|1\rangle = \int \Psi(x)\,\psi^\dagger(x)\,dx\,|0\rangle$ state specified by $\Psi(x)$

$|2\rangle = \frac{1}{\sqrt{2}}\int \Psi(x_1,x_2)\,\psi^\dagger(x_2)\,\psi^\dagger(x_1)\,dx_1\,dx_2\,|0\rangle$

$|n\rangle = \frac{1}{\sqrt{N!}}\int \Psi(x_1\cdots x_N)\,\psi^\dagger(x_{\text{[illegible]}})\cdots\psi^\dagger(x_N)\,dx_1\cdots dx_N\,|0\rangle$

same as for Bose:

This is algorithm for (configuration space) → (occupation number)

$$\frac{1}{N!}\langle 0|\psi(x_N)\cdots\psi(x_1)\int \Psi(x_1'\cdots x_n')\,\psi^\dagger(x_1')\cdots\psi^\dagger(x_N')|0\rangle$$
$$= \frac{1}{\sqrt{N!}}\langle 0|\psi(x_N)\cdots\psi(x_1)|N\rangle$$

For $N=1$ $\langle 0|\psi(x_1)\,\psi^\dagger(x_1')|0\rangle = \langle 0|\,\delta(x_1-x_1') + \underbrace{\cancel{\psi^\dagger(x_1')\psi(x_1)}}_{\text{normal ordering}}|0\rangle$
$$= \delta(x_1-x_1')$$

For $N=2$ $\langle 0|\psi(x_2)\psi(x_1)\,\psi^\dagger(x_1')\psi^\dagger(x_2')|0\rangle = \langle 0|\,\psi(x_2)\left(\delta(x_1-x_1') \pm \psi^\dagger(x_1')\psi(x_1)\right)\psi^\dagger(x_2')|0\rangle$
$$= \delta(x_1-x_1')\,\delta(x_2-x_2')$$
$$\pm\langle 0|\psi(x_2)\,\psi^\dagger(x_1')\left(\delta(x_1-x_2') + \cancel{\psi^\dagger(x_2')\psi(x_1)}\right)|0\rangle$$

For Bosons. $= \delta(x_1-x_1')\,\delta(x_2-x_2') + \delta(x_1-x_2')\,\delta(x_2-x_1')$

For Fermions $= \delta(x_1-x_1')\,\delta(x_2-x_2') - \delta(x_1-x_2')\,\delta(x_2-x_1')$

For General N: $\langle 0|\psi(x_N)\cdots\psi(x_1)\,\psi^\dagger(x_1')\cdots\psi^\dagger(x_N')|0\rangle =$

Bosons $= \sum_{\text{Permutations}}\prod_j \delta(x_j - x_k')$

Fermions $= \sum_{\substack{\text{Permutations}\\ \sigma}} \mathrm{sgn}(\sigma)\prod_j \delta(x_j - x_k')$

Now: $\frac{1}{\sqrt{N!}}\langle 0|\psi(x_1)\cdots\psi(x_N)|N\rangle = \frac{1}{N!}\sum_p \begin{pmatrix}1\\(-1)^p\end{pmatrix}\int\prod_j \delta(x_j-x_k')\,\Psi(x_1'\cdots x_N')\,dx\cdots$

$= \frac{1}{N!}\sum_p \begin{pmatrix}1\\(-1)^p\end{pmatrix}\Psi(\cdots \underset{j\text{th pos.}}{x_k}\cdots)$ This is the configuration space wave function for fermions or bosons.

Thus: $\Psi(x_1\cdots x_N) = \frac{1}{\sqrt{N!}}\langle 0|\psi(x_N)\cdots\psi(x_1)|N\rangle$ for bosons or fermio

This is how you get a configuration space wavefunction from an occupation number representation:

$$\Psi(x_1\cdots x_N) = 1/\sqrt{N!}\,\langle 0|\psi(x_N)\cdots\psi(x_1)|N\rangle$$

multiply by $|0\rangle$ and $\frac{1}{\sqrt{N!}}\psi_1^\dagger\cdots\psi^\dagger(x_N)$

$$1/\sqrt{N!}\int\psi^\dagger(x_1)\cdots\psi^\dagger(x_N)\boxed{|0\rangle\langle 0|}\overset{|0\rangle}{\psi(x_N)\cdots\psi(x_1)}|N\rangle = \frac{1}{\sqrt{N!}}\int \Psi(x_1\cdots x_N)\,\psi^\dagger(x_1)\,\psi^\dagger(x_N)\,dx\,|0\rangle$$

This is the reverse prescription again.

P_0, a projection operator on the empty state.

$P_0\,\psi_{x_N}\cdots\psi_{x_1}|N\rangle = \psi_{x_N}\cdots\psi_{x_1}|N\rangle$, so it is an eigenstate of P_0.

Using this: $P_0|\cdots\rangle = 1$ and we have.

$$\frac{1}{N!}\int\psi^+(x_1)\cdots\psi^+(x_N)\,\psi(x_N)\cdots\psi(x_1)\prod dx_j\,|N\rangle$$

$$= 1/\sqrt{N!}\int\Psi(x_1\cdots x_N)\,\psi^+(x_1)\cdots\psi^+(x_N)\prod dx_j\,|0\rangle$$

$$= \cancel{\frac{1}{N!}}\;N(N-1)\cdots(N-N+1)\,|N\rangle$$

$$= |N\rangle = 1/\sqrt{N!}\int\Psi(x_1\cdots x_N)\,\psi^+(x_1)\cdots\psi^+(x_N)\prod dx_j\,|0\rangle$$

And the reverse route is complete.

Stationary State Schrödinge Eq.

$$H\Psi(x_1\cdots x_n) = E\Psi(x_1\cdots x_N) \qquad H = H(\{p_j\},\{x_j\}) \qquad \vec{p} = -i\hbar\nabla$$

Change to Occupation Numbers:

$$\frac{1}{\sqrt{N!}}\psi^+(x_1)\cdots\psi^+(x_N)\cdot|0\rangle\left(H\frac{1}{\sqrt{N!}}\langle 0|\psi(x_N)\cdots\psi(x_1)|N\rangle = E\frac{1}{\sqrt{N!}}\langle 0|\psi(x_N)\cdots\psi(x_1)|N\rangle\right)$$

$$\frac{1}{N!}\int\psi^+(x_1)\cdots\psi^+(x_N)\,H\underbrace{|0\rangle\langle 0|}_{P_0}\psi(x_N)\cdots\psi(x_1)\prod dx_j\,|N\rangle$$

$$= E\frac{1}{\sqrt{N!}}\int\Psi(x_1\cdots x_N)\,\psi^+(x_1)\cdots\psi(x_N)|0\rangle\prod dx_j$$

We need a $\mathcal{H}$ such that $\mathcal{H}|N\rangle = E|N\rangle$, occupation number Hamiltonian

$$\underline{\mathcal{H} = 1/N!\int\psi^+(x_1)\cdots\psi^+(x_N)\,H\,\psi(x_N)\cdots\psi(x_1)\prod dx_j}$$

If $H = \sum_j h(x_j)$ (no particle interactions)

$+\sum_{j<k}V(x_j,x_k)$, symmetric interaction potential.

+ Many-particle interactions.

For $V=0$: $\mathcal{H}|N\rangle = E|N\rangle$

$$= \sum_j\frac{1}{N!}\int\psi^+(x_1)\cdots\psi^+(x_n)\,h(x_j)\,\psi(x_N)\cdots\psi(x_1)\prod dx_j\,|N\rangle = E|N\rangle$$

$$= \sum_j\frac{1}{N!}\int\psi^+(x_j)\,N(N-1)\cdots(N-(N-1)+1)\,h(x_j)\,\psi(x_j)\,dx_j\,|N\rangle = E|N\rangle$$

$$= \frac{N}{N!}\int\psi^+(x)\,h(x)\,\psi(x)\,dx\;\underbrace{(N-1)\cdots(N-1-(N-1)+1}_{(N-1)!}|N\rangle = E|N\rangle$$

$$= \boxed{\int\psi^+(x)\,h(x)\,\psi(x)\,dx\,|N\rangle = E|N\rangle}$$

This Hamiltonian conserves particle number.

$V\neq 0$: $$\sum_{j<k}\frac{1}{N!}\int\psi^+(x_1)\cdots\psi^+(x_N)\,V(x_j,x_k)\,\psi(x_N)\cdots\psi(x_1)\prod dx_j\,|N\rangle$$

$$= \sum_{j<k}\frac{1}{N!}\int\psi^+(x_j)\,\psi^+(x_k)\,N(N-1)\cdots(N-(N-2)+1)\,V_{jk}\,\psi(x_k)\,\psi(x_j)\prod dx_j\,|N\rangle$$

$$= \frac{N(N-1)}{2N!}\int\psi^+(x)\,\psi^+(x')\,V(x,x')\,\psi(x')\,\psi(x)\,dx\,dx'\,(N-2)\cdots(N-2-N+3)|N\rangle$$

$$= \boxed{1/2\int\psi^+(x)\,\psi^+(x')\,V(x,x')\,\psi(x')\,\psi(x)\,dx\,dx'}$$

for any N

Add these two terms together to get $\mathcal{H}$

The Hamiltonian is independent of the number of particles present.

Example: Identical particles, charge e, interacting with field (A, φ) (external to the particles)

$$h(x) = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\hat{A}\right)^2 + e\varphi(r,t).$$

$$\varphi = 0 \quad \vec{A} = 0 \quad H_0 = \int \psi^\dagger(r)\frac{p^2}{2m}\psi(r)\,d\vec{r} = -\frac{\hbar^2}{2m}\int \psi^\dagger(r)\nabla^2\psi(r)\,d\vec{r}$$

Integrate by parts $= \frac{\hbar^2}{2m}\int (\nabla\psi^\dagger)\cdot\nabla\psi\,d\vec{r}$ provided $\int \psi^\dagger \frac{\partial}{\partial r}\psi\,dS = 0$

The field $\psi(x) = \sum a_j\varphi_j(x)$: complex (not a Hermitian operator) field operator, capable of bearing charge.

Hermitian operators can be used to represent neutral parti

$h_0(x) = p^2/2m$

$H_0 = \int \psi^\dagger(x)\frac{p^2}{2m}\psi(r)\,dr = -\frac{\hbar^2}{2m}\int \psi^\dagger\nabla^2\psi\,dr$

$= \frac{\hbar^2}{2m}\int \nabla\psi^\dagger\cdot\nabla\psi\,dr$ for suitable $\{\varphi_j\}$

Let $h(x) = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\varphi(r t)$

$H = H_0 + H_A + H_\varphi = \int \psi^\dagger \frac{1}{2m}\left\{p^2 + \frac{e}{c}(p\cdot A + A\cdot p) + \frac{e^2}{c^2}A^2\right) + e\varphi\}\psi\,d$

$H_\varphi = e\int \psi^\dagger(r)\psi(r)\varphi(r,t)\,dr = \int \rho(r)\varphi(r t)\,dr$ where $\rho(r) = e\psi^\dagger(r)\psi(r)$ is a charge densi

$H_A = -\frac{e}{2mc}\int \psi^\dagger(r)\{p\cdot A + A\cdot p\}\psi(\vec{r})\,dr +$
$+ \frac{e^2}{2mc^2}\int \psi^\dagger(r)\psi(r)A(r,t)\,dr$

If Lorentz gauge $\nabla\cdot A = 0$, these are equal

$$H_A = -\frac{e}{2mc}\int \psi^\dagger(r)\left\{\frac{\hbar}{i}\nabla\cdot A + A\cdot p\right\}\psi(\vec{r})\,dr$$

$$= -\frac{e\hbar}{2imc}\int (\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi)A(r,t)\,dr + \frac{e^2}{2mc^2}\int \psi^\dagger\psi A^2\,dr$$

Define $j^{(0)}(r) = \frac{e\hbar}{2im}\{\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi\}$

Then $H_A = -\frac{1}{c}\int j^{(0)}(r)\cdot A(r,t)\,dr + \frac{e}{2mc^2}\int \rho(r)A^2(r,t)\,dr$

Better def. $j(r) = \frac{e}{2m}\left\{\psi^\dagger\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\psi + \left[\left(-\frac{\hbar}{i}\nabla - \frac{e}{c}A\right)\psi^\dagger\right]\psi\right\}$

then $H_A = -\frac{1}{c}\int j(r)\cdot A(r,t) - \frac{e}{2mc^2}\int \rho(r)A^2(r,t)\,dr.$

$$H_{int} = \frac{1}{2}\int \psi^\dagger(r)\psi^\dagger(r')V(r,r')\psi(r')\psi(r)\,dr\,dr'$$

e.g. Coulomb interactions $V(r,r') = e^2/|r-r'|$

This does not include electrostatic self energy since $H_{int}|0\rangle = 0$

This interaction assumes an Action at a distance.
This is not characteristic of Field theories.

Note: $H_{int} \neq \int \rho(r) \frac{1}{|r-r'|} \rho(r')\, dr' dr$

$N = \sum a_j^\dagger a_j = \sum N_j$ is conserved: $[N, H] = 0$

Each particle has rest $E = mc^2$
In the Hamiltonian should have term i.e. $H_R = mc^2 N$
but for Non-Relativistic this would have no effect.

$h(x) \rightarrow mc^2 + p^2/2m$

If you have set of operators $O(q(t), t)$, $q(t)$ a set of operators.
Explicit time-dependence: Changes whether or not q is changing.
Implicit " " : changes because $q(t)$ is changing

$$\boxed{\frac{dO}{dt} = \frac{1}{i\hbar}[O, H] + \frac{\partial O}{\partial t}} \qquad H_0 = \int \Psi^\dagger(r)\, h(r)\, \Psi(r)\, dr.$$

Heisenberg picture: $i\hbar \frac{\partial}{\partial t} \Psi(r,t) = [\Psi(r,t), H_0]$, a field equation of motion
There is no explicit time dependence.

So $i\hbar \frac{\partial}{\partial t} \Psi(r,t) = [\Psi(r,t), \int \Psi^\dagger(r',t)\, h(r')\, \Psi(r',t)\, dr']$
$= h(r)\Psi(r,t)$

This would be easiest to compute if you use $\{\varphi_j\}$ that are the eigenfunctions of $h(x)$

If $V(r,r') \neq 0$ $[\Psi, H_{int}] = \frac{1}{2}[\Psi(r), \int \Psi^\dagger(r')\Psi^\dagger(r'')\, V(r',r'')\, \Psi(r'')\Psi(r')]\, dr' dr''$
$= \int \Psi^\dagger(r')\Psi(r')\, V(r'r)\, \Psi(r)\, dr'$

Field eq: $i\hbar \frac{\partial \Psi}{\partial t} = h(r)\Psi + \int \Psi^\dagger(r')\Psi(r')\, V(r'r)\, \Psi(r)\, dr'$

This is a non-linear eq, nothing like Schroedinger (other ex. the Hartree eq), Operator transcription of the Hartree-Fock approximation.

Time translations: $| \rangle \rightarrow | \rangle'$
In $| \rangle'$ we have happening at time t whatever happened at time $t-\tau$ in $| \rangle$

$\Psi'(x_1, \dots x_n, t) = \Psi(x_1, \dots x_n, t-\tau)$

$\Psi(x_1, \dots x_n, t) = \langle x_1 \dots x_n | \Psi(t) \rangle$

$\langle x_1, \dots x_n | \Psi(t) \rangle' = \langle x_1 \dots x_n | \Psi(t-\tau) \rangle$

V unitary $V^{\dagger} = V^{-1}$

$$|\Psi(t)\rangle' = |\Psi(t-\tau)\rangle = V(\tau,t)\,|\Psi(t)\rangle$$

Infinitesimal τ, $V(r,t) = 1 + \frac{i}{\hbar}H\tau$

V is the operator which takes states back in time.

If $|t\rangle = U(t,t_0)|t_0\rangle$, $U = V^{\dagger} = V^{-1}$

$$-i\hbar\frac{\partial}{\partial\tau}V = H(t)\,V(r,t)$$

Transformation of the operator q

$$q' = V(\tau,t)\,q\,V^{-1}(\tau,t) = q - \frac{1}{i\hbar}[H,q]\tau$$

$$= q + \frac{1}{i\hbar}[q,H]\tau = q + \dot{q}\tau \mathrel{\hat{=}} q(t+\tau)$$

$$q' = VqV^{-1} = q(t+\tau)$$

$$\psi'(r,t) = V\psi V = \psi(r,t+\tau)$$

If $\mathcal{H}$ does not depend explicitly on time, eg. $H = \mathcal{H}_0 + \mathcal{H}(m$

Then $\dot{\mathcal{H}} = \frac{1}{i\hbar}[\mathcal{H},H] = 0$, $\mathcal{H}$ is a constant of the motion.

$$V(\tau,t) = e^{\frac{i}{\hbar}\int_0^{\tau}\mathcal{H}(t')dt'} \boxed{= e^{\frac{i}{\hbar}H\tau}}, \quad V(0,t) = 1$$

If $\mathcal{H}$ does depend explicitly on t, integration much more complicated.

Spatial Translations by $\vec{d}$

$$\Psi'(r_1, \dots r_n) = \Psi(r_1-d, \dots r_n-d)$$

Single Particle: $\langle r'|\psi\rangle' = \langle r'-d|\Psi\rangle$, $|\psi\rangle' = D|\Psi\rangle$ $D^{\dagger} = D^{-1}$

$DrD^{-1} = \vec{r}+d$ $\quad D = 1 - \frac{i}{\hbar}\vec{p}\cdot\vec{d}$, d infinitesimal

p the momentum operator.

Finite Transformation: $D(d) = e^{-\frac{i}{\hbar}\vec{p}\cdot d} = e^{-\frac{i}{\hbar}\vec{d}\cdot\vec{p}}$

p does not operate on d

$$|\psi\rangle' = e^{-\frac{i}{\hbar}d\cdot p}|\psi\rangle$$

$$\langle r'|\psi\rangle' = \langle r'|D|\psi\rangle = \langle r'|e^{-\frac{i}{\hbar}p\cdot d}|\psi\rangle$$

$$= e^{-\frac{i}{\hbar}\frac{\hbar}{i}d\cdot\nabla'}\langle r'|\psi\rangle = e^{-d\cdot\nabla'}\langle r'|\psi\rangle$$

$$= \langle r'-d|\psi\rangle$$

Spatial Translations: $\Psi(r_1 \cdots r_n) \rightarrow \Psi'(r_1 \cdots r_n) = \Psi(\vec{r}-\vec{d})$

$$\vec{r}' = \vec{r} - \vec{d} = D(\vec{d})\, \vec{r}\, D(\vec{d})^{-1}$$

Infinitesimal operator : $D = 1 - i\frac{d \cdot p}{\hbar}$

Finite transformation $D(d) = \quad e^{-i\frac{d\cdot p}{\hbar}}$

$$\langle r' | \psi \rangle^{*} = \langle r' | D\psi \rangle = e^{-d\cdot\nabla}\langle r' | \psi \rangle = \langle r' - \vec{d} | \psi \rangle$$

Field: $\psi(\vec{r})$, an operator, $\vec{r}$ is just a labelling parameter.

$$\Psi(r_1 \cdots r_n) = \frac{1}{\sqrt{N!}} \langle 0 | \psi(r_N) \cdots \psi(r_1) | N \rangle$$

$$\Psi'(r_1 \cdots r_n) = 1/\sqrt{N!}\, \langle 0 | \psi(r_N) \cdots \psi(r_1)\, D | N \rangle$$

$$= 1/\sqrt{N!}\, \langle 0 | D\, D^{-1} \psi(r_N) D \cdots | N \rangle$$

Assume $D|0\rangle = |0\rangle$,

$\langle 0 | D \rangle = \langle 0 |$ $\quad = 1/\sqrt{N!}\, \langle 0 | \psi(r_n - d) \cdots \psi(r_1 - d) | N \rangle$

Require $D^{-1}(d)\, \psi(\vec{r})\, D(d)$ $\quad = \Psi(r_1 - d_2 \cdot r_N - d)$

$= \psi(\vec{r} - \vec{d})$

what is the required D?

Infinitesimal ; $D(d) = 1 + i/\hbar\, d \cdot \vec{G}$ $\quad G$ the generator

$$(1 + i/\hbar\, d \cdot G)\, \psi(r)\, (1 - i/\hbar\, d \cdot G) = \psi + i/\hbar\, [d \cdot G, \psi] \doteq \psi(\vec{r} - \vec{d})$$

Solution: $\vec{G} = \vec{P} = \int \psi^{\dagger}(r)\, p\, \psi(r)\, d\vec{r}$

$$= \frac{\hbar}{i} \int \psi^{\dagger}(r)\, \nabla \psi(r)\, d\vec{r}$$

$$D^{-1} \psi D = \psi + i/\hbar \left[\int \psi^{\dagger}(r') \nabla' \psi(r')\, \partial \vec{r}, \psi(r) \right] = \psi - d \cdot \nabla \psi . = \psi(\vec{r} - d)$$

$D(d) = e^{-i/\hbar\, \vec{d} \cdot \vec{P}}$ cause this to define the total momentum of the field.

$$|N\rangle' = D(d)|N\rangle = \frac{1}{\sqrt{N!}} \int \Psi(r_1 \cdots r_n)\, D\, \psi^{\dagger}(r_1) \cdots \psi^{\dagger}(r_n) |0\rangle$$

since $D(d)\, \psi(r)^{\dagger} D(d) = \psi^{\dagger}(r + d)$ $\quad = 1/\sqrt{N!} \int \Psi(r_1 \cdots r_n)\, D \psi^{\dagger}(r_1) D^{-1} \cdots \quad D|0\rangle$

$= 1/\sqrt{N!} \int \Psi(r_1 \cdots r_n)\, \psi^{\dagger}(r_1 + d) \cdots \psi^{\dagger}(r_n + d) |0\rangle \prod dr_j$

Let $\vec{r}_j' = \vec{r}_j + \vec{d}$

$$= 1/\sqrt{N!} \int \Psi(\vec{r}_1' - d \cdots \vec{r}_N - d)\, \psi^{\dagger}(r_1') \cdots \psi^{\dagger}(r_N) |0\rangle \prod r_i$$

If H is invariant under space-translations, then

$D H^{-1} D = H$, $[D, H] = 0$, $[P, H] = 0 \rightarrow P$ is a constant of the motion

(Conservation of momentum)

Plane-Wave Expansions

spin 0: choose complete, orthonormal $\varphi_j(r) = e^{i\vec{k}\cdot\vec{r}}/L^{3/2}$

These are periodic in cube of size L. k can only take on a discrete set of values $k = \frac{2\pi}{L}\vec{n}$, $\vec{n} = (n_1, n_2, n_3)$, ...

$j \to k$ for labelling:

These are eigenfunctions of the momentum operator

$$p\varphi_k(r) = \frac{\hbar}{i}\nabla\varphi_k(r) = \hbar\vec{k}\,\varphi_k$$

For spin $\neq 0$, $\varphi_j = \frac{U e^{-ikr}}{L^{3/2}}$ U a spinor $j \to km$ $\varphi_{km}(r) = \frac{U_m e^{-ikr}}{L^{3/2}}$

and $U^{+}(m)\,U(m) = \delta_{mm'}$

$$\Psi(r) = \sum_{km} a_{km}\varphi_{km}(r)$$

$$P = \int \Psi^{+}(r)\frac{\hbar}{i}\nabla\Psi(r)\,d\vec{r} = \sum a^{+}_{km}a_{k'm'}\int\varphi^{+}_{km}\,\hbar k'\varphi_{k'm'}(r)\,d\vec{r}$$

$$= \sum \hbar k\, a^{+}_{km}a_{km} = \sum_{k,m}\hbar k\, N_{km}$$

Spatial rotation

Rotation by infinitesimal ϵ

$x' = x + \epsilon y$

$y' = y - \epsilon x$, $U = 1 - i\epsilon L_z$, $L_z = (x p_y - y p_x)/\hbar$

$= \frac{1}{i}\partial/\partial\varphi$

Finite ϵ

$x' = x\cos\epsilon + y\sin\epsilon = U x U^{-1}$ $U = e^{-i\epsilon L_z}$

$y' = -x\sin\epsilon + y\cos\epsilon = U y U^{-1}$

Rotation of wavefunction: $|\Psi\rangle' = U|\Psi\rangle$

$\langle r'\varphi'|\Psi\rangle' = \langle r'\varphi'|U|\Psi\rangle = e^{-i\frac{\epsilon}{i}\frac{\partial}{\partial\varphi}}\langle r'\varphi'|\Psi\rangle$

$= \langle r'\varphi' - \epsilon|\Psi\rangle$

In 3-dim: Rotation by angle χ about axis $\hat{n}$

$U(R) = e^{-i\chi(\hat{n}\cdot L)}$, $U^{+} = U^{-1}$, $U(R^{-1}) = U^{-1}(R)$

$(U(R)\,\vec{r}\,U^{-1}(R)) = \vec{r}\,' = R^{-1}\vec{r}$

$r'_j = r_{j'}\,a_{j'j}(R^{-1})$, where $a_{j'j}$ is 3×3, real orthogonal

$a(R^{-1}) = a^{-1}(R)$

$\Psi'(r_1 \ldots r_n)$ a rotated wave functions $= \frac{1}{\sqrt{N!}}\langle 0|\Psi(r_n)\cdots\Psi(r_1)U|N\rangle$

$U|0\rangle = \langle 0|$ $= \frac{1}{\sqrt{N!}}\langle 0|U U^{-1}\Psi(r_n)U \ldots|N\rangle$

~~Require~~: $U^{-1}(R)\Psi(r)U(R) = \Psi(R^{-1}r) = \Psi(\vec{r}\,a(R^{-1}))$

If so, then $\Psi'(r_1 \cdots r_N) = \frac{1}{\sqrt{N!}} \langle 0 | \psi(R^{-1} r_n) \cdots \psi(R^{-1} r_1) | N \rangle$

$= \Psi(R^{-1} r_1, \cdots R^{-1} r_n)$ is the appropriately rotated wave function.

$$U(R) = e^{-i\chi(\hat{n} \cdot \vec{\mathcal{L}})}$$

Total ang. momentum of the field.

$$\vec{\mathcal{L}} = \int \psi^\dagger_{(r)} L \psi_{(r)} \, dr = \tfrac{1}{i} \int \psi^\dagger(r) \, (r \times \nabla) \, \psi(r) \, dr$$

$$|N\rangle' = U(R)|N\rangle = \frac{1}{\sqrt{N!}} \int \Psi(r_1, \cdots r_n) \, U \psi^\dagger(r_1) U^{-1} \cdots U|0\rangle \prod_j dr_j$$

$$U(R)\,\psi(r)\,U^{-1}(R) = \psi(R_r)$$

$$U(R)\,\psi^\dagger(r)\,U^{-1}(R) = \psi^\dagger(R_r) \qquad R\vec{r}_j = \vec{r}_j'$$

$$|N\rangle' = \frac{1}{\sqrt{N!}} \int \Psi(R^{-1} r_1', \ldots R^{-1} r_n') \, \psi^\dagger(r_1') \cdots \psi^\dagger(r_N') |0\rangle \prod_j dr_j'$$

If $U(R)\, H\, U^{-1}(R) = H$

$$[\mathcal{L}, H] = 0 \Rightarrow \mathcal{L} = \text{constant}$$

These transformations leave the spins invariant

How do you transform spins?

single particle $\vec{S} \to \vec{S}' \equiv U(R)\, \vec{S}\, U^{-1}(R) = R^{-1}\vec{S} = S\,a(R^{-1})$

$$|\Psi\rangle' = U(R)|\Psi\rangle$$

$$\langle m | \Psi \rangle' = \langle m | U(R) | \Psi \rangle = \sum_{m'} \langle m | U(R) | m' \rangle \langle m' | \Psi \rangle$$

$$= \sum_{m'} D^{(s)}_{mm'}(R) \, \langle m' | \Psi \rangle$$

a unitary matrix representation of the rotation group.

$$\Psi'_m = \sum D^{(s)}_{mm'}(R) \Psi_{m'}$$

$$D^{(s)}_{mm'}(R) = \langle m | e^{-i\chi(\hat{n}\cdot\vec{S})} | m' \rangle$$

Point Mechanics: $\{q_j\}$

Form $L(q_j, \dot{q}_j, t)$ such that $\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0$ ↗ Integration over a certain path

for variations such that $q_j(t) \rightarrow q_j(t) + \delta q_j(t)$ $\quad \delta q_j(t) = 0$ at $t = t_1, t_2$

This goes to $\int_{t_1}^{t_2} \sum_j \left\{ \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right\} \delta q_j(t)\, dt = 0$

Giving the Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$

$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$

Hamiltonian formulation: $H = \sum p_j \dot{q}_j - L$

$$\partial H = \sum (\partial p_j \dot{q}_j + \cancel{p_j \partial \dot{q}_j}) - \sum \left(\frac{\partial L}{\partial q_j} dq_j + \cancel{\frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j} + \frac{\partial L}{\partial t} dt \right)$$

so $\frac{\partial H}{\partial p_j} = \dot{q}_j$

$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

If $\frac{\partial L}{\partial t} = 0$, then $\frac{\partial H}{\partial t} =$ ~~constant~~ 0.

then $\frac{dH}{dt} = \sum \dot{p}_j \dot{q}_j - \dot{p}_j \dot{q}_j) = 0$ and $H =$ constant

This is eq. to invariance under time displacements.

Fields, Continuum Mechanics $\{q_j\} \rightarrow \varphi(\vec{r}, t)$, real valued

Discrete case: Lattice $\{\vec{r}_j\}$ then $\{q_j(t)\} = \{\varphi(\vec{r}_j, t)\}$

$L = \sum_j L_j(\varphi(r_j, t), \dot{\varphi}(r_j, t))$

This would be a useless Lagrangian, since the L_j's are independent, no propagation of effects.

So must include differences, (differentials): $\varphi(r_{j+1}, t) - \varphi(r_j, t)$

As lattice spacing → 0, we must include $\nabla \varphi(r, t)$

Initial conditions: $\varphi(r,t)$ and $\dot{\varphi}(r,t)$

$$L = \int \mathcal{L}(\varphi(r,t), \dot{\varphi}(r,t), \nabla\varphi, \vec{r}, t)\, d\vec{r}$$

Hamilton principle: $\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} \mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi)\, d\vec{r}\, dt = 0$

$$\boxed{= \delta \int_{\sigma_1}^{\sigma_2} \mathcal{L}(\varphi, \partial_N \varphi, x_N)\, d^4x} = 0$$

where $\partial_N \varphi = (\dot{\varphi}, \nabla\varphi)$,

$x_N = (t, \vec{r}) \qquad d^4x = d\vec{r}\, dt$

σ_1, σ_2 are space-like surfaces.

If we can construct Lorentz invariant $\mathcal{L}$, our equations of motion will be invariant.

Field variations $\delta\varphi(r,t) = 0$ $\begin{matrix} t=t_1 & \text{or } \sigma_1 \\ t=t_2 & \text{or } \sigma_2 \end{matrix}$

$\delta\varphi(r,t) \to 0$ as $|r| \to \infty$

So we can do integrations by parts

$$\int \left\{ \frac{\partial\mathcal{L}}{\partial\varphi(r)} \delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)} \underbrace{\delta\partial_\nu\varphi}_{\partial_\nu\delta\varphi} \right\} d^4x = 0$$

$$\int \left\{ \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\right) \right\} \delta\varphi \, d^4x = 0$$

so $$\boxed{\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}}$$

Define $\pi^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}$ so $\partial_\nu\pi^\nu = \frac{\partial\mathcal{L}}{\partial\varphi}$

$H = \int \pi^0\dot{\varphi}\, d^3r - L = \int \mathcal{H}\, d^3r$ a Hamiltonian density

$\mathcal{H} = \pi^0\dot{\varphi} - \mathcal{L}$ ~~$\mathcal{H}$~~ This is $\boxed{\text{not}}$ invariant, it singles out the time direction.

$= \pi^0 \frac{\partial\varphi}{\partial x_0} - \mathcal{L}$

Invariance: $\boxed{\varphi \to \varphi'(x) = \varphi(\Lambda^{-1}x)}$ This field cannot have spin; since it is real cannot represent charged particles.

for pure rotations $\varphi'(\vec{r}) = \varphi(R^{-1}r)$

$F(\varphi)$ will then be an invariant

$\partial_\nu(\varphi)$, a four-vector.

Choose $\mathcal{L}$ a quadratic form in φ, $\partial_\nu\varphi$

$\mathcal{L} = \frac{1}{2}(a\,\partial_\nu\varphi\,\partial^\nu\varphi + b\,\varphi^2)$, a,b real this is essentially the only invariant quadratic form.

Let $\varphi' = \frac{\varphi}{\sqrt{a}}$ $\mathcal{L} = \frac{1}{2}\left(\overset{=\pm1}{\frac{a}{|a|}}\partial_\nu\varphi'\partial^\nu\varphi' + \frac{b}{|a|}\varphi'^2\right)$

Let $b/a = c$, $\varphi' \to \varphi$

$$\mathcal{L} = \pm\frac{1}{2}(\partial_\nu\varphi\,\partial^\nu\varphi + c\varphi^2)$$

$\pi^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)} = \pm\partial^\nu\varphi$ = covariant momentum density

$= \frac{\partial\mathcal{L}}{\partial\varphi} = \pm c\varphi$

so eq. of motion: $(\partial_\nu\partial^\nu - c)\varphi = 0$ or $(\Box^2 - c)\varphi = 0$

$$\mathcal{H} = \underset{\equiv \pm\partial_0\varphi}{\Pi^0}\,\underset{\partial_0\varphi}{\dot\varphi} \mp \tfrac{1}{2}\{\partial_\mu\varphi\partial^\mu\varphi + c\varphi^2\}$$

$$= \pm\tfrac{1}{2}\{\dot\varphi^2 + (\nabla\varphi)^2 - c\varphi^2\}$$ You must use the positive sign to the energy from being arbitrarily small.

$c \le 0$ By convention: $c = -\mu^2$

so:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2)$$

$$\mathcal{H} = 1/2\{\dot\varphi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\}$$ is positive-definite, non invariant.

and $(\partial_\mu\partial^\mu + \mu^2)\varphi = 0$ The Klein Gordon eq, the eq. of motion.

$[\Box^2 + \mu^2]\varphi = 0$ If you think of φ as a wave-function you run into trouble since you need $\dot\varphi$.

Problems disappear if φ is an operator field.

Plane wave solutions: $\varphi \sim e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ $k^\mu = (\omega, \vec{k})$ a fourvector
$x_\mu = (t, -\vec{r})$ a fourvector

so $\varphi \sim e^{-ik^\mu x_\mu} = e^{-ikx}$ = a scalar invariant

Require: $(-\omega^2 + k^2 + \mu^2)\varphi = 0$ so $\omega^2 = \mu^2 + k^2$ $p^\mu = (p^0, \vec{p}) = (E, \vec{p})$

for a plane wave; $p^\mu = \hbar k^\mu$ so $E^2 = \hbar^2\mu^2 + p^2$

$\hbar^2\mu^2 = m^2$ $\mu = \frac{\hbar}{m}$, if $c \neq 1$, $\mu = \hbar/mc$, the inverse Compton wavelength

or $E^2 - \vec{p}^2 = m^2$

or $p^2 \equiv p^\mu p_\mu = m^2$, the invariant length.

Adding a linear term to the Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2) - g\rho(\vec{r},t)\varphi$
where g is a coupling constant, $\rho(r,t)$ is a source density.

$$\mathcal{H} = \mathcal{H}_0 + g\rho(r,t)\varphi$$

This destroys invariance of Hamiltonian and thus momentum + Energy conservation

$(\Box^2 + \mu^2)\varphi = -g\rho(r,t)$ Similar to electrodynamics except $\mu = 0$

Free Scalar Field: $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2)$

$\mathcal{H}_0 = \frac{1}{2}(\dot\varphi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2)$

$\mathcal{L} = \mathcal{L}_0 - g\rho(r,t)\varphi$ $\rho'(x) = \rho(\Lambda^{-1}x)$

$\mathcal{H} = \mathcal{H}_0 + g\rho\varphi$ $(\Box^2 + \mu^2)\varphi = -g\rho(x)$

Electrodynamics

$\nu = 0$

Analogy

$g\rho(x) \rightarrow$ source density

$\varphi \rightarrow$ Electrostatic potential

Difference: φ as electrostatic potential is part of a four vector,

φ as field is Lorentz invariant, itself.

Static source: $\rho = \rho(\vec{r})$

Particular solution: $\varphi = \varphi(r)$, so $\frac{\partial}{\partial t}\varphi = 0$

$$(\nabla^2 + \nu^2)\varphi(r) = g\rho(r)$$

Define a Green's Function: $(\nabla^2 + \nu^2)G(r, r') = \delta(r - r')$

Because of translational invariance let $G(r, r') = G(r - r')$

~~r'~~r

$r - r' \rightarrow r$

$$(\nabla^2 + \nu^2)G(r \cancel{-r'}) = -\delta(r) = \frac{-1}{2\pi^3}\int e^{-ikr}dk$$

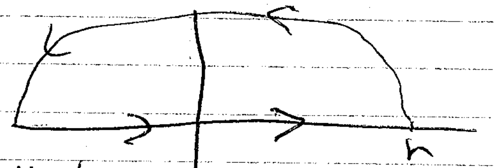
Gives $G(r \cancel{-r'}) = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2 + \nu^2}\,d\vec{k}$ $\rightarrow k^2 dk\, d\Omega_k$

Since $\frac{1}{4\pi}\int e^{ik\cdot r}dr = \frac{\sin kr}{kr}$

so $G(r \cancel{-r'}) = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{\sin kr}{kr(k^2 + \nu^2)}k\,dk$

since $\sin kr = \frac{1}{2i}(e^{ikr} - e^{-ikr})$, so $G(r) = \frac{4\pi}{(2\pi)^3 2ir}\int_{-\infty}^{\infty}\frac{e^{ikr}}{k^2 + \nu^2}k\,dk$

Integrate in k-plane



The value of the integral is the same as that for the contour shown as $r \rightarrow \infty$

two poles. simple poles at $i\nu$ and $-i\nu$.

So the integral $= \frac{1}{(2\pi)^2 2ir}\int_{-\infty}^{\infty} e^{ikr}\left(\frac{1}{k - i\nu} + \frac{1}{k + i\nu}\right)dk = \frac{1}{2\pi i}$ Residue

$$= \frac{1}{(2\pi)^2 ir}(2\pi i)\frac{e^{-\nu r}}{2} = \frac{e^{-\nu r}}{4\pi r}$$

$$\varphi(r) = -g\int G(r - r')\rho(r')dr' = -g\int \frac{e^{-\nu|r - r'|}}{4\pi|r - r'|}\rho(r')dr'$$

This is the Yukawa Potential

1935

Calculate energy of the interaction of ρ with itself.

$$H = \int \mathcal{H}_0 + \mathcal{H}_1 \, d\vec{r} \quad \int \tfrac{1}{2}\left[(\nabla\varphi)^2 + \nu^2\varphi^2 \right\} + g\rho\varphi \right] dr$$

integrate by parts

$$= \int \left\{ \tfrac{1}{2}\left[-\varphi\nabla^2\varphi + \nu^2\varphi \right] + g\rho\varphi \right\} dr$$

$$= \int \left\{ -\tfrac{1}{2}\varphi g\rho + g\rho\varphi \right\} dr = \frac{g}{2}\int \rho\varphi \, dr$$

$$= \frac{g^2}{2}\int \rho(r)\,\frac{e^{-\nu|r-r'|}}{4\pi|r-r'|}\,\rho(r')\,dr\,dr'$$

May be many sources; $\rho(r) = \delta(r-r_1) + \delta(r-r_2) + \cdots$

Here: if point sources, the self-energy is infinite, although the interaction energies are finite.

Use this for nucleon force theory to give the force finite range.

Here like charges attract (because of the sign of the metric?)

$\nu = \frac{\hbar}{mc}$, The π meson: π_+, 1947 $m_{\pi+} = 139.57$ mev

This is the charged version of π_0 we wrote equations for above. $1/\nu = 1.414 \times 10^{-13}$ meters

1936 Street and Stevenson (Harvard) and later, Anderson discovered a meson, thought it was Yukawa's

"Ten - Eleven years of confusion ensued",

This particle had only coulomb interaction with the nucleus

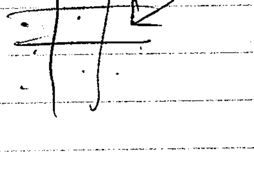
It was the μ-meson, or muon.

~~[illegible]~~ Quantization

If the q_j are coordinates, p_j momenta:

Must have $[q_j, q_k] = 0$ $[p_j, p_k] = 0$ $[q_j, p_k] = i\hbar\delta_{jk}$

This is for point quantum mechanics.



Field Theory: Separate space into Lattice

$q_j \Rightarrow \varphi(r_j, t)$ or $(\varphi(r,t))$ average over cell j.

$$L = \sum \mathcal{L}_j(\dot{\varphi}_j, \varphi_j, \varphi_{j+1}, \varphi_{j-1} \cdots)\Delta V$$

$$p_j = \frac{\partial L}{\partial \dot{\varphi}_j} = \Delta V \frac{\partial \mathcal{L}_j}{\partial \dot{\varphi}_j} \equiv \Delta V \pi_j, \quad \text{so} \quad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_j} = \pi_j^0$$

since we defined $\pi_j^\nu = \frac{\partial \mathcal{L}}{\partial \partial^\nu \varphi_j}$

$[\varphi_j, \varphi_k] = 0$

$[p_j, p_k] = 0 \Rightarrow [\pi_j, \pi_k] = 0$

$[q_j, p_k] = i\hbar\delta_{jk} \Rightarrow [\varphi_j, \pi_k] = \frac{i\hbar\delta_{jk}}{\Delta V}$

Continuous limit: $\Delta V \rightarrow 0$

$[\varphi(r,t), \varphi(r',t)] = 0$

$[\pi(r,t), \varphi(r',t)] = 0$ and $[\varphi(r,t), \pi(r',t)] = i\hbar\, \delta(r-r')$

or $i\hbar\, \delta^3(x-x')$

These are the canonical commutation relations for fields.

How do you make these relations covariant?

$[\varphi(x), \varphi(y)] = 0$ for x, y spacelike, $(x-y)^2 < 0$

Displacement operators:

Dynamics: Hamiltonian is generator of infinitesimal time displacements.

$\dot{\varphi} = \frac{i}{\hbar}[\varphi(r,t), H]$ for all field operators.

$H = \frac{1}{2}\int \{\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\}\, dr$, which is not explicitly time dependent. So $\dot{H} = 0$, $H = \text{constant}$

For displacement by a time τ: $V(\tau) = e^{\frac{iH\tau}{\hbar}}$

Spatial displacements by $\vec{d}$: $D^{-1}(d)\, \varphi(r)\, D(d) = \varphi(r-d)$

For infinitesimal d: $D = 1 - \frac{i d\cdot P}{\hbar}$

Find P that does this:

$\varphi(r) + \frac{i}{\hbar}[d\cdot P, \varphi(r)] = \varphi(r) - d\cdot\nabla\varphi$

$[P, \varphi(r)] = -\frac{\hbar}{i}\nabla\varphi = i\hbar\nabla\varphi$

For π, $[P, \pi(r)] = i\hbar\nabla\pi$ since, for a field F, $[P, F] = i\hbar\nabla F$

Guess $\boxed{\vec{P} = -\int \pi(r')\, \nabla\varphi(r')\, dr'}$ + Constant.

$-[\int \pi(r')\, \nabla'\varphi(r')\, dr', \varphi(r)] = i\hbar\int \delta(r-r')\, \nabla'\varphi(r')\, dr'$

$= i\hbar\nabla\varphi(r)$

π is hermitean, φ is hermitean, but $\vec{P}$ is not hermitean.

So we symmetrize ~~the~~ P

Take $\vec{P} = -\frac{1}{2}\int (\pi\nabla\varphi + (\nabla\varphi)\pi)\, dr \quad P^{\dagger} = P$

This assumes $[P,H]=0$
eg $\vec{p}=0$
$\uparrow$

Finite displacement $D(d) = e^{-i\frac{d \cdot P}{\hbar}}$

Compound space + time displacements: $V(\tau) D(d) = e^{i\frac{H\tau - \vec{d}\cdot P}{\hbar}}$

Define $P^\nu \equiv (H, \vec{P})$, $d^\nu = (\tau, \vec{d})$, $d_\nu = (\tau, -\vec{d})$

Total displacement operator $D(d_\nu) = e^{\frac{iP^\nu d_\nu}{\hbar}}$

$D^{-1}(d)\, \varphi(x)\, D(d) = \varphi(x-d)$, $D(d)\, \varphi(x)\, D^{-1}(d) = \varphi(x+d)$

All this goes through for $F(\varphi, \Pi)$ also.

$$\frac{i}{\hbar}[d_\nu P^\nu, F] = \partial_\nu d^\nu F \quad \text{for all } d_\nu$$

$$[P^\nu, F] = -i\hbar\, \partial^\nu F = -i\hbar \frac{\partial}{\partial x_\nu} F$$

Want to expand φ in basis functions. What is a suitable set of basis functions for relativistic theory? A basis set of solutions to the Klein-Gordon eq.

If f, g are solutions: $(\Box^2+\mu^2) f = 0$ what is $(f,g) \overset{?}{=} \int f^* g \, dr$

But this integral is not a constant, its time derivative can be chosen arbitrarily.

$(\Box^2+\mu^2) f^* = 0 \Rightarrow f^* \Box^2 f - (\Box^2 f^*) f = 0$

$$\boxed{\partial_\nu \{ f^* \partial^\nu f - (\partial^\nu f^*) f \} = 0}$$

So here is something analogous to current conservation. This is not positive definite, can't be a probability density.

<u>Klein Gordon eq.</u> ① $(\Box^2 + \mu^2) f = 0$ f a complex function

② $(\Box^2 + \mu^2) f^* = 0$

multiply ① by f^*, ② by f, subtract

$$f^* \Box^2 f - (\Box^2 f^*) f = 0$$
$$= \partial_\nu (f^* \partial^\nu f - (\partial^\nu f^*) f) = 0$$

or using four vector: $\left(f^* \frac{\partial f}{\partial t} - \frac{\partial f^*}{\partial t} f,\ f^* \nabla f - (\nabla f)^* f\right)$

3 dim matter current: $\frac{\hbar}{2im}(f^* \nabla f - (\nabla f^*) f)$

Analogy to E, M. $\Rightarrow$ $j^\nu = (\rho, \vec{j})$ = density-current 4-vector

Matter density $\overset{?}{=} \frac{\hbar}{2im}\left(f^* \frac{\partial f}{\partial t} - \frac{\partial f^*}{\partial t} f\right)$, but this can be positive or negative; is not positive-definite.

1934: Pauli, Weisskopf rehabilitate the K-G. equation, they reinterpreted it as a field equation, using operators.

Given f, g arbitrary solutions to the K-G. eq:

$\partial_\nu\{f^* \partial^\nu g - (\partial^\nu f^*) g\} = 0$ get this by multiplication + subtraction as before.

$$\frac{\partial}{\partial t}\int \left(f^* \frac{\partial}{\partial t} g - (\frac{\partial}{\partial t} f^*) g\right) d\vec{r} = -\int \nabla\cdot(f^* \nabla g - (\nabla f)^* g)\, d\vec{r}$$

$$= -\int_{\substack{\text{surface}\\ \to\infty}} \underbrace{(f^* \nabla g - (\nabla f)^* g)\cdot d\vec{s}}_{\to 0 \text{ as surface} \to \infty}$$

So $\frac{\partial}{\partial t}\int \left(f^* \frac{\partial}{\partial t} g - (\frac{\partial}{\partial t} f^*) g\right) d\vec{r} = 0$

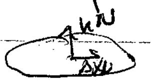
Define $(f,g) = \frac{i}{\hbar}\int \{f^* \frac{dg}{dt} - (\frac{df}{dt})^* g\}\, dr = \frac{i}{\hbar}\int (f^* \overleftrightarrow{\partial_0} g)\, d\vec{r}$

But (f,f) is not positive definite.

Space-like surface: a surface on which for all pairs of points x, y, $(x-y^2) < 0$

Unit normal n^ν such that $n^\nu n_\nu = 1$ $n^\nu \Delta x_\nu = 0$

$n^0 > 0$



Define an element of area $d\sigma^\nu$ parallel to n^ν

$$d\sigma^\nu = \frac{d^4x}{\overleftrightarrow{dx^\nu}} = (d\vec{r}, dt\,dy\,dz, dt\,dx\,dz, dt\,dx\,dy)$$

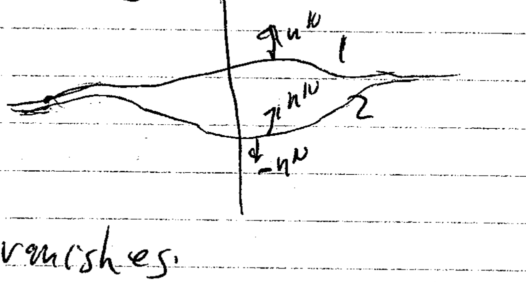
Then $(f,g) = \frac{i}{\hbar}\int f^* \overleftrightarrow{\partial^\nu} g\, d\sigma_\nu$ is a more general definition

Given two space-like surfaces

$$(f,g)_1 - (f,g)_2 = \int_{\text{surf.1}} - \int_{\text{surf}_2} = \int_{\text{Total surf}}$$

$$= \frac{i}{\hbar}\int_{\text{Vol.}} \partial_\nu (f^* \overleftrightarrow{\partial^\nu} g)\, d^4x = 0$$

since, by K-G eq, this vanishes.



Plane-wave solutions: $f \propto e^{i(\vec{k}\cdot\vec{r} - \omega t)} = e^{-ik^\nu x_\nu}$, $x_\nu = (t, -\vec{r})$

For $(\Box^2 + \mu^2) f = 0$ must have $\omega^2 = \mu^2 + k^2$

$\omega = \pm\,\omega_k = \pm\sqrt{\mu^2 + k^2}$ This gives negative energies.

There are two complete sets of solutions.

$f^{(\pm)}_{\vec{k}}(x) = (\text{const})\, e^{\mp ikx}$ where $k^\nu = (\omega_k, \vec{k})$

Any function can be expanded either in $f^+_k(x)$ or $f^-(x)$.

We need this since we have two initial conditions in K-G theory.

$$(f^+_k, f^+_{k'}) = i/\hbar\, |\text{const}|^2 \int e^{ikx}(-i\omega_{k'})\, e^{-ik'x} - (i\omega_k)\, e^{ikx - ik'x}\, d\vec{r}$$

$=$

Box Normalization: Periodicity in $V = L^3$

$\vec{k} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} = (n_1, n_2, n_3) \qquad n_j = -3, -2, -1, 0, 1, 2, 3 \cdots$

$$(f^{+}_{k}, f^{+}_{k'}) = \tfrac{1}{\hbar} |\text{const}|^2 (-2i\omega_k) V \delta_{\vec{k}\vec{k}'} \overset{\text{want}}{=} \delta_{kk'}$$

$|\text{const}|^2 = \frac{\hbar}{2\omega_k} V \qquad \text{const} = \sqrt{\frac{\hbar}{2\omega_k V}}$

so $f^{(\pm)}_{k}(x) = \sqrt{\frac{\hbar}{2\omega_k V}}\; e^{\mp ikx}$

$(f^{(-)}_{k}, f^{(-)}_{k'}) = -\delta_{kk'}$ so $(f^{(-)}_{k}, f^{(-)}_{k}) = -1$ is < 0 so not a scalerproduct

this later turns out to be useful for charge, this expresses negative charge. $(f^{(+)}_{k}, f^{(-)}_{k'}) = 0$

Since the $f^{(-)}_k, f^{(+)}_k$ are complete,

Write: $\varphi(x) = \sum_k \{ a_k f^{+}_k(x) + b_k f^{(-)}_k(x) \}$ where a_k, b_k are appropriate operators

$\varphi(x)$ will be thermitean (for neutral particles)

Thus $b_k = a_k^{+}$ so $\varphi(x) = \sum_k \{ a_k f^{(+)}_k(x) + a^{+}_k f^{(-)}_k(x) \}$

Solve for a_k a_k^{+}:

$a_k = \frac{i}{\hbar} \int f^{(+)}_k(x) \overleftrightarrow{\partial_0} \varphi(x)\, d\vec{r} = (f^{(+)}, \varphi)$

$a_k^{+} = \int f_k^{(-)}(x) \overleftrightarrow{\partial_0} \varphi(x)\, d\vec{r} = -(f^{(-)}, \varphi)$

$$[a_k, a_{k'}] = 0, \quad [a_k, a_k^{+}] = -\left(\tfrac{i}{\hbar}\right)^2 \iint \Big[f^{(+)*}_k(x) \tfrac{\partial}{\partial t} \varphi(x) - \tfrac{\partial f^{(+)*}(x)}{\partial t} \varphi,\; f^{(-)*}_{k'}(x') \tfrac{\partial}{\partial t} \varphi(x') - \tfrac{\partial}{\partial t} f^{(-)*}_{k'}(x') \varphi \Big] d\vec{r}\, d\vec{r}'$$

where $x = (t, r)$ $x' = (t, x')$

since $[\varphi(r,t), \varphi(r',t)] = 0$, $\pi = \frac{\partial}{\partial t}\varphi \Rightarrow [\pi(r,t), \pi(r',t)] = 0$

$[\varphi(r,t), \dot{\varphi}(r',t)] = i\hbar\, \delta(\vec{r} - \vec{r}')$

$$\text{So } [a_k, a^{+}_{k}] = \tfrac{1}{\hbar^2} \iint \Big\{ f^{(+)*}_k(x) \tfrac{\partial}{\partial t} f^{(-)*}_{k'} i\hbar \delta(r - r') - \tfrac{\partial f_k^{(+)*}}{\partial t} f^{(-)*}_{k'}(x') i\hbar \delta(r - r') \Big\} d\vec{r}\, d\vec{r}'$$

note $f^{(-)*} = f^{(+)}$ $$= \tfrac{i}{\hbar} \int f^{(+)*}_k(x) \tfrac{\partial}{\partial t} f^{(-)*}_{k'}(x) - \tfrac{\partial}{\partial t} f^{(+)*}_k f^{(-)*}_{k'} \} dr$$

$[a_k, a_k^{+}] = (f^{(+)}_{k}, f^{(+)}_{k'}) = \delta_{\vec{k}\vec{k}'}$

$[a_k, a_{k'}] = 0$

$[a^{+}_k, a^{+}_{k'}] = 0$ So the a_k are the harmonic oscillator ladder operators

So $\varphi(x) = \sum_k \{ a_k f_k(x) + a_k^{+} f^{*}_k(x) \}$ this eliminates reference to negative E, wave functions.

$\varphi(x)$ is ~~a~~ real or Hermitean operator (since it is for neutral particles, not complex like fields talked about at beginning of course

Hamiltonian: $H = \frac{1}{2}\int (\Pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2)\, d\vec{r}$

$$= \frac{1}{2}\int(\dot{\varphi}^2 + (\nabla\varphi)^2 + \mu^2\varphi^2)\, d\vec{r}$$

$$\vec{P} = -\frac{1}{2}\int(\Pi\nabla\varphi + \nabla\varphi\Pi)\, dr = -\frac{1}{2}\int(\dot{\varphi}\nabla\varphi + \nabla\varphi\dot{\varphi})\, d\vec{r}$$

$$\varphi(x) = \sum_k \{a_k f_k^{(+)}(x) + a_k^+ f_k^{(-)}(x)\}$$

$$f_k(x) = e^{\mp ik\cdot x} \cdot \left(\sqrt{\frac{\hbar}{2\omega_k V}}\right) \qquad k_0 = \omega_k$$

$$[a_k, a_{k'}^+] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0$$

<u>Hamiltonian</u> $H = \frac{1}{2}\int\{\dot{\varphi}^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\}\, d\vec{r}$

<u>Substitute</u>: $= \frac{1}{2}\{-(\sum \omega_k(a_k f_k^{(+)} - a_k^+ f_k^{(-)}))^2 - (\sum \vec{k}(a_k f_k^{(+)} - a_k^+ f_k^{(-)}))^2 + \mu^2(\sum a_k f_k^{(+)} + a_k^+ f_k^{(-)})^2\}\, d\vec{r}$

$$H = \frac{1}{2}\int\{\cancel{\sum_{kk'} -(a_k a_{k'} f_k^{(+)} f_{k'}^{(+)} + a_k^+ a_{k'}^+ f_k^{(-)} f_{k'}^{(-)})(\omega_k\omega_{k'} + \vec{k}\vec{k}' - \mu^2)} + \sum_{kk'}(a_k a_{k'}^+ f_k^{(+)} f_{k'}^{(-)} + a_k^+ a_{k'} f_k^{(-)} f_{k'}^{(+)})(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + \mu^2)\, d\vec{r}$$

Since $\int f_k^{(+)} f_{k'}^{(+)}\, dr = \text{constant}\cdot\delta(k+k') \Rightarrow (\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' - \mu^2 = \omega_k^2 - \omega_k^2) = 0$

$$\int f_k^{(+)} f_{k'}^{(-)}\, dr = \delta_{k,k'} \cdot \frac{\hbar}{2\omega_k V} \times V$$

So $H = \frac{1}{2}\sum_k \frac{\hbar V}{2\omega_k V} \cdot 2\omega_k^2 (a_k a_k^+ + a_k^+ a_k) = \boxed{\frac{1}{2}\sum_k \hbar\omega_k(a_k a_k^+ + a_k^+ a_k)}$

This is a positive definite operator.

<u>Momentum</u>:

$\vec{P} = -\frac{1}{2}\int\{\dot{\varphi}\nabla\varphi + \nabla\varphi\dot{\varphi}\}\, d\vec{r}$, symmetrized to be Hermitian.

After much manipulation: $\vec{P} = \frac{1}{2}\sum_k \hbar\vec{k}(a_k a_k^+ + a_k^+ a_k)$

Four momentum operator: $P^\mu = \sum_k \hbar k^\mu(a_k a_k^+ + a_k^+ a_k)$, $k^0 = \omega_k$

Let $N_k = a_k^+ a_k$

N_k has eigenstates $N_k|n_k\rangle = n_k|n_k\rangle$

It's eigenvalues are $n_k = 0, 1, 2, 3 \text{---}$

$|n_k\rangle = \left((a_k^{+})^{n_k} / n_k!,\right) \cdot |0\rangle_k$

Full field: Construct basis : $|\{n_k\}\rangle = \prod_k |n_k\rangle_k$ → An outer product, in the product space.

$\prod_k |n_k\rangle_k = \prod_k \frac{(a_k^{+})^{n_k}}{\sqrt{n_k!}} \boxed{\prod_{k'} |0\rangle_{k'}}$ — Vacuum state $\equiv |0\rangle$

$P^{\nu} |n_k\rangle = (\sum \hbar k^{\nu} n_k) |\{n_k\}\rangle$

$|vac\rangle = \prod_k |0\rangle_k$, $P^{N} |vac\rangle = 0 = \frac{1}{2} \sum_k \hbar k |vac\rangle$

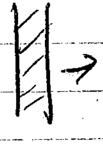
So you have to sum the $\vec{k}$'s in such a way that the sum vanishes, add $-k$ to k.

$H|vac\rangle = \sum_k \frac{1}{2} \hbar \omega_k \overset{\text{by commutation}}{(a_k^{+} a_k + \frac{1}{2})} |vac\rangle = \frac{1}{2} \sum \hbar \omega_k |vac\rangle$
$= \infty$

Zero-point Oscillations: Are there, are observable, amplified as noise

Casimir Polder 1947?

Casimir suggestion: Put two plates, with ~~vac~~ between; pulling them apart, you increase the number of possible modes, increasing the zero-point energy, producing a measureable attractive force.



At high frequencies, the metal plate becomes transparent to the field. So the problem only applies to the lower frequency modes, the zero-point energy is finite

For Full field : $P = 1 = P_0 + P_1 + P_2 + \cdots$

$P_0 = |vac\rangle\langle vac|$

One quantum states : $|0\,0 \cdots 0\,1_k\,0 \cdots 0\rangle = |\vec{k}\rangle = a_k^{+} |vac\rangle$

$\langle \vec{k} | \vec{k}' \rangle = \langle vac | a_k a_{k'}^{+} | vac \rangle = \langle vac | [a_k, a_{k'}^{+}] + \underbrace{a_{k'}^{+} a_k}_{\text{normal ordered.}} | vac \rangle$
$= \delta_{kk'}$
$= \delta_{kk'}$

$P_1 = \sum_k |k\rangle\langle k|$ = unit operator within 1-quantum subspace,

$P_1 |k\rangle = |k'\rangle$, $\langle r | k \rangle = \frac{e^{ik \cdot r}}{\sqrt{V}}$ to secure $\langle k | k' \rangle = \delta_{kk'}$

Take: $H = \frac{1}{2} \int :\left(H^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2\right): $, i.e, H is to be normal-ordered

This eliminates zero-point energy problems,

Here this is equivalent to adding a constant to the Hamiltonian

Then $H|vac\rangle = 0$

Continuum: 1 particle, non-relativistic state:

$$\langle r' | k \rangle = \frac{e^{ik\cdot r'}}{(2\pi)^{3/2}} \qquad \langle k'|k'\rangle = \int \langle k|r'\rangle\langle r'|k'\rangle d\vec{r} = \delta(k-k$$

Completeness: $1 = \int d\vec{k}\, |k\rangle\langle k|$

$$\frac{1}{\sqrt{V}} \to \frac{1}{2\pi^{3/2}}, \quad \sum_k \to \int d\vec{k}$$

Relativistic: $f_k^{(\pm)}(x) = \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}}\; e^{\mp ikx}$

$$(f_k^{(\pm)}, f_{k'}^{(\pm)}) = \frac{i}{\hbar}(\mp i)(\omega_k + \omega_{k'}) \frac{\hbar}{2\sqrt{\omega_k\omega_{k'}}\,(2\pi)^3}(2\pi)^3\, \delta(k-k')\, e^{i(\omega_k - \omega_{k'})t}$$

$$= \pm\delta(k-k')$$

$$\varphi(x) = \int d\vec{k}\, \{ a(k) f_k^{(+)}(x) + a^\dagger(k) f_k^{(-)}(x) \}$$

$$a(k) = (f_k^{(+)}, \varphi), \quad a^\dagger(k) = -(f_k^{(-)}, \varphi)$$

$$[a(k), a^\dagger(k')] = (f_k^{(+)}, f_{k'}^{(+)}) = \delta(k-k')$$

$$H = \frac{1}{2}\int \hbar\omega_k \{ a(k)a^\dagger(k) + a^\dagger(k)a(k)\}\, d\vec{k}$$

$$P^\mu = \frac{1}{2}\int \hbar k^\mu \{ a(k)a^\dagger(k) + a^\dagger(k)a(k)\}\, d\vec{k}$$

One quantum states: $|k\rangle = a^\dagger(k)|vac\rangle$

$$\langle k|k'\rangle = \langle vac| [a(k), a^\dagger(k')] |vac\rangle = \delta(k-k')$$

$|k\rangle$ are now normalized.

Projection operator onto one-particle states: $P_1 = \int d\vec{k}\, |k\rangle\langle k|$

$P_1 |k'\rangle = |k'\rangle$

$f_k^{(\pm)}(x)$ is *not* a scalar invariant.

$N = \int a^\dagger(k) a(k) d\vec{k}$ should be invariant.

$a^\dagger, a$ are *not* scalar invariants.

$e^{\pm ikx}$ is scalar invariant, but ω_k in $f^{(\pm)} = \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}}\, e^{\mp ikx}$ is not so $f^\pm$ is not

$N = \int a^\dagger(k) a(k)\, d\vec{k}$ should be invariant although neither $a^\dagger(k)a(k)$ nor $d\vec{k}$ are invariant.

$$N = \int 2\omega_k a^\dagger(k) a(k) \frac{dk}{2\omega_k} = \int_{k_0>0} 2\omega_k a^\dagger(k) a(k)\, \delta(\overset{k_0=\pm\omega_k}{k_0^2} - k^2 - \mu^2)\, dk_0\, d\vec{k}$$

$\delta(k_0 - k^2 - \mu^2) = \delta(2\omega_k(k_0 - \omega_k))$ for k_0 near ω_k

so $N = \int_{k_0>0} 2\omega_k\, a^{\dagger}(k)\, a(k)\, \overbrace{\boxed{\delta(k^2-\nu^2)}}^{\text{invariant}}\overbrace{\boxed{d^4k}}^{\text{invariant}}$

So we suspect that $\sqrt{2\omega_k}\; a_k$ is an invariant.

Take as basic solutions to the K-G equation. $F_k^{(\pm)}(x) = e^{\mp ikx}$

$$(F_k^{(\pm)}, F_{k'}^{(\pm)}) = \delta(\vec{k}-\vec{k}') \cdot \frac{2\omega_k}{\hbar}(2\pi)^3$$

$$\varphi(x) = \int \sqrt{\frac{\hbar}{2\omega_k(2\pi)^3}} \{ a(k) F_k^{(+)}(x) + a^{\dagger}(k) F_k^{(-)}(x) \} d\vec{k}$$

Define a new set of $a(k)$: $\alpha(k) = \sqrt{2\hbar\omega_k(2\pi)^3}\; a_k$

$\alpha^{\dagger}(k) = \sqrt{2\hbar\omega_k(2\pi)^3}\; a^{\dagger}_k$

$$[\alpha_k, \alpha^{\dagger}_{k'}] = 2\hbar\,\omega_k (2\pi)^3\, \delta(\vec{k}-\vec{k}')$$

$$\varphi(x) = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\omega_k} \{ \alpha(k) F_k^{(+)}(x) + \alpha^{\dagger}(k) F_k^{(-)}(x) \}$$

$$= \frac{1}{(2\pi)^3} \int_{k_0>0} \{ \alpha(k) F_k^{(+)}(x) + \alpha^{\dagger}(k) F_k^{(-)}(x) \} \delta(k^2-\nu^2)\, d^4k$$

Every term here is an invariant.

Should get same answer using any of these definitions of normalization. Most books don't use the Planck's constant this far.

<u>Lorentz transformations</u> $\varphi'(x) = \varphi(\Lambda^{-1}x)$

Should exist a unitary operator for each Λ, $U(\Lambda)$, such that

$$\varphi'(x) = U^{-1}(\Lambda)\, \varphi(x)\, U(\Lambda) = \varphi(\Lambda^{-1}x)$$

These are same equations as for $\Lambda = R$, since R's are a subgroup.

$$\frac{1}{(2\pi)^3} \int \delta(k^2-\nu^2) \{ U^{-1}\alpha(k) U F^{+}(x) + U^{-1}\alpha^{\dagger}(k) U F_k^{(-1)} \} d^4k = \varphi(\Lambda^{-1}x)$$

this will hold iff: $\boxed{U^{-1}(\Lambda)\alpha(k)U(\Lambda) = \alpha(\Lambda^{-1}k)}$

Assuming this, $\frac{1}{(2\pi)^3} \int \delta(k^2-\nu^2) \{ \alpha(\Lambda^{-1}k)\, e^{-ikx} + \alpha^{\dagger}(\Lambda^{-1}k)\, e^{ikx} \} d^4k =$

Let $k' = \Lambda^{-1}k$, $k = \Lambda k'$

$$= \frac{1}{(2\pi)^3} \int \delta(k'^2-\nu^2) \{ \alpha(k')\, e^{-i(\Lambda k')x} + \alpha^{\dagger}(k')\, e^{+i(\Lambda k')x} \} d^4k'$$

$$= \varphi(\Lambda^{-1}x)$$

Since $(\Lambda k')\cdot x = k' \cdot (\Lambda^{T}x) = k' \cdot (\Lambda^{-1}x)$

since $\Lambda^{T} = \Lambda^{-1}$, Lorentz trans. are orthogonal)

So $\alpha(k)$ is a scalar operator since it transforms like a scalar

<u>One Quantum states:</u> $|\vec{k}\rangle = \alpha^{\dagger}(k)|0\rangle$

so $\langle \vec{k}|\vec{k}'\rangle = \langle 0|\alpha(k)\alpha^{\dagger}(k')|0\rangle = \langle 0|[\alpha(k), \alpha^{\dagger}(k')]|0\rangle$

$= 2\hbar\omega_k (2\pi)^3\, \delta(k-k')$

So $P_1 = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\hbar\omega_k} |\vec{k}\rangle\langle\vec{k}|$

Field Commutators: $[\varphi(x), \varphi(y)]$

We know time-dependence only for free-fields

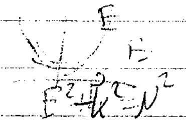
For Freefields, $[\varphi(x), \varphi(y)] = [\sum_k (a_k f_k^{(+)}(x) + a_k^\dagger f_k^{(-)}(x)), \sum_{k'} (a_{k'} f_{k'}^{+}(y) + a_{k'}^\dagger f_{k'}^{-}(y))]$

$$= \sum_k (f_k^{(+)}(x) f_k^{(-)}(y) - f_k^{(-)}(x) f_k^{(+)}(y))$$

Since $\sum_k \to \int \frac{d\vec{k}}{(2\pi)^3} V$, $\equiv \int \frac{d\vec{k} V}{(2\pi)^3} \frac{\hbar}{2\omega_k V} \{e^{-ik(x-y)} - e^{ik(x-y)}\}$

so $[\varphi(x), \varphi(y)] = i\hbar \Delta(x-y)$, where $\Delta(x) = \frac{1}{(2\pi)^3 i} \int \frac{d\vec{k}}{2\omega_k} (e^{-ikx} - e^{ikx}$

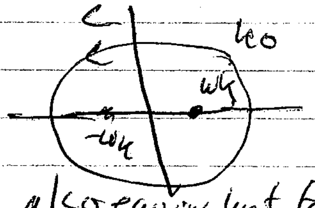
Or, $\Delta(x) = \frac{1}{i(2\pi)^3} \int_{k^0>0} (e^{-ikx} - e^{ikx}) \delta(k^2 - \mu^2)\, d^4k$



This is an invariant function.

Define $\epsilon(k^0) = \frac{k^0}{|k_0|} = \begin{matrix} +1 & \text{for } k^0>0 \\ -1 & \text{for } k^0<0 \end{matrix}$ then $\Delta(x) = \frac{1}{i(2\pi)^3} \int e^{-ikx} \delta(k^2 - \mu^2) \epsilon(k_0)\, d^4k$

Also $\Delta(x) = -\frac{1}{(2\pi)^4} \int_C \frac{e^{-ikx}}{k^2 - \mu^2} d^4k$



Think of this as integration in k_0 plane:

Two poles: $k_0 = \pm\omega_k$, evaluate the residues and you can see this formula is also equivalent to $\Delta(x)$

★ Δ must be such that $\Delta = \Delta(x^2, \mu^2)$ since x, is not invariant, only x^2

Evaluate: $\Delta(r,t) = \frac{1}{(2\pi)^3 i} \int \frac{d\vec{k}}{2\omega_k} (e^{i(\vec{k}\cdot\vec{r} - \omega t)} - e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$

Letting $k \to -k$

$$\Delta(r,t) = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{\omega_k} e^{i(\vec{k}\cdot\vec{r})} \sin \omega_k t \qquad \omega_k = \sqrt{k^2 + \mu^2}$$

This shows:

$\frac{\partial}{\partial t} \Delta(r,t) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \cos \omega_k t$ These functions are expressible in Bessel functions + their derivatives

$\Delta(\vec{r}, 0) = 0$, $-\frac{\partial}{\partial t} \Delta|_{t=0} = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}} d\vec{k} = \delta(\vec{r})$

equal time commutator

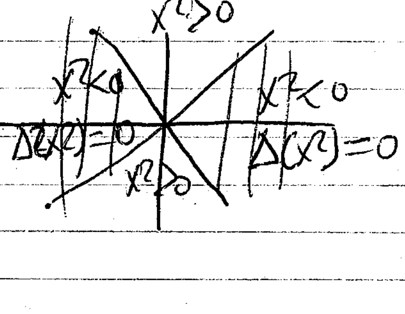
$$[\varphi(r,t), \Pi(r',t)] = [\varphi(r,t), \dot{\varphi}(r',t)] = i\hbar\, \delta(r - r')$$

$x^2 = -r^2$ or x^2 is neg. on surf. $t=0$

so $\Delta(x^2) = 0$ for all $x^2 < 0$

or

This comes out this way since effects cannot propagate faster than speed of light.



For coupled fields:

Local coupling: Then causality is preserved, commutator vanishes outside light cone, but the $\Delta(x)$ is completely changed inside the light cone.

Thus this has been used as an axiom by Axiomatists.

$$\langle vac|\varphi(x)|vac\rangle = \langle 0|\varphi(x)|0\rangle = 0$$

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \langle 0|\sum_k a_k f_k^{(+)}(x) + a_k^{+} f_k^{(-)}(x) \cdot \sum_{k'}(a_{k'} f_{k'}^{(+)}(y) + a_{k'}^{+} f_{k'}^{(+)}(y))|0\rangle$$

Since $a_k|0\rangle = 0$, $\langle 0|a_k^{+} = 0$

$$= \langle 0|\sum a_k f_k^{(+)}(x) \sum a_{k'}^{+} f^{(-)}(y)|0\rangle$$

$$= \langle 0|\sum_{kk'} [a_k, a_k^{+}] f_k^{(+)}(x) f_k^{(-)}(y)|0\rangle$$

$$= \sum f_k^{(+)}(x) f_k^{(-)}(y) = \frac{\hbar}{(2\pi)^3}\int \frac{d\vec{k}}{2\omega_k} e^{-ik(x-y)}$$

$$= \frac{\hbar}{(2\pi)^3}\int_{k^0>0} e^{-ik(x-y)}\delta(k^2+\mu^2)\,d^4 4$$

$$= i\hbar\,\Delta^{(+)}(x-y)$$

Since $\Delta^{(+)}(x) = \frac{1}{(2\pi)^3 i}\int \frac{dk}{2\omega_k} e^{-ik(x-y)}$

Let $\Delta^{+}(x) = [\Delta^{(+)}(x)]^{*} = -\Delta^{(+)}(-x)$

$$\langle 0|[\varphi(x), \varphi(y)]|0\rangle = i\hbar\,\Delta(x-y)$$

$$= i\hbar\{\Delta^{+}(x-y) - \Delta^{(+)}(y-x)\}$$

So $\Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = 2\,\mathrm{Re}\,\Delta^{(+)}(x)$

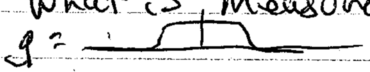
$\Delta^{+}(x)$ is complex-valued, does not, in general, vanish at $t=0$

Let $x=y$ $\langle 0|\varphi^2(x)|0\rangle = \frac{\hbar}{(2\pi)^3}\int \frac{dk}{2\omega_k} = \frac{\hbar}{(2\pi)^3}\int \frac{4\pi k^2 dk}{2\sqrt{\mu^2+k^2}}$

For $k \gg \mu$, this goes to $\int \frac{k^2 dk}{k} = \int_0^\infty k\,dk \Rightarrow$ quadratically divergent

To get rid of this problem, you have to regularize the field, realize that the field at a point is not measureable.

What is measureable is $\int g(r,t)\varphi(r,t)\,d\vec{r}$, g a weight function this removes the divergence.

$g =$ 

Have been dealing with one hermitean field $\varphi(x)$

2 Hermitian fields $\varphi_1(x), \varphi_2(x)$

Independent, no interaction, then $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$

If, $\mu_1 = \mu_2$ $\mathcal{L} = \frac{1}{2}\{\partial^\nu\varphi^{(1)}\partial_\nu\varphi^{(1)} + \partial^\nu\varphi^{(2)}\partial_\nu\varphi^{(2)} - \mu^2(\varphi^{(1)^2} + \varphi^{(2)^2})\}$

$\varphi^{(1)'} = \cos\lambda(\varphi^{(1)}) - \sin\lambda\,\varphi^{(2)}$

$\varphi^{(2)'} = \sin\lambda(\varphi^{(1)}) + \cos\lambda\,\varphi^{(2)}$ $\Rightarrow$ this transformation preserves the form of the Lagrangian.

Euler-Lagrange Eqs: $\frac{\partial \mathcal{L}}{\partial \varphi^{(j)}} = \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(j)}}$, $j=1,2$

$\boxed{\text{Noethers Theorem}}$: Give $\mathcal{L}(\varphi^{(j)}, \partial_\nu \varphi^{(j)})$

Assume this in invariant under a variation of the $\varphi^{(j)}$

$\varphi^{(j)} \to \varphi^{(j)} + \delta\varphi^{(j)}$ $\partial_\nu \varphi^{(j)} \Rightarrow \partial_\nu \varphi^{(j)} + \partial_\nu \delta \varphi^{(j)}$

$$\delta \mathcal{L} = 0 = \sum_j \frac{\partial \mathcal{L}}{\partial \varphi^{(j)}} \delta\varphi^{(j)} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\nu \delta \varphi^{(j)}\}$$

$$0 = \sum_j \{ \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(j)}} \delta \varphi^{(j)} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\nu \delta \varphi^{(j)}\}$$

$$0 = \partial_\nu \sum_j \{ \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(j)}} \delta \varphi^{(j)} \}$$, a fourdivergence

In this case: Take λ infinitesimal: $\delta\varphi^{(1)} = -\lambda \varphi^{(2)}$

the Group here is $SO(2)$ $\delta\varphi^{(2)} = \lambda \varphi^{(1)}$

Then $0 = \partial_\nu \{ \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(2)}} \varphi^{(1)} - \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(1)}} \varphi^{(2)} \}$

let $j_\nu = \text{const} \cdot \{ \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(2)}} \varphi^{(1)} - \frac{\partial \mathcal{L}}{\partial \partial_\nu \varphi^{(1)}} \varphi^{(2)} \}$

and $\partial_\nu j^\nu = 0$

$j_\nu = \text{const} \cdot \{ \partial^\nu \varphi^{(2)} \cdot \varphi^{(1)} - \partial^\nu \varphi^{(1)} \cdot \varphi^{(2)} \}$

$\partial_\nu j^\nu = 0$, a strange conservation law.

j^ν is arbitrary to within a divergenceless quantity.

for instance $j \to j^\mu + \partial^\nu F$, where $\partial_\nu \partial^\nu F = 0$

For n real scalar fields $\varphi^{(j)}$, $j = 1 \ldots n$

Then $\mathcal{L}$ is invariant under $SO(n)$

there are $\frac{n(n-1)}{2}$ independent infinitesimal rotations.

Given $\frac{n(n-1)}{2}$ conservation laws, conserved currents.

$n=3 \Rightarrow 3$ currents. Situation for Isospin

Conserved "Charge" $Q = \int j^0 d\vec{r} = \text{const} \int (\dot{\varphi}^{(2)} \varphi^{(1)} - \dot{\varphi}^{(1)} \varphi^{(2)}) d\vec{r}$

$Q = \text{const} \cdot \int \sum_k i\omega_k (a_k^{(2)} f_k^{(+)} - a_k^{(2)\dagger} f_k^{(+)}) \sum_{k'} (a_{k'}^{(1)} f_{k'}^{(+)} + a_{k'}^{(1)\dagger} f_{k'}^{(-)}) +$ something $1 \rightleftarrows 2$ interchanges

$Q = \text{const} \cdot \int (-i) \sum \omega_k \, a_k^{(2)} a_{k'}^{(1)} f_k^{(+)} f_{k'}^{(+)} + a_k^{(2)} a_{k'}^{(1)\dagger} f_k^{(+)} f_{k'}^{(-)} - a_k^{(2)\dagger} a_{k'}^{(1)} f_k^{(-)} f_{k'}^{(+)}$

$+ a_k^{(2)\dagger} a_{k'}^{(1)\dagger} f_k^{(-)} f_{k'}^{(+)}$

$+$ interchanges.

Since $[a_k^{(j)}, a_{k'}^{(\ell)}] = 0 = [a_k^{\dagger(j)}, a_{k'}^{\dagger(\ell)}]$

$[a_k^{(j)}, a_{k'}^{\dagger(\ell)}] = \delta_{kk'}\delta_{j\ell}$.

$$Q = \text{const} \int (-i) \sum_{kk'}^{\omega_k \omega_{k'}} (a_k^{(2)} a_{k'}^{(1)\dagger} f_k^{(+)} f_{k'}^{(-)} - a_k^{(2)\dagger} a_{k'}^{(1)} f_k^{(-)} f_{k'}^{(+)} - a_k^{(1)} a_{k'}^{(2)\dagger} f_k^{(+)} f_{k'}^{(-)} - a_k^{(1)\dagger} a_{k'}^{(2)} f_k^{(-)} f_{k'}^{(+)}$$

since $\int f_k^{(+)} f_{k'}^{(-)} = \delta_{kk'} \cdot \hbar/2\omega_k$

$$Q = \text{const} \cdot \frac{\hbar}{2} (-i) \cdot 2 \sum_k \{ a_k^{(2)} a_k^{(1)\dagger} - a_k^{(2)\dagger} a_k^{(1)} \}$$

Add to $\mathcal{L}$: $\text{const} \cdot j^\mu A_\mu$ $\quad\quad$ $\mathcal{H} \to \mathcal{H} - j^\mu A_\mu$

The two fields would become coupled, numbers of type 1 and 2 particles would not be individually conserved.

<u>Complex fields</u> Define complex field (Non-Hermitean)

$\varphi(x) = \frac{1}{\sqrt{2}}(\varphi^{(1)}(x) + i\varphi^{(2)}(x))$

$\varphi^\dagger(x) = 1/\sqrt{2}(\varphi^{(1)}(x) - i\,\varphi^{(2)}(x))$

$\varphi^{(1)}(x) = 1/\sqrt{2}(\varphi(x) + \varphi^\dagger(x))$

$\varphi^{(2)}(x) = 1/\sqrt{2}(\varphi(x) - \varphi^\dagger(x))$

commutation rules for fixed t

$[\varphi(r,t), \varphi(r',t)] = [\varphi(r,t), \varphi^\dagger(r',t)] = 0$

$[\varphi(r,t), \dot\varphi(r',t)] = \frac{1}{2}[\varphi^{(1)}(r,t) + i\varphi^{(2)}(r,t), \dot\varphi^{(1)}(r',t) + i\dot\varphi^{(2)}(r,t)] = 0$

since they cancel

$[\varphi(r,t), \dot\varphi^\dagger(r',t)] = \frac{1}{2} \cdot 2i\hbar\,\delta(\vec r - \vec r') = i\hbar\,\delta(r-r') = [\varphi^\dagger(r,t), \dot\varphi(r',t)]$

$[\dot\varphi(r,t), \dot\varphi(r',t)] = 0$

$[\dot\varphi(r,t), \dot\varphi^\dagger(r',t)] = 0$

$$\mathcal{L} = \frac{1}{2}\{\partial^\mu \varphi^\dagger \partial_\mu \varphi + \partial^\mu \varphi \partial_\mu \varphi^\dagger - \mu^2(\varphi^\dagger\varphi + \varphi\varphi^\dagger)\}$$

Because of commutation rules,

$$\boxed{\mathcal{L} = \partial^\mu \varphi^\dagger \partial_\mu \varphi - \mu^2 \varphi^\dagger \varphi}$$

$j^\mu = \text{const} \cdot \{ (\partial^\mu \varphi \cdot \varphi^\dagger - \partial_\mu \varphi^\dagger \cdot \varphi) \}$

$\quad = \text{const}(\varphi^\dagger \partial^\mu \varphi - \varphi\, \partial^\mu \varphi^\dagger)$

Now $\mathcal{L}(\varphi, \varphi^\dagger, \partial_\mu\varphi, \partial_\mu\varphi^\dagger)$ $\quad\quad$ $\delta \int L\, dt = \delta \int \mathcal{L}\, d^4x = 0$

$\varphi \to \varphi + \delta\varphi$, $\varphi^\dagger \to \varphi^\dagger + \delta\varphi^\dagger$

$$\int \{ [\quad] \delta\varphi(x) + [\quad]^\dagger \delta\varphi^\dagger \} d^4x = 0$$

$\delta\varphi$ is complex variation, so real and complex parts must vanish $[\quad] = 0 \Rightarrow$ 2 real eqs. $\Rightarrow [\quad]^{\dagger} = 0$

Effect is same as if $\delta\varphi, \delta\varphi^{\dagger}$ were independent variations.

Euler Lagrange: $\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi^{\dagger}}\right) = \frac{\partial\mathcal{L}}{\partial\varphi^{\dagger}}$ $\quad \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}$

Define $\pi^{\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}$, $\pi^{\nu\dagger} = \frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi^{\dagger}}$

$$\pi_\nu = \partial^\nu\varphi^{\dagger}, \quad \pi = \pi^0 = \partial_t\varphi^{\dagger} \quad \pi^{\dagger} = \pi^{0\dagger} = \partial_t\varphi$$

Equal-time commutation relations:

$$[\varphi(r,t), \varphi(r',t)] = [\varphi^{\dagger}(r,t), \varphi^{\dagger}(r',t)] = [\varphi(r,t), \varphi^{\dagger}(r',t)] = 0$$

$$[\varphi(r,t), \pi(r',t)] = i\hbar\,\delta(r'-r) = [\varphi(r,t), \dot{\varphi}(r',t)] = [\varphi^{\dagger}(r,t), \pi^{\dagger}(r',t)]$$

$$[\varphi(r,t), \pi^{\dagger}(r',t)] = 0$$

Eq. of motion:

$$(\Box^2 + \nu^2)\,\varphi = 0$$
$$(\Box^2 + \nu^2)\,\varphi^{\dagger} = 0$$

The SO(2) group: $\varphi' = e^{i\lambda}\varphi$
$\varphi^{\dagger} = e^{-i\lambda}\varphi^{\dagger}$ so $\varphi^{\dagger}\varphi$ is an invariant

Conserved current: $0 = \partial_\nu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi^{\dagger}}\delta\varphi^{\dagger} + \frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}\delta\varphi\right\}$

$$= -i\,\partial_\nu\left\{\frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi^{\dagger}}\varphi^{\dagger} \cancel{+} - \frac{\partial\mathcal{L}}{\partial\partial_\nu\varphi}\varphi\right\}$$

$$= -i\,\partial_\nu\{\partial^\nu\varphi\varphi^{\dagger} - \partial^\nu\varphi^{\dagger}\varphi\} = \text{const.}\ \partial_\nu j^\nu$$

$$j^\nu = \boxed{ie\{\partial^\nu\varphi\varphi^{\dagger} - \partial^\nu\varphi^{\dagger}\varphi\}}$$

Suppose $\varphi(x) = \sum\{b_k f_k^{(+)}(x) + c_k f^{(-)}(x)$

$$j^\nu = \text{const.}\{(\partial^\nu \varphi^{(2)})\varphi^{(1)} - \partial^\nu \varphi^{(1)}\varphi^{(2)}\}$$

$$Q = \int j^0 d\vec{r} = \text{const}: (-i\hbar)\sum_k \{a_k^{(2)} a_k^{(1)\dagger} - a_k^{(2)\dagger} a_k^{(1)}$$

$a_k^{(1)}$ typeparticle, when scattered off a potential gives $a_k^{(2)}$ particle
So $\varphi^{(1)}, \varphi^{(2)}$ particles are not physical
Superselection rules charge is precisely defined, is conserved.
So must use: $\frac{1}{\sqrt{2}}(\varphi^{(1)} + i\varphi^{(2)}) = \varphi$

$$j^\nu = \text{const.}\frac{1}{i}\{\partial^\nu \varphi\varphi^\dagger - \partial^\nu \varphi^\dagger \varphi\}$$

$$\varphi(x) = \sum\{b_k f_k^{(+)}(x) + c_k^\dagger f_k^{(-)}(x)\}$$ since $c_k \neq b_k$, this doesn't have to be Hermitian.

$$b_k = (f_k^{(+)}, \varphi) \qquad c_k^\dagger = -(f_k^{(-)}, \varphi)$$

$$\varphi^\dagger(x) = \sum(b_k^\dagger f_k^{(-)}(x) + c_k f_k^{(+)}(x)) \quad \text{so } b_k^\dagger = -(f_k^{(-)}, \varphi^\dagger),\ c_k = (f_k^{(+)}, \varphi^\dagger)$$

$$[b_k, b_{k'}^\dagger] = (f_k^{(+)}, f_{k'}^{(+)}) = \delta_{kk'} \qquad [b_k, c_{k'}] = -(f_k^{(+)}, f_{k'}^{(+)*}) = -(f_k^{(+)}, f_{k'}^{(-)}) = 0$$

$$[c_k, c_{k'}^\dagger] = \quad '' \quad '' \quad = \delta_{kk'}$$

So the b_k and the c_k are two sets of independent Harmonic oscillator operators. $b_k^\dagger b_k$, $c_k^\dagger c_k$ both have non-neg. integer eigenvalues

Alternate method: $b_k = \frac{1}{\sqrt{2}}(a_k^{(1)} + i a_k^{(2)})$ $c_k^\dagger = \frac{1}{\sqrt{2}}(a_k^{(1)\dagger} + i a_k^{(2)\dagger})$

$$[b_k, b_{k'}^\dagger] = \frac{1}{2}[a_k^{(1)} + i a_k^{(2)}, a_{k'}^{(1)\dagger} - i a_{k'}^{(2)\dagger}] = \delta_{kk'}$$

$$[b_k, c_{k'}^\dagger] = \frac{1}{2}[a_k^{(1)} + i a_k^{(2)}, a_{k'}^{(1)\dagger} + i a_{k'}^{(2)\dagger}] = \frac{1}{2}(\delta_{kk'} - \delta_{kk'}) = 0$$

$$\mathcal{H} = \pi\dot{\varphi} + \pi^\dagger \dot{\varphi}^\dagger - \mathcal{L} = \dot{\varphi}^\dagger\dot{\varphi} + \dot{\varphi}\dot{\varphi}^\dagger - \mathcal{L} = 2\dot{\varphi}^\dagger\dot{\varphi} - \mathcal{L}$$

$$H = \int \mathcal{H}\, d\vec{r} = \int\{\dot{\varphi}^\dagger\dot{\varphi} + \nabla\varphi^\dagger \cdot \nabla\varphi + \nu^2 \varphi^\dagger\varphi\}\, d\vec{r}$$
$$= \int\{\sum_{kk'} \omega_k(b_k^\dagger f_k^{(-)} - c_k f_k^{(+)})\, \omega_{k'}(b_{k'} f_{k'}^{(+)} - c_{k'}^\dagger f_{k'}^{(-)})$$
$$+ \sum_{kk'} \vec{k}(\quad '' \quad)\, \vec{k}'(\quad '' \quad)$$
$$+ \sum_{kk'} \mu^2(b_k^\dagger f_k^{(-)} + c_k f_k^{(+)})(b_{k'} f_{k'}^{(+)} - c_{k'}^\dagger f_{k'}^{(-)})\}\, d\vec{r}$$

$$H = \int \sum_{kk'}\{(b_k^\dagger c_{k'}^\dagger f_k^{(-)} f_{k'}^{(-)} + c_k b_{k'} f_k^{(+)} f_{k'}^{(+)})(-\omega_k\omega_{k'} \mp \vec{k}\cdot\vec{k}') + \nu^2)$$
$$(b_k^\dagger b_{k'} f_k^{(-)} f_{k'}^{(+)} + c_k^\dagger c_{k'} f_k^{(+)} f_{k'}^{(-)})(\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + \nu^2)\}\, d\vec{r}$$

$$\int f_k^{(+)} f_{k'}^{(+)} d\vec{r} = \frac{\hbar}{2\omega_k}\delta_{kk'}$$

so $$H = \sum_k \hbar\omega_k (b_k^+ b_k + c_k c_k^+) = \sum_k \hbar\omega_k (b_k^+ b_k + c_k^+ c_k + \overset{\frac{1}{2}+\frac{1}{2}}{1})$$

↓ zero-point energ for two oscillators.

Field $\varphi(r,t)$: Must satisfy K-G equation, this forces us to expand using negative frequency as well as possitive frequency plane waves.

$$\varphi(r,t) = \sum\{b_k f^{(+)}_k(r,t) + c_k^{(+)} f^{(-)}(r,t)\}$$

thus relativity forces charge, Energy symmetry,

$$Q = \int j^0 d\vec{r} = \text{const}\frac{1}{i}\int(\partial^0\varphi\varphi^+ - \partial^0\varphi^+\varphi)d\vec{r}$$

$$= \frac{\text{const}}{i}\int\Big\{-i\sum_{kk'}\omega_k(b_k f_k^{(+)} - c_k^+ f_k^{(-)})(b_{k'}^+ f_{k'}^{(-)} + c_{k'} f_{k'}^{(+)})$$
$$-i\sum_{kk'}\omega_k(b_k^+ f_k^{(-)} - c_k f_k^{(+)})(b_{k'} f_{k'}^{(+)} + c_{k'}^{\cancel{+}} f_{k'}^{(-)})\Big\}d\vec{r}$$

$$= -\text{const}\int\sum_{kk'}\omega_k\Big\{\cancel{(b_k c_{k'} - c_k b_{k'}) f_k^{(+)} f_{k'}^{(+)} - (c_k^+ b_{k'}^+ - b_k^+ c_{k'}^+) f_k^{(-)} f_{k'}^{(-)}}$$
$$(b_k b_{k'}^+ - c_k c_{k'}^+) f_k^{(+)} f_{k'}^{(-)} - (c_k^+ c_{k'} - b_k^+ b_{k'}) f_k^{(-)} f_{k'}^{(+)}\, dr$$

$$= -\text{const}\cdot\frac{\hbar}{2}\sum_k\{b_k b_k^+ - c_k c_k^+ - c_k^+ c_k + b_k^+ b_k\}$$

$$= -\text{const}\,\frac{\hbar}{\cancel{2}}\sum_k\{(b_k^+ b_k - c_k^+ c_k)\}$$

If $N_{+k} = b_k^+ b_k$, $N_{-k} = c_k^+ c_k$, then $Q = -\text{const}\,\hbar\{\sum_k N_{+k} - N_{-k}\}$

$$Q = e\sum_k(N_{+k} - N_{-k}) \quad \text{so const} = -e/\hbar$$

So $$j^\nu = i\frac{e}{\hbar}\{\partial^\nu\varphi\varphi^+ - \partial^\nu\varphi^+\varphi\} = \frac{ie}{\hbar}\{\varphi^+\partial^\nu\varphi - \varphi\partial^\nu\varphi^+\}$$

$$Q = ie\sum(a_k^{(2)} a_k^{(1)+} - a_k^{(2)+} a_k^{(1)})$$

By the complex transformation $a_k^{(1)} = \frac{1}{\sqrt{2}}(b_k + c_k)$
we are diagonalizing this. $a_k^{(2)} = \frac{1}{\sqrt{2}i}(b_k - c_k)$

$$Q = \frac{e}{2}\sum(bb^+ - cc^+ + bc^+ - cb^+ + b^+b - c^+c + b^+c - c^+b\}$$

$$= e\sum_k(b_k^+ b_k - c_k^+ c_k)$$

Electromagnetic Field: $A^{\mu}(\vec{r},t) = (\Phi(r,t), \vec{A}(r,t))$
If you take A^{μ} of frequency $\hbar\omega = 2\mu$, pair production will occur

Vacuum Polarization: Assume we have point source for the field
$\nabla^2\Phi = -\delta(r)$ $\Phi = \frac{1}{4\pi r}$
$j^0(x) = \rho(r,t)$ this induces changes in vacuum fluctuations making them non-zero, giving a vacuum polarization.

External Electromagnetic Field $A^{\mu}(x) = (\Phi, \vec{A})$

Classical: $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$ $\quad$ $E \to E - e\Phi$ $\quad$ $c=1$

Q.M.: $\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - e\vec{A}$, $i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - e\Phi$

This substitution is only one of many. It is the one necessary for gauge invariance.
This algorithm is known as "minimal" coupling.

In covariant terms: $\boxed{\frac{\hbar}{i}\partial_{\mu} \to \frac{\hbar}{i}\partial_{\mu} + eA_{\mu}}$ $(i\hbar\partial_0 \to i\hbar\partial_0 - e\Phi$

$$\partial_{\mu}\varphi \to \left(\partial_{\mu} + \frac{ie}{\hbar}A_{\mu}\right)\varphi \qquad \partial_{\mu}\varphi^{+} \to \left(\partial_{\mu} - \frac{ie}{\hbar}A_{\mu}\right)\varphi^{+}$$

Gauge transformation: $A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\Lambda(x)$
Field Strength Tensor: $F^{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ → invariant under gauge transformation

$$\left(\partial_{\mu} + \frac{ie}{\hbar}A_{\mu} + \frac{ie}{\hbar}\partial_{\mu}\Lambda(x)\right)\varphi(x) = e^{-\frac{ie}{\hbar}\Lambda(x)}\left(\partial_{\mu} + \frac{ie}{\hbar}A_{\mu}\right)e^{\frac{ie}{\hbar}\Lambda(x)}\varphi(x)$$

Also Adjoint expression to this:

Under a gauge transformation: $\varphi(x) \to \varphi'(x) = e^{\frac{ie}{\hbar}\Lambda(x)}\varphi(x)$

$$\mathcal{L} = \left(\partial^{\mu} - \frac{ie}{\hbar}A^{\mu}\right)\varphi^{+}\left(\partial^{\nu} + \frac{ie}{\hbar}A_{\mu}\right)\varphi - \mu^2\varphi^{+}\varphi$$

$$= \partial^{\mu}\varphi^{+}\partial_{\mu}\varphi - \mu^2\varphi^{+}\varphi - \frac{ie}{\hbar}\left(\varphi^{+}\partial_{\mu}\varphi - \partial_{\mu}\varphi^{+}\cdot\varphi\right)A^{\mu} + \frac{e^2}{\hbar^2}\varphi^{+}\varphi A^{\mu}A_{\mu}$$

$$= \mathcal{L}_0 - \frac{ie}{\hbar}\left(\varphi^{+}\overleftrightarrow{\partial_{\mu}}\varphi\right)\Lambda^{\mu} + \frac{e^2}{\hbar^2}\varphi^{+}\varphi A^{\mu}A_{\mu}$$

Since $\overset{\text{for } A=0}{j_{\mu}} = \frac{ie}{\hbar}\left\{\partial^{\mu}\varphi\varphi^{+} - \partial^{\mu}\varphi^{+}\varphi\right\} \longrightarrow \langle 0|Q|0\rangle = \langle 0|\int j^0 dr|0\rangle = 0$

At this point, ordering of operators is arbitrary since commuting operators just gives c-numbers which don't affect the dynamics

For $A^\nu \neq 0$ $j^\nu(x) = \frac{ie}{\hbar}\left\{\left(\partial^\nu + \frac{ie}{\hbar}A^\nu\right)\varphi\cdot\varphi^\dagger - \left(\partial^\nu - \frac{ie}{\hbar}A^\nu\right)\varphi^\dagger\cdot\varphi\right\}$

this is the gauge invariant current:

$$j^\nu(x) = \frac{ie}{\hbar}\left(\partial^\nu\varphi\varphi^\dagger - \partial^\nu\varphi^\dagger\varphi\right) - \underbrace{\frac{e^2}{\hbar^2}\left(\varphi\varphi^\dagger + \varphi^\dagger\varphi\right)A^\nu}_{\frac{2e^2}{\hbar^2}(\varphi^\dagger\varphi)A^\nu}$$

Normally ordered Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 - :j_\nu: A^\nu - \frac{e^2}{\hbar^2} :\varphi^\dagger\varphi: A^\nu A_\nu$$

$\pi^\nu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi(x))} = \left(\partial^\nu - \frac{ie}{\hbar}A^\nu\right)\varphi^\dagger$ instead of $\partial^\nu\varphi^\dagger$

$\pi^0(x) = \left(\partial^0 - \frac{ie}{\hbar}A^0\right)\varphi^\dagger$ . . . . .

$j^\nu(x) = \frac{ie}{\hbar}\left\{\pi^{\nu\dagger}\varphi^\dagger - \pi^\nu\varphi\right\}$

$\rho\Phi - \vec{j}\cdot\vec{A}$ contains no A^0 terms

$$\mathcal{H} = \pi\dot\varphi + \pi^\dagger\dot\varphi^\dagger - \mathcal{L} = \boxed{\mathcal{H}_0 + j^\nu A_\nu - \frac{e^2}{\hbar^2}\varphi^\dagger\varphi(\vec{A})^2}$$

This is not a scalar invariant.

Non zero couplings:

For Hermitean φ (neutral particles): $H = H_0 + H_i$

Ex: $\mathcal{H}_1 = g\rho(x)\varphi$: Interaction freely creates and annihilates massive quanta

Ex: $\mathcal{H}_1 = g\rho(x)\varphi^2(x)$: Absorb, or emit pairs of quanta. Also absorb + emit one quanta (scattering)

In these theories lose momentum and energy conservation if $\rho(x)$ varies with space + time

For Complex φ: $\mathcal{H}_1 = g\rho(x)\varphi^\dagger(x)\varphi(x)$

Must conserve charge: φ lowers charge
$\varphi^\dagger$ raises charge

So $\mathcal{H}_1 = g\rho(x)(\varphi^\dagger + \varphi)$ doesn't conserve charge.

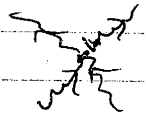
Pure field couplings: $\mathcal{H}_1 = g\varphi^3$ (φ Hermitian)

This applies in three particle processes: esp. virtual processes

Virtual processes don't have to obey conservation laws.

But adding a cubic term to quadratic form makes you lose positive definiteness, so you cannot have a ground state.

$\mathcal{H}_1 = g\,\varphi^4$

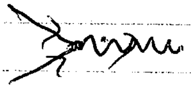
This applies to scattering  processes.

$\mathcal{H}_1 = g_4\varphi^4 + g_3\varphi^3$ for $g_4 > 0$ will have a ground state.

Coupling Different Fields $\varphi_1\, \varphi_2$.

$\mathcal{H}_1 = g\varphi_1\varphi_2$ This is useless, means that by making linear combinations of $\varphi_1\, \varphi_2$ you'll get uncoupled eqs.

$\mathcal{H}_1 = g\varphi_1^2\varphi_2$

In QED, Ψ complex $\mathcal{H} = g\,\Psi^+\Psi\,\varphi$ 

QED Analogy: $\Psi \to$ charged part. $\varphi \to A$

Nuclear Analogy: $\Psi \to$ Nucleon $\varphi \to$ Pion Field.

Time-dependence: $|\ \rangle_H$ time-independent.

$\varphi(r,t) = V(t)\,\varphi(\vec{r},0)\,V^{+}(t)$ $V(t)$ unitary $\varphi(\vec{r},0) = \varphi_{\text{schrodinger}}$

Define $|\Psi(t)\rangle_s = V^{+}(t)\,|\Psi\rangle_H$

$${}_s\langle \chi(t)|\,\varphi_s\,|\Psi(t)\rangle_s = {}_H\langle \chi|\varphi|\Psi\rangle_H$$

$$i\hbar \frac{\partial}{\partial t} V^{+}(t) = V^{+}(t)\,\mathcal{H}(t) = [V^{+}(t)\,H(t)\,V(t)]\,V^{+}(t)$$
$$= H_s\,V^{+}(t)$$

so $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle_s = H_s\,|\Psi(t)\rangle_s$

For H_s time-independent: $|\Psi(t)\rangle_s = e^{-i\frac{H_s t}{\hbar}}|\Psi(0)\rangle_s = e^{-i\frac{H_s t}{\hbar}}|\Psi\rangle_H$

$H_s = H_0 + H_1$ Know eigenstates of H_0.

Hope that H_1 is small so we can do perturbation theory on it.

$$\boxed{\text{Set } \hbar = 1}$$

Schrodinger: $i\frac{\partial}{\partial t}|\Psi(t)\rangle_s = H_s|\Psi(t)\rangle_s$

For H_s time-independent: $|\Psi t\rangle_s = e^{-iH_s t}|\Psi(0)\rangle_s = e^{-iH_s t}|\Psi\rangle_H$

Let $H_s = H_{0_s} + H_{1_s}$ we know the eigenstates of H_{0s}.

$|\Psi(t)\rangle_s = e^{-iH_{0s}t}\,|\Psi(t)\rangle_{int}$, its time dependence is due to interaction, If $H_{1s} = 0$, $|\Psi(t)\rangle_I = |\Psi\rangle_H$

$$\cancel{H_0 e^{-iH_0 t}|\Psi(t)\rangle_I} + e^{-iH_0 t}\, i\frac{\partial}{\partial t}|\Psi(t)\rangle_I = (\cancel{H_{0s}}+H_{1s})e^{-iH_0 t}|\Psi(t)\rangle_I$$

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle_I = e^{iH_{0s}t}H_{1s}e^{-iH_{0s}t}|\Psi(t)\rangle_I$$

Interaction Hamiltonian: $H_I(t) = e^{iH_0 t}H_{1s}e^{-iH_{0s}t}$

Fields: $e^{iH_0 t}\varphi_s(\vec{r})e^{-iH_0 t} = \varphi_I(\vec{r},t)$, has free field time dependence.

$$H_I(t) = e^{iH_0 t}H_1[\varphi_s(r) \cdots]e^{-iH_0 t} = H_1[\varphi_I(r,t), \cdots]$$

$$\boxed{i\frac{\partial}{\partial t}|\Psi(t)\rangle_I = H_I(t)|\Psi(t)\rangle_I}$$

Define $U(t,t_0)$ such that $|\Psi(t)\rangle_I = U(t,t_0)|\Psi(t_0)\rangle_I$

$U(t,t_0)$ is Unitary, $U(t_0,t_0)=1$

Then $\boxed{i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)\,U(t,t_0)}$

$$i\frac{\partial}{\partial t_0}U(t,t_0) = -U(t,t_0)H_I(t)$$

$i\frac{\partial}{\partial t}(U^\dagger U) = U^\dagger(-H+H)U = 0$, $U^\dagger U = \text{const} = 1$ since $U(t_0,t_0)=1$

~~[illegible]~~

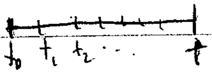
$$i\frac{\partial}{\partial t_0}U(t,t_0)U(t_0,t_1) = U(t,t_0)(-H_I(t_0)+H_I(t_0))U(t_0,t_1) = 0$$

So $U(t,t_0)U(t_0,t_1)$ is independent of t_0, so $U(t,t_0)U(t_0,t_1) = U(t,t$

If $t_1 = t$, $U(t,t_0)U(t_0,t) = 1$

$$U(t_0,t) = U^{-1}(t,t_0) = U^\dagger(t,t_0)$$

$$U(t,t_0) - 1 = -i\int_{t_0}^{t} H_I(t')\,U(t',t_0)\,dt'$$



Assume $t_{j+1}-t_j = \Delta t$

$$|\Psi_{t_j}\rangle_I = (1 - iH_I(t_j)\Delta t)|\Psi_{t_{j-1}}\rangle_I$$

$$|\Psi t\rangle_I \underset{\lim_{\Delta t\to 0}}{=} \prod_{j=t_1}^{t_n}(1 - iH_I(t_j))\Delta t\,|\Psi t_0\rangle_I$$

Making sure you keep (time) ordering

$$U(t,t_0) = \lim_{\Delta t\to 0}\left(1 - i\sum_{j=1}^{n}H_I(t_j)\Delta t + (-i)^2\sum_{j>k}H_I(t_j)H_I(t_k)(\Delta t)^2 + O(H_I^3)\right)$$

$$U(t,t_0) = 1 - i\int_{t_0}^{t}H_I(t')\,dt' + (-i)^2\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1\,H_I(t_2)H_I(t_1) + \cdots$$

Can also get this by iterating $U(t,t_0) = 1 + i\int_{t_0}^{t} H_I(t')\,U(t',t_0)\,dt'$
Start first term by assuming $U = 1$,
then

$$U = 1 - i\int H_I(t')\,dt' + (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_2} H_I(t_2)\,H_I(t_1)\,dt_1\,dt_2$$

$$+ (-i)^3\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{t_0}^{t_2} H_I(t_3)\,H_I(t_2)\,H_I(t_1)\,dt_1\,dt_2\,dt_3$$

where $t_3 \geq t_2 \geq t_1$

Time-ordering Symbol:

$T\{A(x)\,B(y)\,C(z)\cdots\}$ = product of all factors writte in time-order such that time-arguments increase from right to left.

$T\{B(x), B(y)\cdots\}$ is a symmetric operator for $x, y, z\cdots$)

$$U = 1 - i\int H_I(t')\,dt' + (-i)^2\iint_{t_0}^{t\;\;t_2} T\{H_I(t_2)H_I(t_1)\}\,dt_1\,dt_2$$

$$+ (-i)^3\iiint T\{H_I(t_3), H_I(t_2), H_I(t_1)\}\,dt_1\,dt_2\,dt_3$$

This makes the equation symmetric in time variables, we no can integrate over all space not just $t_3 \geq t_2 \geq t_1$

$$U = 1 - i\int H_I(t')\,dt' + \frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} T\{H_I(t_1), H_I(t_2)\}\,dt_1\,dt_2$$

the area of the restricted region is $1/n!$ the area of the n-dim. hypercube.

$$U = 1 - \cdots\cdots + \frac{(-i)^n}{n!}\underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{t}}_{n\,\text{fold}} T\{H_I(t_1)\cdots H(t_n)\}\,dt_1\cdots dt_n$$

$$\boxed{U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} H_I(t')\,dt'}\right\}} \quad \rightarrow U(t_0,t_0) = 1$$

Usually compute S-matrix $= U(\infty, -\infty)$

$$i\frac{d}{dt}U(t,t_0) = T\left\{H_I(t)\,e^{-i\int_{t_0}^{t} H_I(t')\,dt'}\right\} = H_I(t)\,T\{\cdots$$

$$= H_I(t)\,U(t,t_0)$$

So this is a sol. to our diff eq.

$\varphi(x)$ is the free field.

Letting $\mathcal{H}_{Is} = g\rho(r,t)\varphi_s(r,t)$ $\quad \mathcal{H}_I(t) = g\rho(r,t)\varphi(r,t)$

$\boxed{H_I(t) = g\int\rho(x)\varphi(x)\,dx}$ $\qquad = g\rho(x)\varphi(x)$

$$U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t}\mathcal{H}_I(x)\,d^4x}\right\} = T\left\{e^{-i\int_{t_0}^{t} g\rho(x)\varphi(x)\,d^4x}\right\}$$

$$U(t,t_0) = 1 - i\int_{t_0}^{t} g\rho\varphi\,d^4x + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} g^2\rho(x)\rho(y)\,T\{\varphi(x),\varphi(y)\}\,d^4x\,d^4y$$
$$+\frac{(-i)^3}{3!}\iiint g^3\rho(x)\rho(y)\rho(z)\,T\{\varphi(x),\varphi(y),\varphi(z)\}\,d^4x\,d^4y\,d^4z$$

<u>Matrix Elements</u> of $\langle f|\,U(t,t_0)\,|i\rangle$

We are interested in only few-particle states, states which are essentially vacuum with small deviations.

$\langle 0|U(t,t_0)|0\rangle$ $\qquad$ since $\langle 0|\varphi(x)|0\rangle = 0$

For odd number of field operators $\langle 0|\varphi(x)\varphi(y)\varphi(z)|0\rangle = 0$

You always have unbalance annihilation, creation operators.

Proof: Let $W = -1\,\Sigma a^\dagger_k a_k$

$$W\varphi(x)W^{-1} = -\varphi(x)$$

$$\langle 0|\varphi(x)\varphi(y)\varphi(z)|0\rangle = \langle 0|W^{-1}W\varphi W^{-1}W\varphi W^{-1}W\varphi W^{-1}W|0\rangle$$
$$= (-1)^3\langle 0|W^{-1}\varphi(x)\varphi(y)\varphi(z) = 1|0\rangle$$
$$\Rightarrow -1\langle 0|\varphi(x)\varphi(y)\varphi(z)|0\rangle = 0$$

$$\langle 0|U(t,t_0)|0\rangle = 1 + \frac{(-i)^2}{2}g^2\iint\rho(x)\rho(y)\,\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle\,d^4x\,d^4y$$
$$+\frac{(-i)^4}{4}g^4\iiiint\rho(x_4)\cdots\rho(x_1)\,\langle 0|T\{\varphi(x_4)\cdots\varphi(x_1)\}|0\rangle\prod_{i}^{4}d^4x_i$$

$$T\{\varphi(x)\varphi(y)\} = \begin{cases}\varphi(x)\varphi(y) & x^0>y^0\\ \varphi(y)\varphi(x) & x^0<y^0\end{cases} = \varphi(x)\varphi(y)\,\theta(x_0-y_0) + \varphi(y)\varphi(x)\,\theta(y_0-x_0)$$

$$\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle = i\Delta^+(x-y)\,\theta(x^0-y^0) + i\Delta^+(y-x)\,\theta(y^0-x^0)$$

$$\equiv i\Delta_F(x-y)$$, the Feynman propagator

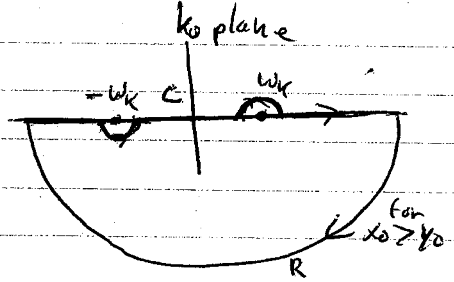
First user: E.C.G. Stueckelberg

$$\Delta^+(x-y) = \frac{1}{(2\pi)^3 i}\int\frac{d\vec{k}}{2\omega_k}\,e^{-ik(x-y)}, \quad k_0=\omega_k$$

$$= \frac{1}{(2\pi)^4}\int d\vec{k}\,dk_0\,\frac{1}{2\omega_k}\left\{\frac{1}{k^0-\omega_k} - \frac{1}{k^0+\omega_k}\right\}e^{-ik(x-y)}$$

If $x_0 > y_0$ $\quad e^{-ik(x-y)} \to e^{-ik^0(x^0-y_0)}$

so close contour downward

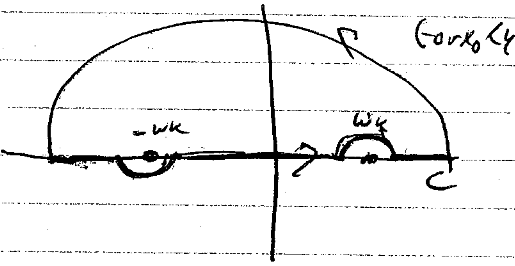


$$\Delta^+(x-y) = \frac{1}{(2\pi)^4}\int_C d\vec{k}\, dk_0 \frac{1}{2w_k}\left\{\frac{1}{k_0-w_k} - \frac{1}{k_0+w_k}\right\} e^{-ik(x-y)}$$

$$= \frac{1}{(2\pi)^4}\overset{\text{change integ-dir.}}{(-2\pi i} \cdot \text{Residue at } w_k$$

$$= \frac{1}{(2\pi)^3 i}\int \frac{d\vec{k}}{2w_k} e^{-ik(x-y)} = \frac{1}{(2\pi)^4}\int_C d^4k \frac{e^{-ik(x-y)}}{k^2-\nu^2}$$

For $x_0 < y_0$ Calculate same integral



$$\frac{1}{(2\pi)^4}\int_C d^4k \frac{e^{-ik(x-y)}}{k^2-\nu^2}$$

$$= \frac{1}{(2\pi)^4} -2\pi i \int \frac{e^{-ik(x-y)}}{2w_k} d\vec{k}, \quad k_0 = -w_k$$

$$= \frac{1}{(2\pi)^3 i}\int \frac{1}{2w_k} e^{iw_k(x^0-y^0) + i\vec{k}(\vec{x}-\vec{y})} d\vec{k}, \quad \text{let } k \to -\vec{k}$$

$$= \frac{1}{(2\pi)^3 i}\int \frac{e^{ik(x-y)}}{2w_k} d\vec{k}, \quad k^0 = w_k \quad = \underline{\Delta^+(y-x)}$$

So $\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int_C d^4k \frac{e^{-ik(x-y)}}{k^2-\nu^2}$

To evaluate, let $w_k \to w_k - i\eta$, $\eta > 0$, infinitesimal
Then the integral contour becomes a straight line on x-axis

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ikx}}{k^2 + 2iw\eta - \nu^2} \qquad \text{Let } 2w\eta = \epsilon$$

$$\text{then } \Delta_F(x-y) = \frac{1}{(2\pi)^4}\lim_{\epsilon \to 0}\int dk^4 \frac{e^{-ikx}}{k^2 - \nu^2 + i\epsilon}$$

$$\theta(x) = \frac{1}{2}(1 + \epsilon(x)) \qquad \epsilon(x) = \frac{x}{|x|} = \begin{matrix} 1 & x > 0 \\ -1 & x < 0 \end{matrix}$$

$$T\{\varphi(x), \varphi(y)\} = \frac{1}{2}\varphi(x)\varphi(y)(1 + \epsilon(x^0-y^0)) + \frac{1}{2}\varphi(y)\varphi(x)(1 - \epsilon(x^0-y^0))$$

$$= \frac{1}{2}\{\varphi(x), \varphi(y)\} + \frac{1}{2}[\varphi(x), \varphi(y)]\,\epsilon(x^0-y^0)$$

$$\partial_{x^0} T\{\varphi(x)\varphi(y)\} = \frac{1}{2}\{\partial_0\varphi(x), \varphi(y)\} + \frac{1}{2}[\partial_0\varphi(x), \varphi(y)]\,\epsilon(x^0-y^0)$$

$$\cancel{+\frac{1}{2}\underbrace{[\varphi(x), \varphi(y)]}_{\text{vanishes for } t=0}\,2\delta(x^0-y^0)}$$

so it goes to 0

$$\partial_{x_0}^2 T\{\varphi(x),\varphi(y)\} = \tfrac{1}{2}\{\partial_0^2\varphi(x),\varphi(y)\} + \tfrac{1}{2}[\partial_0^2\varphi(x),\varphi(y)]\,\epsilon(x^0-y^0)$$

No problem with spatial differentiations ↓

$$+ \underbrace{[\partial_0\varphi(x),\varphi(y)]}_{-i\delta^3(\vec{x}-\vec{y}) \text{ for } x^0=y^0}\,\delta(x^0-y^0)$$

$$(\Box^2+\mu^2)T\{\varphi(x)\varphi(y)\} = T\{(\Box^2+\mu^2)\varphi(x)\varphi(y)\} - i\delta^4(x-y)$$

$$(\Box^2+\mu^2)\,i\Delta_F(x-y) = -i\delta^4(x-y)$$

Helmholtz eq.: $(\Box^2+\mu^2)\Delta_F(x) = -\delta^4(x)$

$\Delta_F(x)$ is a type of green's function for the Inhomogenous Helmholtz eq.

$$(\Box^2+\mu^2)\frac{1}{(2\pi)^4}\int\frac{e^{-ikx}}{k^2-\mu^2}d^4k = \frac{-1}{(2\pi)^4}\int e^{-ikx}d^4k$$

$\Delta_F(x)$ is only one of many Green's Functions for Helmholtz eq., depending on boundary conditions:

Retarded solution

Advanced Solution

$$\boxed{\Delta_F(-x) = \Delta_F(x)}$$

$$\langle 0|\{\varphi(x),\varphi(y)\}|0\rangle \equiv \Delta^{(1)}(x-y) \equiv i(\Delta^{(+)}(x-y) + \Delta^{(+)}(y-x))$$

$$= i(\Delta^{(+)}(x-y) - \Delta^{(-)}(x-y))$$

$$\langle 0|T\{\varphi(x),\varphi(y)|0\rangle = \tfrac{1}{2}\Delta^{(1)}(x-y) + \tfrac{1}{2}i\Delta(x-y)\,\epsilon(x^0-y^0)$$

$$= i\Delta_F(x-y)$$

$$\text{So } \Delta_F = \tfrac{1}{2}\{\underset{\substack{\uparrow\\ \text{real part}}}{\epsilon\Delta} - \underset{\substack{\uparrow\\ \text{Imag. part}}}{i\Delta^{(1)}}\} \qquad \Delta(x-y) = -i(\Delta^{(+)}(x-y) - \Delta^{(+)}(y-x))$$

$$\langle 0|U(t,t_0)|0\rangle = 1 + \frac{(-ig)^2}{2!}\iint\rho(x)\rho(y)\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle\,d^4x\,d^4y$$

$$+ \frac{(-ig)^4}{4!}\iint\prod_{j=1}^{4}\rho(x_j)\langle 0|T\{\varphi(x_1)\cdots\varphi(x_j)\}|0\rangle\prod_j d^4x_j$$

$$\varphi = \sum_k\{a_k f_k^{(+)} + a_k^\dagger f_k^{(-)}\} = \underset{\substack{\text{pos. freq.}\\ \text{part}\\ \text{annihilation}\\ \text{terms}}}{\varphi^{(+)}} + \underset{\substack{\text{neg. freq.}\\ \text{part}\\ \text{Creation}\\ \text{terms}}}{\varphi^{(-)}}$$

For $x^0 > y^0$ $\langle 0|\varphi(x)\varphi(y)|0\rangle = \langle 0|(\varphi^{(+)}(x)+\cancel{\varphi^{(-)}(x)})(\cancel{\varphi^{(+)}(y)}+\varphi^{(-)}(y))|0\rangle$

$$= \langle 0|\varphi^{(+)}(x)\,\varphi^{(-)}(y)|0\rangle$$

$$= \sum_k \langle 0|a_k a_k^\dagger|0\rangle f^{(+)}(k) f^{(-)}(k)$$

For $x_1^0 > x_2^0 > x_3^0 > x_4^0$: $\langle 0|\varphi(x_1)\cdots\varphi(x_4)|0\rangle = \langle 0|\varphi^{(+)}(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi^{(-)}(x_4)|0\rangle$

Creating at x_4, annihilating at x_3 $= \langle 0|\varphi(x_1)\varphi(x_2)|0\rangle\langle 0|\varphi(x_3)\varphi(x_4)|0\rangle$

'' '' '' '' at x_2 $+ \langle 0|\varphi(x_1)\varphi(x_3)|0\rangle\langle 0|\varphi(x_2)\varphi(x_4)|0\rangle$

'' at x_1 $+ \langle 0|\varphi(x_2)\varphi(x_3)|0\rangle\langle 0|\varphi(x_1)\varphi(x_4)|0\rangle$

These are contractions

Coleman's $\overbrace{\quad}$

As long as these are fields with diff. momenta no commutation problems. Problem if the momenta are same, pairings break down but theorem still holds.

<u>Contraction</u>: $T(\varphi(x)\varphi(y)) = :\varphi(x)\varphi(y): +$ c-number function.

$\langle 0|T(\varphi(x)\varphi(y))|0\rangle = 0 +$ c-number function

since $:\;:$ always has 0 vacuum expectation.

so: $T\{\varphi(x)\varphi(y)\} = :\varphi(x)\varphi(y): + \langle 0|T(\varphi(x)\varphi(y))|0\rangle$ $= i\Delta_F(x-y)$

$= :\varphi(x)\varphi(y): + \overbrace{\varphi(x)\varphi(y)}$, ← the contraction of $\varphi(x)\varphi(y)$

Wick's Theorem:

$$T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\} = :\varphi(x_1)\cdots\varphi(x_n): + :\overbrace{\varphi(x_1)\varphi(x_2)}\varphi(x_3)\cdots\varphi(x_n):$$
$$+ :\overbrace{\varphi(x_1)\varphi(x_2)\varphi(x_3)}\cdots\varphi(x_n): \text{ etc. (single contractions)}$$
$$+ :\overbrace{\varphi(x_1)\varphi(x_2)}\overbrace{\varphi(x_3)\varphi(x_4)}: + \text{etc. (double contractions.}$$

+ triple contractions etc.

+ Either linear terms or terms containing no more φ's

$\langle 0|T(\varphi(x)\varphi(y))|0\rangle = \overbrace{\varphi(x)\varphi(y)}$

~~$\langle 0|T(\varphi(x)$~~ $\langle 0|T(\varphi_1\varphi_2\varphi_3\varphi_4)|0\rangle = \overbrace{\varphi_1\varphi_2}\overbrace{\varphi_3\varphi_4} + \overbrace{\varphi_1\varphi_2\varphi_3}\varphi_4 + \overbrace{\varphi_1\overbrace{\varphi_2\varphi_3}\varphi_4}$

as in above example.

Proof by induction; true for $n=1$, $n=2$.

Production Process: $\langle \vec{k} | U(t,t_0) | 0 \rangle = \langle 0 | a_k U(t,t_0) | 0 \rangle$

Proof of Wick's Thm. for $n=3$

$t_1 > t_2 > t_3$

$$T(\varphi(x_1)\varphi(x_2)\varphi(x_3)) = \varphi_1\varphi_2\varphi_3 = (\varphi_1^{(+)} + \varphi_1^{(-)})\varphi_2\varphi_3$$

$$= \varphi_1^{(-)}\varphi_2\varphi_3 + (\varphi_2\varphi_3]\varphi_1^{(+)} + [\varphi_1^{(+)}, \varphi_1\varphi_2]$$

By Wick's thm for $n=2$ $= \varphi_1^{(-)}\{:\varphi_2\varphi_3: + \overparen{\varphi_2\varphi_3}\} + \{:\varphi_2\varphi_3: + \overparen{\varphi_2\varphi_3}\}\varphi_1^{(+)} + [\varphi_1^{(+)}, \varphi_2\varphi_3]$

$$= :\varphi_1\varphi_2\varphi_3: + \varphi_1\overparen{\varphi_2\varphi_3} + \overparen{\varphi_1\varphi_2}\varphi_3 + \varphi_2\overparen{\varphi_1\varphi_3}$$

since $[\varphi_1^{+}, \varphi_2] = \langle 0 | [\varphi_1^{(+)}, \varphi_2] | 0 \rangle = \langle 0 | [\varphi_1^{(+)}, \varphi_2^{(-)}] | 0 \rangle$

$= \langle 0 | \varphi_1^{(+)}\varphi_2^{(-)} - \varphi_2^{(-)}\varphi_1^{(+)} | 0 \rangle$

$= \langle 0 | \varphi_1^{(+)}\varphi_2^{(+)} | 0 \rangle = \overparen{\varphi_1\varphi_2}$

So $T(\varphi_1\varphi_2,\varphi_3) = :\varphi_1\varphi_2\varphi_3: + \varphi_1\overparen{\varphi_2\varphi_3} + \overparen{\varphi_1\varphi_2}\varphi_3 + \varphi_2\overparen{\varphi_1\varphi_3}$

In general: assume $n-1, n-2$, can then prove n.

$\boxed{\text{Wicks Thm:}}$ $T(\varphi_1 \cdots \varphi_n) = :\varphi_1 \cdots \varphi_n: + \sum_{\text{single contractions}} :\overparen{\varphi_1\varphi_2}\varphi_3 \cdots \varphi_n: \cdots$

$+ \sum_{\text{double}} :\overparen{\varphi_1\varphi_2}\,\overparen{\varphi_3\varphi_4} \cdots : + \cdots$

+ Either ↗ n-even case: a c-number function. (completely contracted)
↘ n-odd: a function linear in φ_j's.

There are: $\frac{n(n-1)}{2}$ single contractions.

$\frac{1}{2}\frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}$ double contractions ....

$$\langle 0 | T(\varphi_1 \cdots \varphi_{(2n)}) | 0 \rangle = \overline{\overparen{\varphi_1\varphi_2}\,\overparen{\varphi_3\varphi_4} \cdots} \overparen{\varphi_{2n-1}\varphi_{2n}}$$
$$+ \overparen{\varphi_1\varphi_2\varphi_3}\varphi_4 \cdots \varphi_{2n-1}\varphi_{2n}$$

Total number of pairs: $(2n-1)(2n-3)\cdots 1 = (2n-1)!!$
$= \boxed{\frac{(2n)!}{2^n n!}}$

Look at term of order g^{2n} in $\langle 0 | U | 0 \rangle$

$$\frac{(-ig)^{2n}}{(2n)!}\int\int\cdots\int \prod_{j=1}^{2n} d^4_{x_j}\varphi(x_j) \langle 0 | T(\varphi(x_1)\cdots\varphi(x_{2n})) | 0 \rangle$$

since there are $\frac{(2n)!}{2^n n!}$ Δ_F terms from the contractions.

$$= \frac{(-ig)^{2n}}{\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{2^n n!} \iint \cdots \prod \rho(x_j) d^4x_j \; \overbrace{\varphi_1 \varphi_2} \overbrace{\varphi_3 \varphi_4} \cdots \overbrace{\varphi_{2n-1} \varphi_{2n}}$$

$$= -\left(\frac{g}{2}\right)^n \frac{1}{n!} \left[\iint \rho(x_1) \overbrace{\varphi_1 \varphi_2} \rho(x_2) \, d^4x_1 d^4x_2 \right]^n$$

$$= \left(-\frac{ig^2}{2}\right)^n \frac{1}{n!} \left[\iint_{t_0}^{t} \rho(x_1) \Delta_F(x_1 - x_2) \rho(x_2) \, d^4x_1 d^4x_2 \right]^n$$

So $\langle 0 | U(t, t_0) | 0 \rangle = \exp\left[-\frac{ig^2}{2} \iint_{t_0}^{t} \rho(x) \Delta_F(x-y) \rho(y) \, d^4x \, d^4y \right]$

Δ_F has an imaginary part, otherwise this would just be a phase-factor, nothing would happen.
This doesn't conserve energy, it produces quanta.

More generally: $U(t, t_0) = 1 - ig \int \rho(x) \varphi(x) d^4x + \frac{(-ig)^2}{2} \iint \rho(x_1) \rho(x_2) :\varphi(x_1) \varphi(x_2): d^4x_1 d^4x_2$

$$+ \frac{(-ig)^2}{2} \iint \rho(x_1) \rho(x_2) \overbrace{\varphi(x_1) \varphi(x_2)} \, d^4x_1 \, d^4x_2$$

$$+ \frac{(-ig)^3}{3!} \iint \rho(x_1) \rho(x_2) \rho(x_3) :\varphi(x_1) \varphi(x_2) \varphi(x_3): d^4x_1 d^4x_2 d^4x_3$$

$$+ \cdots$$

$$U(t, t_0) = :e^{-ig \int \rho(x) \varphi(x) d^4x}: \left\{ 1 - \frac{ig^2}{2!} \iint \rho(x_1) \Delta_F(x_1 - x_2) \rho(x_2) \, d^4x_1 d^4x_2 + \cdots \right\}$$

Problem set ① Show $U(t, t_0) = :e^{-ig \int_{t_0}^{t} \rho(x) \varphi(x) d^4x}: \; \underbrace{e^{-\frac{ig^2}{2} \int_{t_0}^{t} \int_{t_0}^{t} \rho(x) \Delta_F(x-y) \rho(y) d^4x d^4y}}_{\langle 0 | U(t - t_0) | 0 \rangle}$

② Demonstrate the Unitarity of this.

Pair theory: $\mathcal{H}_I = g \underbrace{V(x)}_{\text{externally specified, "potential"}} \psi^\dagger(x) \psi(x)$, $\psi(x)$ complex, spin-zero field,

$\langle 0 | \psi(x) | 0 \rangle = 0$, since $\psi(x) | 0 \rangle$ is state with lower charge, $\neq | 0 \rangle$
$\langle 0 | \psi(x) \psi(y) | 0 \rangle = 0 = \langle 0 | \psi^\dagger(x) \psi^\dagger(y) | 0 \rangle$

$$\langle 0 | \psi^\dagger(x) \psi(y) | 0 \rangle = \langle 0 | \sum_{k'} \left(b_{k'}^\dagger f_{k'}^{(-)}(x) + c_{k'} f_{k'}^{+}(x) \right) \cdot \sum_{k} \left(b_k f_k^{(+)}(y) \, c_k^\dagger f_k^{(-)}(y) \right) | 0 \rangle$$

$$= \sum_k f_k^{+}(x) f_k^{(-)}(y) = i \Delta^+(x-y)$$

$$\langle 0 | \psi^\dagger(x) \psi(y) | 0 \rangle = \sum_k f_k^{(+)}(x) f_k^{(-)}(y) = i \Delta^+(x-y)$$

$$\langle 0 | [\psi(x), \psi^\dagger(y)] | 0 \rangle = i\left(\Delta^{(+)}(x-y) - \Delta^{(+)}(y-x)\right) = i \left(\Delta^+(x-y) + \Delta^{(-)}(x-y)\right) = i \Delta(x-y)$$

So $\langle 0|T(\psi^{+}(x)\psi(y))|0\rangle = i\Delta_F(x-y)$

$$U(t,t_0) = 1 - ig\int_{t_0}^{t} v(x)\psi^{+}(x)\psi(x) + \frac{(-ig)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} v(x_1)v(x_2)T\{\psi_1^{+}\psi_1\ \psi_2^{+}\psi_2\}\,d^4x_1\,d^4x_2$$

Should have define $\mathcal{H}_I = g\,v(x) : \psi^{+}(x)\psi(x):$

But then we have to deal with products such as

$T\{:\psi_1^{+}\psi_1::\psi_2^{+}\psi_2:\}$

Use extension of Wick's theorem.

Define the fields $\psi'(x) = \psi^{(+)}(\vec{r}, t-\epsilon) + \psi^{(-)}(\vec{r}, t+\epsilon)$

$\psi^{+\prime}(x) = \psi^{+(+)}(r, t-\epsilon) + \psi^{+(-)}(\vec{r}, t+\epsilon)$

Instead of $T\{:\psi_1^{+}\psi_1:\cdots\}$, $T\{\psi_1^{+\prime}\psi_1^{\prime}, \cdots\}$

Then Use Wick's thm: $T\{\psi_1^{+\prime}\psi_1^{\prime}\cdots\} = :\psi_1^{+}\psi_1^{\prime}\cdots: + \sum_{\text{contractions}}$

But $\overbrace{\psi_i^{\prime}\psi_j^{\prime}} = \langle 0|T(\psi(x_i),\psi(x_j))|0\rangle = 0 = \overbrace{\psi_i^{+\prime}\psi_j^{\prime +}}$

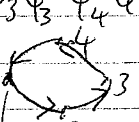
$\overbrace{\psi_j^{+\prime}\psi_j^{\prime}} = \langle 0|T(\psi^{+}(x_j)\psi^{\prime}(x_j))|0\rangle = \langle 0|:\psi^{+\prime}(x_j)\psi^{\prime}(x_j):|0\rangle = 0$

Let $\epsilon \to 0$

$$T(:\psi_1\psi_1^{+}::\psi_2^{+}\psi_2:) = :\psi_1^{+}\psi_1\psi_2^{+}\psi_2: + :\psi_1^{+}\psi_2:\overbrace{\psi_1\psi_2} + :\psi_1\psi_2^{+}:\overbrace{\psi_1^{+}\psi_2}$$
$$+ \overbrace{\psi_1^{+}\psi_2}\overbrace{\psi_1\psi_2^{+}}$$
$$= :\psi_1^{+}\psi_1\psi_2^{+}\psi_2: + \{:\psi_1^{+}\psi_2: + :\psi_2^{+}\psi_1:\}\,i\Delta_F(x_1-x_2)$$
$$+ (i\Delta_F(x_1-x_2)^2)$$

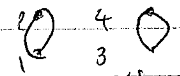
$\langle 0|T\{:\psi_1^{+}\psi_1:\cdots:\psi_4^{+}\psi_4:\}|0\rangle$ Typical term: $\psi_1^{+}\psi_1^{*}\psi_2^{+}\psi_2^{*}\psi_3^{+}\psi_3^{*}\psi_4^{+}\psi_4$

+ 5 other "single-cycle" terms.



① creating change at, arrows give direction of change transfer

$+\ \psi_1^{+}\psi_1\psi_2^{+}\psi_2\psi_3^{+}\psi_3\psi_4^{+}\psi_4$



+ 2 other 'double-cycle terms'

Assignment 3 due Dec. 14

$$\mathcal{H}_I = g\, v(x) :\psi^\dagger(x)\psi(x):$$

$$U(t,t_0) = 1 - ig\int_{t_0}^{t} v :\psi^\dagger\psi: d^4x + \frac{1}{2}(-ig)^2 \iint v_1 v_2\, T\{:\psi_1^\dagger\psi_1: :\psi_2^\dagger\psi_2:\}\, d^4x_1\, d^4x_2$$

$$+\frac{1}{3!}(-ig)^3 \iiint v_1 v_2 v_3\, T\{ \cdots \}\prod d^4x_j$$

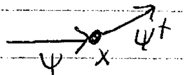
$$+ 1/4!\,(-ig)^4 \iiiint v_1 v_2 v_3 v_4\, T\{ \cdots \}\prod d^4x_j + \cdots$$

$$\langle 0|U(t,t_0)|0\rangle \approx \cdots + \frac{1}{3!}\iiint v_1 v_2 v_3 \{:\psi^\dagger\psi_1\, \psi_2^\dagger\psi_3\, \psi_3^\dagger\psi_3: + :\psi_1^\dagger\psi_1\, \psi_2^\dagger\psi_2\, \psi_3^\dagger\psi_3:\}$$

$$+\frac{1}{4!}(-ig)^4 \iiiint v_1 v_2 v_3 v_4 \{:\psi_1^\dagger\psi_1\, \psi_2^\dagger\psi_2\, \psi_3^\dagger\psi_3\, \psi_4^\dagger\psi_4: + 5 \text{ other cycles}$$

$$+ :\psi_1^\dagger\psi_1\, \psi_2^\dagger\psi_2\, \psi_3^\dagger\psi_3\, \psi_4^\dagger\psi_4: + 2 \text{ other double cycles}$$

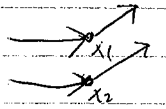
Diagrams:



1rst order

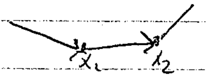
No reference to time sequence.
ψ: either annihilation of particle or creation of antiparticle,

2nd order:

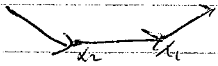


S=2 corresponds to $:\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2:$ $v(x_1)v(x_2)$ term, disconnected graph

Two diff. terms in expansion

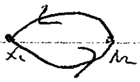


S=1 $:\psi_1^\dagger\psi_1\,\psi_2^\dagger\psi_2:\; v(x_1)v(x_2) = :\psi_1\psi_2^\dagger: \, i\Delta_F(x_1-x_2)\, v(x_1)v(x_2)$



S=1 $:\psi_1^\dagger\psi_1\,\psi_2^\dagger\psi_2:\; v(x_1)v(x_2) =$

Interchange of x_1, x_2 useless only one way of labelling, so one term

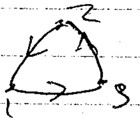


$\frac{2!}{2}$ graphs S=2 $\quad :\psi_1^\dagger\psi_1\,\psi_2^\dagger\psi_2:\; v(x_1)v(x_2) = i\Delta_F(x_1-x_2)^2$

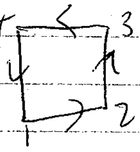
In QED by Furry's theorem these go to 0

There are 6 possible permutations $\frac{3!}{③}$ but only are distinct → symmetry #

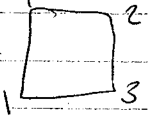
Vac Expectation: Third order terms:



Fourth order:



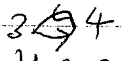
+



+ ... + 4 others. $\frac{4!\text{ permutations}}{④}$ symmetry # S=4

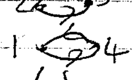
Symmetry is due to arbitrariness of the starting point. S = Symmetry #

+



+

+

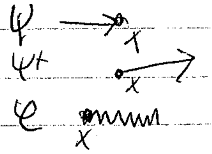


these are disconnected graphs

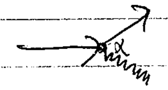
$\frac{4!\text{ permutations}}{\boxed{2\cdot2\cdot2}}$ symmetry # S=8

$\psi(x)$:- charged, mass m, spin zero field
$\varphi(x)$:- neutral, hermitian, spin zero field, mass μ

$\mathcal{H}_I = g :\psi^\dagger(x)\,\psi(x)\,\varphi(x):$ Diagrams:



"vertex":



This is analogous to a meson-coupling theory.
ψ a "nucleon" field, φ a "meson" field.
Then this would represent $N \to N + \varphi$ or $N + \varphi \to N$
or $\bar{N} \to \bar{N} + \varphi$ or $N + \bar{N} \to \varphi$
$\bar{N} + \varphi \to \bar{N}$ or $\varphi \to N + \bar{N}$
vacuum $\to N + \bar{N} + \varphi$
$N + \bar{N} + \varphi \to$ vacuum.

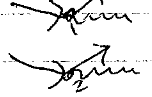
$H_I = \int \mathcal{H}_I = g \int :\psi^\dagger(x)\psi(x)\varphi(x): d\vec{r}$
this is invariant under spatial translation, so momentum is conserved. No explicit time-dependence $\Rightarrow$ Energy is conserved.
So the First order processes are forbidden by P^μ = constant for $t \to \infty$, $t_0 \to -\infty$

$$U(t,t_0) = 1 - ig \int :\psi^\dagger\psi\varphi: d^4x$$

$$+ \frac{(-ig)^2}{2!} \iint T\{:\psi_1^\dagger\psi_1\varphi_2\ \psi_2^\dagger\psi_2\varphi_2:\}\, d^4x_1\, d^4x_2 \ + \cdots$$

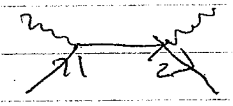
The contraction $\overbrace{\psi_1\varphi_2} = \langle 0 | T(\psi(x_1)\varphi(x_2)) | 0 \rangle = 0$ since operators commute

$$\frac{(-ig)^2}{2!} \iint :\psi_1^\dagger\psi_1\varphi_1\psi_2^\dagger\psi_2\varphi_2:\, d^4x_1\, d^4x_2 \Rightarrow S=2$$



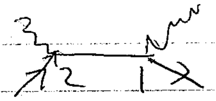
etc
a single disconnected diagram.
Repetition of a lowest-order graph so $= 0$

$$\frac{(-ig)^2}{2!} \iint :\overbrace{\psi_1^\dagger\psi_1\varphi_1\psi_2^\dagger\psi_2}\varphi_2:\, d^4x_1\, d^4x_2 \Rightarrow$$



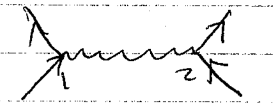
$S = 1$

$$\frac{(-ig)^2}{2!} \iint :\psi_1^\dagger\overbrace{\psi_1\varphi_1\psi_2^\dagger}\psi_2\varphi_2:\, d^4x_1\, d^4x_2 \Rightarrow$$

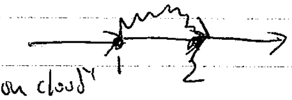


$S = 1$

These can describe: Meson-Nucleon scattering: $N + \varphi \to N + \varphi$
or $\bar{N} + \varphi \to \bar{N} + \varphi$ or $N + \bar{N} \to 2\varphi$ or $\varphi + \varphi \to N + \bar{N}$
These processes are permitted by conservation laws, so they have non-vanishing matrix elements.

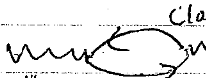
$\left(\frac{-ig}{2!}\right)^2 \iint :\psi_1^+\psi_1\varphi_1\psi_2^+\psi_2\varphi_2: d^4x_1\, d^4x_2 \Rightarrow$  s=2

For ex: $N+N \to N+N$, $N+\bar{N} \to N+$
$\bar{N}+\bar{N} \to \bar{N}+\bar{N}$

$\frac{(-ig)^2}{2!} \iint :\psi_1^+\psi_1\varphi_1\psi_2^+\psi_2\varphi_2: d^4x_1\, d^4x_2 \Rightarrow$  S=1

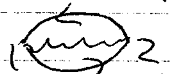
"meson cloud"

$\frac{(-ig)^2}{2!} \iint :\psi_1^+\psi_1\varphi_1\psi_2^+\psi_2\varphi_2: d^4x_1\, d^4x_2 \Rightarrow$  S=1

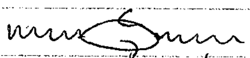
$\frac{(-ig)^2}{2!} \iint :\psi_1^+\psi_1\varphi_1\psi_2^+\psi_2\varphi_2: d^4x_1\, d^4x_2 \Rightarrow$  S=2

closed loops bubbles

"nucleon cloud": vacuum polarization.

$\frac{(-ig)^2}{2!} \iint :\psi_1^+\psi_1\varphi_1\psi_2^+\psi_2\varphi_2: d^4x_1\, d^4x_2 \Rightarrow$  s=2

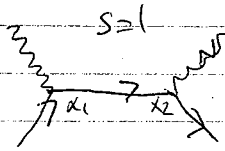
This becomes infinite, affects vaccuum exp., and diagonal matrix elemen

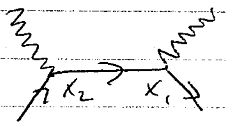
 Charge renormalization: $q_{eff} = q(1+O(g^2))$ $\varphi^0 \cdot g^2$

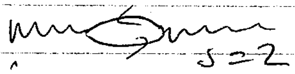
Vacuum polarization graph doesn't show up until third order, therefore this is a charge-renormalization graph.

mass of meson: $\mu^2_{eff} = \mu^2(1+O(g^2))$

S=1





 s=2

The integral Operators for the two above are the same.

Corresponding to a diagram D. of order n(D) i.e., N(D) vertices.
number of occurences $= \frac{n(D)!}{S(D)}$

Integral operator is $:O(D):$

$U = \sum U^{(n)}$, $U^{(n)}$ contains $\frac{1}{\cancel{n(D)!}}\frac{\cancel{n(D)!}}{S(D)} :O(D):$

For higher orders, Disconnected diagrams consist of connected ones.
Take a basis set of connected diagrams D_j $j=1,\dots\infty$
A general diagram D will contain D_j n_j times.
So D will be specified by $\{n_j\}$

So $:O(D): = :\prod_{j=1}^{\infty}\{:O(D_j):\}^{n_j}:$

U contains in order n(D) $\frac{1}{\cancel{n(D)}}\frac{\cancel{n(D)}}{S(D)} :O(D):$

What is $S(D)$? $S(D) = \prod_j \{S(D_j)\}^{n_j'} n_j!$

So U contains in order $n(D)$ $\frac{:\Theta(D):}{\prod_j \{S(D_j)\}^{n_j'} n_j!} = :\prod_{j=1}^{\infty} \left\{\frac{\Theta(D_j)}{S(D_j)}\right\}^{n_j} \cdot \frac{1}{n_j!}$

$$U(t,t_0) = :\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \prod_{j=1}^{\infty} \frac{1}{n_j!} \left\{\frac{\Theta(D_j)}{S(D_j)}\right\}^{n_j'}:$$
$$= \prod_{j=1}^{\infty} \sum_{n_j'=0}^{\infty} \frac{1}{n_j!} \left[\frac{\Theta(D_j)}{S(D_j)}\right]^{n_j'}$$
$$= :\prod_{j=1}^{\infty} e^{\frac{\Theta(D_j)}{S(D_j)}}: = :\exp\left(\sum_j \frac{\Theta(D_j)}{S(D_j)}\right):$$

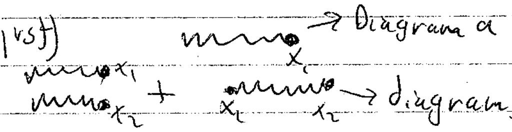
Main application: Statistical Mechanics.

See problem set.

Examples: ① $H_I = g\,\rho(x)\,\varphi(x)$

Lowest order: (1rst)

2nd order:



a) $S(a) = 1$ Dagram is: $(-ig)\int \rho(x)\,\varphi(x)\,d^4(x)$

b) $(-ig)^2 \int \rho(x_1)\, i\Delta_F(x_1 - x_2)\,\rho(x_2)\, d^4x_1\, d^4x_2$ $S(b) = 2$

Sol: $U(t,t_0) = :\exp\left(-ig\int_{t_0}^{t} \rho(x)\,\varphi(x)\,d^4x\right): \exp\left(-i\frac{g^2}{2}\iint_{t_0\,t_0}^{t\;\;t} \rho(x_1)\,\Delta_F(x_1-x_2)\,\rho(x_2)\,d^4x_1 d^4x_2\right)$

② Pair theory $H_I(x) = g\,V(x)\,\psi^{+}(x)\,\psi(x)$

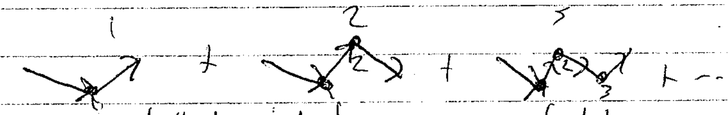
$\langle 0|U(t,t_0)|0\rangle = \langle 0|:e^{\sum_j \frac{\Theta(D_j)}{S(D_j)}}:|0\rangle$

Need only to return diagrams with no external legs (closed loops), the others give 0.

$v =$	2	3	4
Integral =	I_2	I_3	I_4
$S_j =$	2	3	4

$$\langle 0|U(t,t_0)|0\rangle = e^{\sum_{j=2}^{\infty} \frac{I_j}{j}}$$

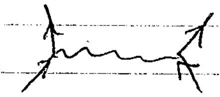
$\langle \vec{p}'|U(t,t_0)|p\rangle$ #



all contribute but vacuum bubble contribute also.

Nucleon ~~N~~ucleon scattering $N + N \to N + N$

~~of~~ order g^2



Initial state $|\vec{p}_1 \vec{p}_2\rangle$ $\vec{p}_1 \neq \vec{p}_2$ $|\vec{p}_1 \vec{p}_2\rangle = b^\dagger_{p_1} b^\dagger_{p_2} |0\rangle$

Final state $|p_1' p_2'\rangle$ $\underbrace{|p_1' p_2'\rangle = b^\dagger_{p_1'} b^\dagger_{p_2'} |0\rangle}$

Identical nucleons, can't tell which is which

these are eigenstates of non-interacting field Hamiltonian.

Want to find, to order g^2 $\langle p_1' p_2' | U(t, t_0) | p_1 p_2\rangle$ this is not really right unless coupling to meson field is infinitely weak.

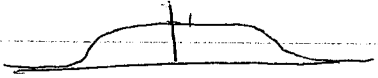
Let $t_0 \to -\infty$

$t \to \infty$

Other problem: we are turning on the interaction abruptly. This causes undamped oscillations in field.

Avoid by ~~[illegible]~~ turning on interaction adiabatically.

$g \to g\, f(t)$



$f(t)$:

sometimes: $f(t) = e^{-\epsilon|t|}$

$f(t) = \lim_{\epsilon \to 0^+} e^{-\epsilon|t|}$

Since this doesn't change integrals, forget about it.

Consider $\mathcal{H}_I(x) = g\, \overset{\text{static source.}}{\rho(\vec{r})}\, \varphi(x)\, f(t)$

This must leave the meson number unchanged.

$$U(t, t_0) = e^{-ig\int \rho(r) f(t) \varphi(x)\, d^4x} \cdot e^{\underbrace{-\frac{ig^2}{2}\iint \rho(r) f(t)\, \Delta_F(r-r', t-t')\, \rho(r') f(t')\, dr\, dt\, dr'\, dt}_{c\text{-number}}}$$

$$\varphi = \varphi^{(+)}_{\text{anni}} + \varphi^{(-)}_{(\text{creat})}$$

$$U(t, t_0) = e^{-ig\int \rho f \varphi^{(-)}(x) d^4x}\, e^{-ig\int \rho f \varphi^{(+)} d^4x} \quad \times\ c\text{-number}$$

$$U(t, t_0)|0\rangle = e^{-ig\int \rho f \varphi^{(-)}(x) d^4x} |0\rangle \quad \text{since } \varphi^{(+)}|0\rangle = 0$$

Powers of $\int \rho f \varphi^{(-)} d^4x$ create mesons.

For $t \to \infty$ $t_0 \to -\infty$ $\int_{-\infty}^{\infty} dt\, f(t) \int d\vec{r}\, \rho(r) \varphi(x) = \sum_k \int_{-\infty}^{\infty} dt\, e^{-\epsilon|t|} a^\dagger_k \int \frac{e^{-i(k \cdot r - \omega}}{\sqrt{2\omega_k V}}$

$\times\ \rho(r) dr$

If $\tilde{\rho}(k) \int e^{ik \cdot r} \rho(r)\, dr.$

$$~~Bd~~ = \sum_k \frac{\tilde{\rho}(k)^*}{\sqrt{2\omega_k V}} a^\dagger_k \int_{-\infty}^{\infty} dt\, e^{-\epsilon|t| + i\omega_k t}$$

$$= \sum_k \frac{\hat{\rho}^*(k)}{\sqrt{2\omega_k V}} a_k^\dagger \left\{ \overset{\int_{-\infty}^{0}}{\frac{1}{\epsilon - i\omega_k}} + \overset{\int_{0}^{\infty}}{\frac{1}{\epsilon + i\omega_k}} \right\}$$

$$= \sum_k \frac{\hat{\rho}^*(k}{\sqrt{2\omega_k V}} a_k^\dagger \frac{2\epsilon}{\epsilon^2 + \mu^2 + k^2} \to 0 \text{ as } \epsilon \to 0$$

So there is no meson production.

Without adiabatic switching $f(t) = 1$

Let $t_0 = -t$

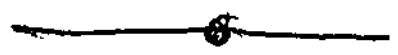
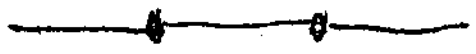
$$\text{Integral} = \sum_k \frac{\hat{\rho}^*(k)}{\sqrt{2\omega_k V}} a_k^\dagger \int_{-t}^{t} e^{i\omega_k t'} dt' = \sum_k \frac{\hat{\rho}^*(k)}{\sqrt{2\omega_k V}} a_k^\dagger \frac{2\sin\omega_k t}{\omega_k} \neq 0$$

So the switch-on produces these unphysical oscillations: In order for perturbation theory to make sense, have to use adiabatic switch on.

$\boxed{\text{Problem 3}}$: $U = :e^{\frac{\sum O(D_i)}{S(D_i)}}:$

As $t \to \infty$, $t_0 \to -\infty$, find $\langle k' | U(\infty, -\infty) | k \rangle$

so we need not bother with diagrams having more than two legs.

One-particle part =

lowest order
+

+ etc. - -

For every closed diagram, $O(D_j)$ will be a c-number, will factor away:

$$\langle k' | U(\infty, -\infty) | k \rangle = \langle k' | U_{1\,\text{particle}} | k \rangle \cdot \underbrace{\langle 0 | U(-\infty, \infty) | 0 \rangle}_{\text{for closed loops.}}$$

If we are careful about adiabatic switching this* must have modulus unity.

$U(\infty, -\infty)|0\rangle = \text{const}|0\rangle = e^{i\chi}|0\rangle$, χ real

So, in problem 3, just leave this as an undetermined phase factor.

Transition Probabilities

$\langle f|i\rangle = 0$

Initial state $|i\rangle$, final state $|f\rangle$ $\langle f|\mathcal{H}_I|i\rangle \neq 0$

Assume no diagonal matrix elements: $\langle i|\mathcal{H}_I|i\rangle = 0$

If not, carry over into $\mathcal{H}_0$

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathcal{H}_I(t')\,dt' + \cdots\cdots$$

$$\langle f|U(t,t_0)|i\rangle = \underset{=0}{\langle f|i\rangle} - i\int \langle f|\mathcal{H}_I(t')\,dt'|i\rangle\,dt'$$

$$W = \sum_{f\neq i} |\langle f|U(t,t_0)|i\rangle|^2 = \iint \langle i|\mathcal{H}_I(t)\sum_{f\neq i}|f\rangle\langle f|\mathcal{H}_I(t')|i\rangle\,dt\,dt'$$

= Total probability of transition.

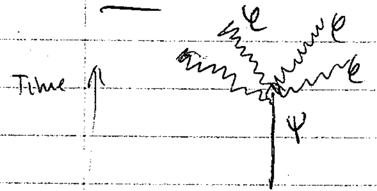
since $\langle i|\mathcal{H}_I|i\rangle = 0$ $\quad \sum_{\text{all } f} |f\rangle\langle f| = 1$

and

$$W = \langle i|\iint_{t_0}^{t} \mathcal{H}_I(t)\,\mathcal{H}_I(t')\,dt\,dt'|i\rangle$$

$$W = \langle i|\iint \mathcal{H}_I(x)\,\mathcal{H}_I(x')\,d^4x\,d^4x'|i\rangle$$ is invariant for $\mathcal{H}_I$ invariant. $|i\rangle$ invariant.

Ex:



Let $\mathcal{H}_I(x) = g\,\psi(x)\,\varphi^{(1)}(x)\,\varphi^{(2)}(x)\cdots\varphi^{(n)}(x)$

$\varphi^{(1)}\cdots\varphi^{(n)}$ are $\boxed{\text{indep. fields}}$ masses N_j

$m_\psi = m$

Decay of a ψ into n φ's

Initial state $|i\rangle = |1,0,0\cdots\cdots0\rangle$

Final state $|f\rangle = |0,1,1,1,1,1\rangle$

$\langle f|\mathcal{H}_I(x)|i\rangle$ contains a factor $\langle 0|\psi(x)|p\rangle$

$$W = g^2\iint d^4x\,d^4x'\,\langle i|\psi(x)\prod_j \varphi^{(j)}(x)\,\varphi^{(j)}(x')\,\psi(x')|i\rangle$$

$$= g^2\iint \langle p|\psi(x)\prod_j \underbrace{\langle 0|\varphi^{(j)}(x)\,\varphi^{(j)}(x')|0\rangle}\,\psi(x')|p\rangle\,d^4x\,d^4x'$$

$$= g^2\iint \langle p|\psi(x)\prod_{j=1}^{n} i\Delta_j^{(+)}(x-x')\,\psi(x')|p\rangle\,d^4x\,d^4x'$$

In order to have conservation laws, only annihilation part of ψ contributes $(\psi^{(+)}(x))$

$\psi^{(+)}|p\rangle$ must be a multiple of $|0\rangle$

$$\psi^{+}(x)\, a_p^{+}|0\rangle = [\psi^{(+)}(x), a^{+}]\,|0\rangle \pm a^{+}\cancel{\psi^{(+)}}(x)|0\rangle$$

$$= |0\rangle\langle 0|\psi(x)|p\rangle = \frac{e^{-ipx}}{\sqrt{2\omega_p V}}\,|0\rangle$$ for box Normalizat

$$W = \frac{g^2}{2\omega_p V}\iint e^{ip(x-x')}\prod_{j=1}^{n}\int_{k_j^0>0}\delta(k_j^2-\mu_j^2)\,e^{-ik_j(x-x')}\,d^4k_j\; d^4x\, d^4x'$$

The integration variable k_j is chosen to represent the final momenta.

$$W = \frac{g^2}{2\omega_p V}\int_{k_j^0>0}\int e^{i(p-\Sigma k_j)x}\,d^4x \cdot \int e^{-i(p-\Sigma k_j)x'}\,d^4x' \;\frac{1}{(2\pi)^{3n}}\prod_j \delta(k_j^2-\mu_j^2)\,d^4k_j$$

Since $\int e^{iPx}d^4x \int e^{-iPx'}d^4x' \rightsquigarrow [(2\pi)^4\delta^4(P)]^2$ for $V, T \to \infty$.

If you leave V, T finite.

$$\int e^{iPx}d^4x \int e^{-iPx'}d^4x' \to (2\pi)^4\,\delta^4(P)\,(V,T)$$

$P = p - \sum_j k_j$ This is what forces 4-energy conservation.

So $\frac{W}{T} \equiv$ Transition Rate $= w = \frac{g^2}{2\omega_p}\frac{1}{(2\pi)^{3n-4}}\int_{k_j^0>0}\delta^4(p-\Sigma k_j)\prod\delta(k_j^2-\mu^2)\,d^4k_j$

$w \propto \frac{1}{\omega_p}$, the particle energy, this expresses time-dilation since an increase in $\omega_p(E)$ decreases the transition rate.

$$w = \frac{g^2}{2\omega_p}\frac{1}{(2\pi)^{3n-4}}\int \underset{\uparrow\ \text{momentum conservation}}{\delta^3(\vec{p}-\Sigma\vec{k}_j)}\;\underset{\uparrow\ \text{Energy Cons.}}{\delta(\omega_p-\Sigma\omega_{k_j})}\;\underset{\uparrow\ \text{all possible decay particles}}{\prod_{j=1}^{n}\frac{d^3k_j}{2\omega_{k_j}}}$$

Changes in above:

Define $P_\Omega = \sum_\Omega |f\rangle\langle f|$

Then $W_\Omega = \langle i|\iint_{T_0}^{t} H_I(t)\,P_\Omega\,H_I(t')\,dt\,dt'\,|i\rangle$

$H_I = g\int \psi(x)\,\varphi^{(1)}(x)\cdots\varphi^{(n)}(x)\,d\vec{r}$

In order to have conservation laws, the $\psi(x)$ field must be annihilated.

If P is a projection on the ψ-vacuum, what occurs on the expression is,

$$\langle 0|\psi(x')|p\rangle$$

$$\int e^{iPx} d^4x \int e^{-iPy} d^4y = [\delta(P)]^2 \text{ not defined}$$

But for V, T finite

$$\int d^4P \int e^{iPx} d^4x \int e^{-iPy} d^4y = (2\pi)^4 \int \delta^4(x-y)\, d^4x\, d^4y = (2\pi)^4 V \cdot T$$

$$\text{so } \left|\int e^{iPx} d^4x\right|^2 = (2\pi)^4\, \delta^4(P)\, V \cdot T$$

For continuum normalization

$$\langle 0|\,\psi(x)\,|p\rangle = \langle 0|\,\psi(x)\, a^\dagger(p)\,|0\rangle \qquad \text{since } \psi = \psi^{(+)} + \psi^{(-)}$$

$$\langle 0|\,\psi^{(-)} = 0$$

$$\text{so} \qquad = \langle 0|\,\psi^{(+)}(x)\, a^\dagger(p)\,|0\rangle$$

$$= \langle 0|\,[\psi^{(+)}(x), a^\dagger(p)]\ \cancel{+a^\dagger}\psi^{(+)}\,|0\rangle$$

$$= [\psi^{(+)}(x), a^\dagger(p)]$$

invariant normalization

$$\psi^{(+)}(x) = \int \frac{1}{(2\pi)^3\, 2\omega_{p'}}\, a(\vec{p}')\, d\vec{p}' \cdot \underbrace{e^{-ip'x}}_{F^{(+)}_{p'}(x)}$$

$$[a(p), a^\dagger(p')] = 2\omega_p (2\pi)^3\, \delta(\vec{p} - \vec{p}')$$

$$\text{So } \langle 0|\,\psi(x)\,|p\rangle = F_p^{(+)}(x) = e^{-ipx}$$

Box normalization: $\langle 0|\,\psi(x)\,|p\rangle = \dfrac{e^{-ipx}}{\sqrt{2\omega_p V}}$

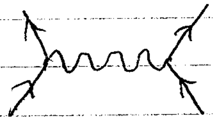
For continuum normalization: $\dfrac{W}{VT} = g^2 \dfrac{1}{(2\pi)^{3n-1}} \displaystyle\int_{k_j^0 > 0} \delta^4\!\left(\vec{p} - \Sigma \vec{k}_j\right) \prod_{j=1}^{n} \delta(k_j^2 - \mu_j^2)\, d^4k_j$

You have to find the density of ψ's, $2\omega_p$, 4th component of current-density 4-vector.

Decay rate $= \dfrac{W}{VT}\dfrac{1}{2\omega_p} = \dfrac{W}{VT}\dfrac{1}{2m}\left(\dfrac{m}{\omega_p}\right) \rightarrow$ time dilation factor.

Nucleon-Nucleon Scattering Cross-Sections

$N + N \rightarrow N + N$



$$U^{(2)}_{NN}(\infty,-\infty) = \frac{(-ig)^2}{2}\iint :\psi_1^{\dagger}\psi_1\psi_2^{\dagger}\psi_2: \, i\Delta_F(x_1-x_2)\, d^4x_1\, d^4x_2$$

Initial state: $|\vec{p}_1\,\vec{p}_2\rangle$, $\vec{p}_1 \neq \vec{p}_2$ $\quad |\vec{p}_1{}', \vec{p}_2{}'\rangle$, $\vec{p}_1{}'\vec{p}_2{}' \neq \vec{p}_1, \vec{p}_2$

$$\langle p_1' p_2' | U^{(2)}_{NN} | p_1 p_2\rangle = \frac{(-ig)^2}{2}\iint \langle p_1' p_2' | :\psi_1^{\dagger}\psi_1\psi_2^{\dagger}\psi_2: | p_1 p_2\rangle \, i\Delta_F(x_1-x_2)\, d^4x_1\, d^4x_2$$

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_p}\{b(p)F_p^{(+)}(x) + c^{\dagger}(p)F_p^{(-)}(x)\}\, d\vec{p}$$

$$\psi^{\dagger}(x) = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_p}\{b^{\dagger}(p)F_p^{(-)}(x) + c(p)F_p^{(+)}(x)\}\, d\vec{p}$$

$b(p)$, $c(p)$ are independent, nucleon anti-nucleon.

For the c's, since there are no anti-nucleons in the final or initial states, we are just taking the vac expectation of a normal-ordered expression of them, which is 0.

So: $\psi(x) \rightarrow \psi^{(+)}(x)$ $\quad \psi^{\dagger}(x) \rightarrow \psi^{\dagger(-)}(x)$

$\psi^{(+)}$ = annihilator

So we want:

$$\langle p_1' p_2' | :\psi_1^{\dagger(-)}\psi_1^{(+)}\psi_2^{\dagger(-)}\psi_2^{(+)}: | p_1 p_2\rangle$$

$$= \langle p_1' p_2' | \psi_1^{\dagger(-)}\psi_2^{\dagger(-)}\psi_1^{(+)}\psi_2^{(+)} | p_1 p_2\rangle$$

But $\psi_1^{(+)}\psi_2^{(+)}|p_1 p_2\rangle = \psi_1^{(+)}\psi_2^{(+)}b^{\dagger}(p_1)b^{\dagger}(p_2)|0\rangle$

$$= \psi_1^{(+)}\{[\psi_2^{(+)}, b^{\dagger}(p_1)b^{\dagger}(p_2)] + b^{\dagger}(p_1)b^{\dagger}(p_2)\cancel{\psi_2^{(+)}}\}|0\rangle$$

$$= \psi_1^{(+)}\{b^{\dagger}(p_1)[\psi_2^{(+)}, b^{\dagger}(p_2)] + [\psi_2^{(+)}, b^{\dagger}(p_1)]b^{\dagger}(p_2)\}|0\rangle$$

$$= \{[\psi_1^{(+)}, b^{\dagger}(p_1)][\psi_2^{(+)}, b^{\dagger}(p_2)] + [\psi_1^{(+)}, b^{\dagger}(p_2)][\psi_2^{(+)}, b^{\dagger}(p_1)]$$

$$= \{F^{+}_{p_1}(x_1)F^{+}_{p_2}(x_2) + F^{(+)}_{p_2}(x_1)F^{(+)}_{p_1}(x_2)\}|0\rangle$$

$$\langle p_1' p_2' | \psi_1^{+(-)} \psi_2^{+(-)} = \langle 0 | \{ F_{p_1'}^{(-)}(x_1) F_{p_2'}^{(-)}(x_2) + F_{p_2'}^{(-)}(x_1) F_{p_1'}^{(-)}(x_2) \}$$

$$S_0 ; \langle p_1' p_2' | U_{Nn}^{(2)} | p_1 p_2 \rangle = \frac{(-ig)^2}{2} \iint \{ F_{p_1'}^{(-)}(x_1) F_{p_2'}^{(-)}(x_2) \oplus F_{p_2'}^{(-)}(x_1) F_{p_1'}^{(-)}(x_2)$$

$$\times F_{p_1}^{(+)}(x_1) F_{p_2}^{(+)}(x_2) + F_{p_2}^{(+)}(x_1) F_{p_1}^{(+)}(x_2) \} \, i\Delta_F(x_1 - x_2) \, d^4x_1 \, d^4$$

Since x_1, x_2 symmetry:

$$\langle p_1' p_2' | U_{NN}^{(2)} | p_1 p_2 \rangle = 2 \times \frac{(-ig)^2}{2} \iint F_{p_1'}^{(-)}(x_1) F_{p_2'}^{(-)}(x_2) \, i\Delta_F(x_1 - x_2) F_{p_1}^{(+)}(x_1) F_{p_2}^{(+)}(x_2) \, d^4x_1 \, d^4x_2$$

$$+ 2 \times \frac{(-ig)^2}{2} \iint F_{p_2'}^{(-)}(x_1) F_{p_1'}^{(-)}(x_2) \, i\Delta_F(x_1 - x_2) F_{p_1}^{(+)}(x_1) F_{p_2}^{(+)}(x_2) \, d^4x_1 \, d^4x_2$$

The firstterm is the "direct integral"

$$= \int F^{(-)} F^{(-)} i\Delta_F F^{(+)} F^{(+)} d^4x_1 \, d^4x_2$$

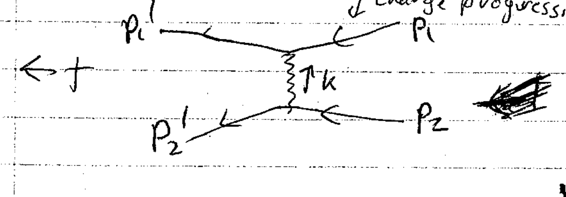
$$= \int e^{i(p_1' x_1 + p_2' x_2)} \left[\frac{i}{(2\pi)^4} \int \frac{e^{-ik(x_1 - x_2)}}{k^2 - \mu^2 + i\epsilon} \right] e^{-i(p_1 x_1 + p_2 x_2)} \, d^4x_1 \, d^4x_2 \, d^4k$$

$$= \int (2\pi)^4 \delta^4(p_1' - k - p_1) \frac{i}{(2\pi)^4} \frac{1}{k^2 - \mu^2 + i\epsilon} (2\pi)^4 \delta^4(p_2' + k - p_2) \, d^4k$$

$$= (2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) \frac{i}{(p_1' - p_1)^2 - \mu^2 + i\epsilon}$$

"Exchange" Integral: $= (2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) \dfrac{i}{(p_2' - p_1)^2 - \mu^2 + i\epsilon}$

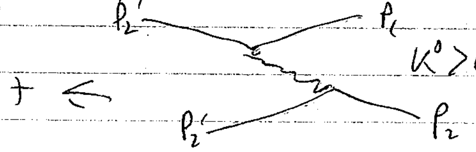
Diagrams in momentum space: p's are associated with legs, not vertices.



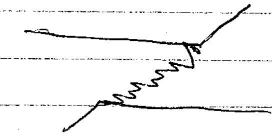
Energy-Momentum conservation holds at each vertex.

But there is no conservation of rest mass. So the meson is virtual, it fails to conserve energy. The way the meson is going is undetermined.

Two possibilities

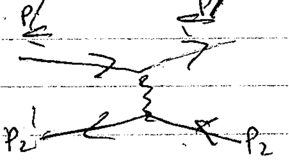


or

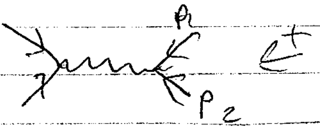


Both possibilities are being integrated over.

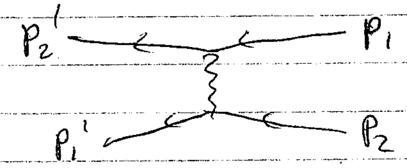
For Nucleon-Anti-Nucleon scattering:



also, instead of exchange diagram, have

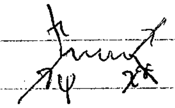


Exchange Diagram:



$$\boxed{\langle p_1' p_2' | U(\infty,-\infty) | p_1 p_2 \rangle = (2\pi)^4 i\, \delta^4(p_1'+p_2'-p_1-p_2) \times (-ig)^2 \left\{ \frac{1}{(p_1-p_1')^2-\mu^2+i\epsilon} + \frac{1}{(p_1-p_2')^2-\mu^2+i\epsilon} \right\}}$$

If there were two different types of nucleons here:



$$\mathcal{H}(x) = g(\psi^\dagger\psi + \chi^\dagger\chi)\varphi$$

$$U^{(2)} = \frac{(-ig)^2}{2!}\iint :\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2 + \chi_1^\dagger\chi_1\psi_2^\dagger\psi_2: i\Delta_F(x_1-x_2)\, d^4x_1\, d^4x_2$$

$$= \frac{(-ig)^2}{\cancel{2!}}\iint :\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2: i\Delta_F(x_1-x_2)\, d^4x_1\, d^4x_2$$

Initial state $= |p_\psi\, p_\chi\rangle$
So only 1 of the four cross terms can contribute, unlike previous example.

So the answer is:

$$\langle p_\psi' p_\chi' | U(\infty,-\infty) | p_\psi p_\chi \rangle = (2\pi)^4 i\, \delta^4(p_\psi'+p_\chi'-p_\psi-p_\chi) \times (ig)^2 \left\{ \frac{1}{(p_\psi-p_\psi')^2-\mu^2+i\epsilon} \right\}$$

: No Counter-term.

$S = U(\infty, -\infty)$: Heisenberg, 30's: Only S-matrix elements can be measured.

$S_{fi} = \langle f | U(\infty, -\infty) | i \rangle$

$$H_I = \frac{1}{2} \int \psi^\dagger(r,t)\, \psi(r,t)\, V(\vec{r} - \vec{r}\,')\, \psi^\dagger(r',t)\, \psi(r',t)\; dr\, dr'$$

Effective direct interaction (nonlocal – action at a distance)

For the Yukawa potential $V(r) = -g^2 \frac{e^{-\mu r}}{4\pi r}$ → Very likely asymptotic form for nuclear force.

$\langle p_1' p_2' | U(\infty, -\infty) | p_1 p_2 \rangle =$

$$S_{fi} = \langle f | U(\infty, -\infty) | i \rangle = \underset{\langle f|i\rangle}{\delta_{fi}} + (2\pi)^4 i\, \underset{\text{total final + initial 4-moment}}{\delta^4(P_f - P_i)}\, \underset{\text{T matrix elemen}}{\boxed{T_{fi}}}$$

Before the limit $V \to \infty$, $T \to \infty$ $\delta^4(p) = \frac{1}{(2\pi)^4} \int_{VT} e^{-iPx}\, d^4x$

Transition probability to a set of final states $\{|f\rangle\}$, $\langle f|i\rangle = 0$

$$W = \sum_{f\in i} |S_{fi}|^2 = \sum_{f\in i} [(2\pi)^4 \delta^4(P_f - P_i)]^2\, |T_{fi}|^2$$

$$\frac{W}{VT} = \sum_{f\in i} (2\pi)^4 \delta^4(P_f - P_i)\, |T_{fi}|^2 \to \text{must be an invariant.}$$

$$= \frac{\text{Reaction rate}}{\text{unit volume and time}} \quad \text{(an invariant)}$$

Kinematics Cross sections: $\xrightarrow{\text{a particle}} \boxed{b}$

$$\underset{\substack{\text{density of}\\ \text{particles a}}}{\ } \frac{\text{Reaction rate}}{\text{particle b}} = \text{Flux of } a \cdot \sigma \to \text{cross section "target area".}$$

Flux of a $= \rho_a V$ (density of particles a) → relative velocity.

(# of Reactions) ~~rate~~ / unit vol · unit time $= \rho_a \rho_b V \sigma = W/VT$

In the ~~frame~~ : b stationary $\xrightarrow[\rho_a,\, \vec{j}_a]{} \boxed{b} \to \rho(b)$

In the frame : a stationary $\rho_a \to \rho_a'$ $\rho_b \to \rho_b'$

$(\rho, \vec{j}/c)$, a 4-vector

$\rho_b \to \rho_b' = \frac{\rho_b}{(1 - V^2)}$ $\rho_a \to \rho_a' = \frac{\rho_a - V j_a}{\sqrt{1 - V^2}} = \frac{\rho_a (1 - V^2)}{\sqrt{1 - V^2}} = \rho_a \sqrt{1 - V^2}$

Since $j_a = \rho_a \cdot V$

So $\rho_a' \rho_b' = \rho_a \rho_b$ $V' = V$ So: $\sigma' = \sigma$

σ ~~must~~ [illegible] ~~for~~ is a Lorentz Invariant. (in this case)

Note Invariance of $(j_a j_b) = j_a^\nu j_{b\nu} = \rho_a \rho_b - \vec{j}_a \cdot \vec{j}_b$

<u>General definition</u> $\frac{W}{VT} =$ [illegible] $j_a j_b \, v \, \sigma$

Where v is an invariant number which is equal to the relative velocity in the two rest systems, but not so in general.

$p_a^\nu p_{b\nu} =$ invariant $= \frac{m_a}{\sqrt{1-v^2}} m_b$

so $1 - v^2 = \left(\frac{m_a m_b}{p_a p_b}\right)^2$ $\quad v = \sqrt{1 - \left(\frac{m_a m_b}{p_a p_b}\right)^2}$

<u>an invariant</u>

$p_a p_b v = \sqrt{(p_a p_b)^2 - m_a^2 m_b^2}$

$$\boxed{\sigma = \frac{1}{(j_a j_b) v} \overbrace{\sum_{f \neq i} (2\pi)^4 \delta^4(P_f - P_i) |T_{fi}|^2}^{W/VT}}$$

Invariant normalization: $j_{a\nu} = i F_{p_a}^* \overleftrightarrow{\partial_\nu} F_{p_a}$ $\to$ positive energy sols to K-G eq.

$$\boxed{j_{a\nu} = 2 p_{a\nu}}$$

So $j_a j_b = 4 (p_a \cdot p_b)$, $(j_a j_b) v = 4 \sqrt{(p_a p_b)^2 - m_a^2 m_b^2}$

In the Center of Mass system: $\vec{p}_a = -\vec{p}_b$ $\quad E_a = p_a^0 = \sqrt{p^{*2} + m_a^2}$

$\downarrow \qquad E_b = p_b^0 = \sqrt{p^{*2} + m_b^2}$

$= \vec{p}^* =$ internal momentum in CM system

$(j_a j_b) v = \cancel{4} \sqrt{(E_a E_b + p^{*2})^2 - m_a^2 m_b^2}$

$= 4 \sqrt{(p^{*2} + m_a^2)(p^{*2} + m_b^2) + 2 p^{*2} E_a E_b + p^{*4} - m_a^2 - m_b^2}$

$= 4 p^* \sqrt{p^{*2} + m_a^2 + p^{*2} + m_b^2 + 2 E_a E_b}$ $\quad \nearrow$ total E in c.m. system.

$= 4 p^* \sqrt{E_a^2 + E_b^2 + 2 E_a E_b} \quad = 4 p^* (E_a + E_b) =$

Construct: Mandelstam variables:

$s = (p_a + p_b)^2 = (E_a + E_b)^2 - \cancel{(\vec{p}_a + \vec{p}_b)}^2$ $\nearrow$ 0 in C.M.

$s = m_a^2 + m_b^2 + 2 (p_a p_b)$

so $(j_a j_b)v = 4p^*\sqrt{s}$ an invariant.

$$\sigma = \frac{1}{4p^*\sqrt{s}} \sum_{f\neq i} (2\pi)^4 \delta^4(P_f - P_i)\, |\mathcal{T}_{fi}|^2$$

* * *

$S \equiv U(\infty, -\infty) = 1 + R$ $\qquad$ P_f = sum of final moment

$S_{fi} \equiv \langle f|S|i\rangle = \langle f|R|i\rangle$ for $i \neq f$

Write $\langle f|R|i\rangle \equiv i(2\pi)^4 \delta^4(P_f - P_i)\, \mathcal{T}_{fi}$

$$\sigma = \frac{1}{(j_a\cdot j_b)V} \sum_{f\neq i} (2\pi)^4 \delta^4(P_f - P_i)\, |\mathcal{T}_{fi}|^2$$ where j_a, j_b are the two colliding currents.

$j_a\cdot j_b = 4(p_a\cdot p_b)$

$(p_a p_b)v = \sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2} = \overset{\nearrow \text{(C.M. momentum)}}{p^*} \cdot (E_a + E_b)_{cm}$

Using $s = (p_a + p_b)^2 = (E_a + E_b)^2_{cm}$, since spatial components are 0 in C.M. frame.

$$\sigma = \frac{1}{4p^*\sqrt{s}} \sum_{f\neq i} (2\pi)^4 \delta^4(P_f - P_i)\, |\mathcal{T}_{fi}|^2$$

P_1 = Projection operator on 1-particle subspace $= \frac{1}{(2\pi)^3} \int |\vec p\rangle\langle \vec p|\frac{d\vec p}{2\omega_p}$

$= \frac{1}{(2\pi)^3} \int_{p^0>0} |\vec p\rangle\langle\vec p|\,\delta(p^2 - m^2)\, d^4p$

Remember:
$\langle p|p'\rangle = (2\pi)^3 \cdot 2\omega_p\, \delta(p - p')$

$|i\rangle = |\vec p_a \vec p_b\rangle \qquad |f\rangle = |\vec q_1, \cdots \vec q_n\rangle$ = n. particle state

Assume, for now, all particles are different.

$\sum_{f\neq i} \delta^4(P_f - P_i)\, |\mathcal{T}_{fi}|^2 = \langle i|\mathcal{T}^+ \delta^4(P_{op} - P_i) \sum_f |f\rangle\langle f|\mathcal{T}|i\rangle$

→ should be $f\neq i$, but we ignore this.

$= \langle i|\mathcal{T}^+ \delta^4(P_f - P_i) \prod_{j=1}^{n} P_{1j}\, \mathcal{T}|i\rangle$

$= \langle i|\mathcal{T}^+ \delta^4(P_{op} - P_i) \frac{1}{(2\pi)^{3n}} \prod_{j=1}^{n} \int_{q_j^0>0} |q_j\rangle\langle q_j|\,\delta(q_j^2 - m_j^2)\, d^4q_j\, \mathcal{T}|i\rangle$

$= \frac{1}{(2\pi)^{3n}} \int_{\{q_j^0>0\}} \cdots \int \delta^4(\Sigma q_j - p_a - p_b)\, |\langle \vec q_1 \cdots \vec q_n|\mathcal{T}|\vec p_a\, p_b\rangle|^2 \prod_j \delta(q_j^2 - m_j^2)\, d^4q_j$

So $$\boxed{\sigma = \frac{1}{4p^*\sqrt{s}} \frac{1}{(2\pi)^{3n-4}} \iint_{\{q_j^0>0\}} \delta^4(\Sigma q_j - p_a - p_b)\, |\langle \vec q_1 \cdots \vec q_n|\mathcal{T}|\vec p_a\, p_b\rangle|^2 \prod_j \delta(q_j^2 - m_j^2)\, d^4q_j}$$

<u>Two-Body Reactions</u>: $a + b \to c + d$

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{4p^*\sqrt{s}} \iint \delta^4(p_c + p_d - p_a - p_b)\, |\langle p_c\, p_d|\mathcal{T}|p_a\, p_b\rangle|^2\, \underbrace{\frac{d\vec p_c}{2\omega_{p_c}} \frac{d\vec p_d}{2\omega_{p_d}}}_{2p_c^0 \;\, 2p_d^0}$$

In the CM frame: $\vec{p}_a + \vec{p}_b = 0$ $\quad \vec{p}_a = \vec{p}^*$ ↓ due to change in rest mass.

$p_d^0 = \sqrt{p_c^2 + m_d^2}$ $\quad \vec{p}_c + \vec{p}_d = 0$ $\quad \vec{p}_c = p'^*$

$$\sigma = \frac{1}{(2\pi)^2}\frac{1}{4p^*\sqrt{s}} \iint \delta(p_c^0 + p_d^0 - p_a^0 - p_b^0)\, |\langle \vec{p}_c, -\vec{p}_c | \mathcal{T} | \vec{p}_a, -\vec{p}_a\rangle|^2 \frac{p_c^2\, d\Omega_c\, dp_c}{4p_c^0 p_d^0}$$

We can either ~~[illegible]~~ get an integral over p_c, giving final-momentum distribution, or integral over $d\Omega_c$, giving angular distribution.

$$\sigma = \frac{1}{(2\pi)^2}\frac{1}{4p^*\sqrt{s}}\frac{1}{4p_c^0 p_d^0}\frac{p_c^2}{4p_c^0 p_d^0}\frac{1}{\frac{p_c}{p_c^0}+\frac{p_c}{p_d^0}} \int |\langle p_c, -p_c | T | p_a, -p_a\rangle|^2\, d\Omega_c$$

Using the identity: $\int \delta(f(x))\,dx = \int \delta(f'(x-x_0))\,dx$ where $f(x_0)=0$ a root

$= 1/|f'|_{x=x_0}$

since $p_c = p'^*$

$$\sigma = \frac{1}{(2\pi)^2}\frac{1}{4p^*\sqrt{s}}\frac{p'^*}{4\underbrace{(p_c^0+p_d^0)}_{\sqrt{s}}}\int |\langle p_c - p_c | \mathcal{T} | p_a - p_a\rangle|^2\, d\Omega_c$$

$$= \frac{1}{64\pi^2 s}\frac{p^{*\prime}}{p^*}\int |\langle \vec{p}_c, -\vec{p}_c | \mathcal{T} | \vec{p}_a, -\vec{p}_a\rangle|^2\, d\Omega_c$$

$$\text{or } \frac{d\sigma}{d\Omega_c} = \frac{1}{64\pi^2 s}\frac{p^{*\prime}}{p^*} |\langle \vec{p}_c - \vec{p}_c | \mathcal{T} | \vec{p}_a, \vec{p}_a\rangle|^2$$

Consider the case $a=c$, $b=d$: Elastic Scattering

Then $\frac{d\sigma}{d\Omega_c} = \frac{1}{64\pi^2 s} |\langle p_c - p_c | \mathcal{T} | p_a - p_a\rangle|^2$ ← this is not invariant since $d\Omega_c$ is not an invariant.

Analogy to non-relativistic fixed potential problem: phase factor $= \pm 1$

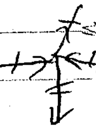
there $\frac{d\sigma}{d\Omega} = |f|^2$ so here $f(p,\theta) = \frac{1}{8\pi\sqrt{s}}\langle \vec{p}_c - \vec{p}_c | T | \vec{p}_a - \vec{p}_a\rangle$ ~~i~~ x real

Define $t = (p_a' - p_a)^2$ an invariant version of the momentum transfer.

$= \cancel{(p_a'^0 - p_a^0)^2} - (\vec{p}_a' - \vec{p}_a)^2$

$-(\text{p transfer})^2$ $\quad 0$

So $t = -(\vec{p}_a' - \vec{p}_a)^2$ since


← must be equal

$= -(p_a'^2 + p_a^2 - 2p_a^2\cos\theta_a)$ $\qquad \theta_a$ = scattering angle

$= -2p_a^2(1 - \cos\theta_a)$

$dt > 2p_a^2\, d\cos\theta_a = -2p_a^2 \sin\theta_a\, d\theta_a$

$-2\pi\, dt = 2p_a^2(2\pi \sin\theta_a\, d\theta_a) = 2p_a^2\, d\Omega_a$

Here $\frac{d\sigma}{d\Omega_a} = \frac{p_a^2}{\pi} \frac{d\sigma}{d|t|}$

$$\frac{d\sigma}{d|t|} = \frac{\pi}{p_a^2}\frac{d\sigma}{d\Omega} = \frac{1}{64\pi p^{*2} s} \left| \langle p_a', -p_a' | \mathcal{T} | p_a, -p_a \rangle \right|^2$$

S matrix is unitary: $SS^\dagger = 1$

$$\sum_\beta S^\dagger_{\alpha\beta} S_{\beta\gamma} = \delta_{\alpha\gamma} \quad \text{or} \quad \sum_\beta S^*_{\beta\alpha} S_{\beta\gamma} = \delta_{\alpha\gamma}$$

where $S_{\beta\gamma} = \delta_{\beta\gamma} + i(2\pi)^4 \delta^4(P_\beta - P_\gamma)\mathcal{T}_{\beta\gamma}$

$$\text{so } \sum_\beta S^*_{\beta\alpha} S_{\beta\gamma} = \cancel{\delta_{\alpha\gamma}} + (2\pi)^4 i \delta^4(P_\alpha - P_\gamma)\{T_{\alpha\gamma} - \mathcal{T}_{\gamma\alpha}^*\}$$

$$+ (2\pi)^8 \sum_\beta \underbrace{\delta^4(P_\beta - P_\alpha)\delta^4(P_\beta - P_\gamma)}_{\delta^4(P_\alpha - P_\gamma)\,\delta^4(P_\beta - P_\gamma)} \mathcal{T}^*_{\beta\alpha} T_{\beta\alpha} = \cancel{\delta_{\alpha\gamma}}$$

If we take only states where $P_\alpha = P_\gamma$

then $$\boxed{-i\{T_{\alpha\gamma} - \mathcal{T}^*_{\gamma\alpha}\} = (2\pi)^4 \sum_\beta \delta^4(P_\beta - P_\alpha)\mathcal{T}^*_{\beta\alpha} T_{\beta\gamma}}$$

This is the Unitarity identity for the $\mathcal{T}$-matrix. $\rightarrow$ conservation of orthogonality.

Special case $\gamma = \alpha$

$$-i\{T_{\alpha\alpha} - T^*_{\alpha\alpha}\} = 2\,\mathrm{Im}\,T_{\alpha\alpha} = (2\pi)^4 \sum_\beta \delta^4(P_\beta - P_\alpha) |\mathcal{T}_{\beta\alpha}|^2$$

$$2\,\mathrm{Im}\,T_{\alpha\alpha} = \frac{W}{VT} = \frac{\text{Total number of events}}{\text{Vol}\cdot\text{Time}}$$

$$= (j_a j_b) V \sigma = 4(p_a p_b) V \sigma$$

So $$\boxed{\sigma_T = \frac{1}{2(p_a p_b)V}\,\mathrm{Im}\,\mathcal{T}_{\alpha\alpha} = \frac{1}{2p^*\sqrt{s}}\,\mathrm{Im}\,\mathcal{T}_{\alpha\alpha}}$$: The Optical Theorem

This expresses the conservation of probability (unitarity)

Since, from last page: $f(p,\theta) = \frac{1}{8\pi\sqrt{s}} \langle p_a' - p_a' | \mathcal{T} | p_a, -p_a \rangle$

$$\mathrm{Im}\,f(p\theta) = \tfrac{1}{8\pi\sqrt{s}}\,\mathrm{Im}\langle p_a, -p_a | \mathcal{T} | p_a, -p_a \rangle$$

So $\sigma = \frac{1}{2p_a\sqrt{s}} 8\pi\sqrt{s}\,\mathrm{Im}\,f(p,\theta) = \frac{4\pi}{p_a}\,\mathrm{Im}\,f(p,\theta)$ same as in non-relativistic scattering.

Final states with identical particles:

2 particles: $|\vec{k}_1\vec{k}_2) = a^\dagger(k_1)a^\dagger(k_2)|0\rangle = |k_2 k_1\rangle$

$\langle k_1 k_2 | k_1' k_2'\rangle = \langle 0 | a(k_1) a(k_2) a^\dagger(k_1') a^\dagger(k_2') | 0\rangle$ $\quad$ exchange term

$$= (2\pi)^6\, 2\omega_{k_1} 2\omega_{k_2} \{\delta(k_1-k_1')\delta(k_2-k_2') + \delta(k_1-k_2')\delta(k_2-k_1')\}$$

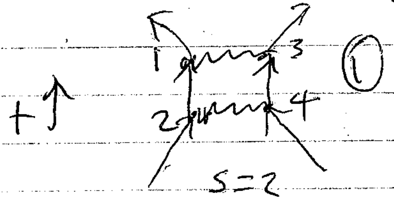
Projection operator on 2-particle subspace:

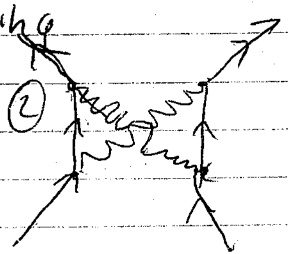
$$P_2 = \frac{1}{(2\pi)^6}\int \frac{d\vec{k}_1 d\vec{k}_2}{2\omega_{k_1} 2\omega_{k_2}} |k_1 k_2\rangle\langle k_1 k_2| \cdot \textcircled{\tfrac{1}{2}}$$ to correct for double counting.

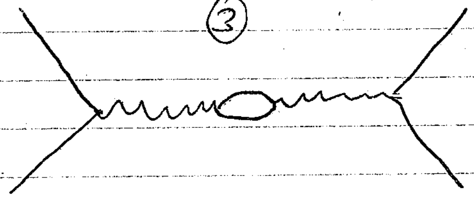
$1/n!$ for n-particles.

The $1/2$ can be avoided by use of a convention to avoid multiple counting. For instance, take only $|p_1| \leq |p_2|$

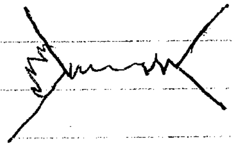
Second-Order N–N scattering



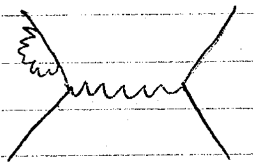




④ Radiative Correction



or ⑤



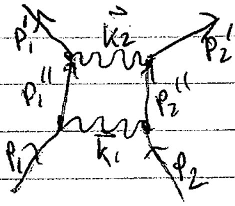
For Graph #1 $U^{(2)}(\infty,-\infty) =$

Fourth order correction

$$S_1^{(4)} = \frac{(-ig)^4}{4!}\int \frac{4!}{2} :\psi_1^\dagger \psi_2 \psi_3 \psi_4: \, i\Delta_F(x_1-x_2, m^2)\, i\Delta_F(x_3-x_4, m^2)\, i\Delta_F(x_1-x_3, N^2)\, i\Delta_F(x_2-x_4, N^2) \prod d^4x_j$$

numbered permutations of 1, 2, 3, 4 ; symmetry ; Nucleon ; meson

Diagram #1 in momentum space



Want a matrix element? So we want to work directly with integrals in momentum space, by passing construction of the configuration space operator:

$\langle p_1' p_2' | S_1^{(2)} | p_1 p_2 \rangle$

$$= \qquad \int$$

General Recipe:

① A factor of the fourier transform of $i\Delta_F(x_i - x_j', m^2)$ for each internal nucleon line"
i.e. An integration $\int d^4 p_j''$ and a factor $\frac{i}{(2\pi)^4} \frac{1}{p_j''^2 - m^2 + i\epsilon}$ in the integrand."

② For each internal meson line: An integration $\int d^4 k_j'$, a factor $\frac{i}{(2\pi)^4} \frac{1}{k_j^2 - \mu^2 + i\epsilon}$

③ For each vertex: a factor $-ig\,(2\pi)^4\,\delta^4(p_{out} - p_{in})$ since there is momentum conservation at each vertex,

④ For each external line: Do nothing (since no normalization constant $f_k^{(+)}(x) = e^{ikx}$)

So $\underbrace{\langle p_1' p_2' | S_1^{(4)} | p_1 p_2 \rangle}_{\text{to within a constant}} = (-ig)^4 (2\pi)^{16} \int \left(\frac{i}{(2\pi)^4}\right)^4 \frac{1}{p_1''^2 - m^2 + i\epsilon} \frac{1}{p_2''^2 - m^2 + i\epsilon}$

$\times \frac{1}{k_1^2 - \mu^2 + i\epsilon} \frac{1}{k_2^2 - \mu^2 + i\epsilon} \delta^4(p_1 - p_1'' - k_1)\,\delta^4(p_2 - p_2'' + k_1)\,\delta^4(p_1'' - p_1' - k_2)$

$\times \delta^4(p_2'' - p_2' + k_2)\, dp_1''\, dp_2''\, dk_1\, dk_2$

The factor of 1/2 in conf. space is canceled by a factor of two since Ψ_2^* can annihilate either p_1 or p_2
~~so we get a~~
So the surviving coefficient is 1

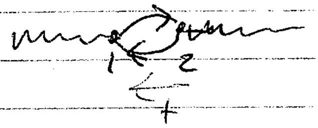
Here, to evaluate this, you can factor out $\delta^4(p_1 + p_2 - p_1' - p_2')$ still multiplying three other δ functions.

So $\langle p_1' p_2' | S_1^{(4)} | p_1 p_2 \rangle = i(2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')\, \underbrace{\langle p_1' p_2' | T | p_1 p_2 \rangle}_{\text{given by a single 4-dim momentum integral (a "loop" integral).}}$

The integral we have to do will look like

"$\int (4 \text{ denominators}) \, d^4 p_i$"

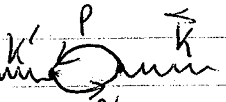
Simpler example: a loop such as



The "Nucleon bubble": the meson self-energy

$$S^{(2)}_{n.b.} = \frac{(-ig)^2}{2!} \int :\varphi_1 \varphi_2: \left[i \Delta_F(x_1 - x_2; m^2) \right]^2 d^4x_1 \, d^4x_2$$

In momentum space



$$\langle k' | S^{(2)}_{n.b.} | k \rangle = (-ig)^2 \int \cancel{(2\pi)^8} \, \delta^4(k - p + p') \, \delta^4(k' - p + p') \frac{i}{\cancel{(2\pi)^4}} \frac{1}{p^2 - m^2 + i\epsilon}$$

$$\times \frac{i}{\cancel{(2\pi)^4}} \frac{1}{p'^2 - m^2 + i\epsilon} \, d^4p \, d^4p'$$

Factor of 1/2! is again cancelled.

$$\langle k' | S^{(2)}_{n_b} | k \rangle = -(-ig)^2 \, \delta(k - k') \int \frac{d^4p}{(p^2 - m^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]}$$

$$= -(-ig)^2 \, \delta^4(k - k') \, I(k^2 - m^2)$$

"Feynman's famous formula":

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}$$

Will let a, b, be the denominator~~s~~ in I, Then switch integration ord ~~variables~~ of d^4p, dx

$$\frac{1}{\prod_{j=1}^{n} a_j} = \cancel{\Sigma} \prod_{j=1}^{n} \int_0^\infty e^{-a_j y_j} \, dy_j = \int e^{-\sum_{j=1}^{n} a_j y_j} \, \pi \, dy_j$$

$$= \int e^{-\sum a_j y_j} \, \delta(\lambda - \Sigma y_j) \, d\lambda \, \pi \, dy_j \qquad \text{Let } y_j = \lambda x_j$$

Interchange integration order.

$$= \int d\lambda \int \delta(\lambda - \lambda \Sigma x_j) \, e^{-\lambda \Sigma a_j x_j} \lambda^n \, \pi \, dx_j$$

$$= \int \Pi \, dx_j \, \delta(1 - \Sigma x_j) \int_0^\infty \lambda^{n-1} e^{-\lambda \Sigma a_j x_j} \, d\lambda$$

$$\boxed{\frac{1}{\prod_{j=1}^{n} a_j} = (n-1)! \int \frac{\delta(\sum_{j=1}^{n} x_j - 1)}{(\Sigma a_j x_j)^n} \prod_{j=1}^{n} dx_j}$$

$$I(k^2, m^2) = \int \frac{\partial^4 p}{\underbrace{(p^2 - m^2 + i\epsilon)}_{b}\underbrace{((p-k)^2 - m^2 + i\epsilon)}_{a}}$$

$$I(k^2, m^2) = \int \partial^4 p \int_0^1 \partial x \frac{1}{[(p-k)^2 x + p^2(1-x) - m^2 + i\epsilon]^2}$$

$$I(k^2, m^2) = \int_0^1 \partial x \int \partial^4 p \frac{1}{[p^2 - 2pkx + k^2x - m^2 + i\epsilon]^2}$$

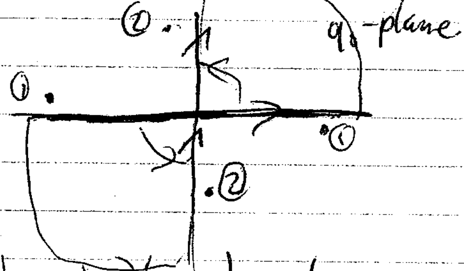
Let $q = p - kx$, this is just shifting origin of integration

$$I(k^2, m^2) = \int_0^1 dx \int \frac{\partial^4 q}{[q^2 + k^2 x(1-x) - m^2 + i\epsilon]^2}$$

Define: $I_n(a) = \int \frac{\partial^4 q}{(q^2 + a)^n}$ $n = 0, 1, 2 \ldots$

$$= \int \frac{\partial^3 q \, \partial q_0}{[q_0^2 - \vec{q}^2 + a]^n}$$

Require $\text{Im}\, a > 0$



Singularities at $q_0 = \pm\sqrt{\vec{q}^2 - a}$

① If $\vec{q}^2 - \text{Re}\, a > 0$,

② If $q^2 - \text{Re}\, a < 0$

So the integrand is analytic in the third quadrant, doesnt sweep over any singularities when rotated by 90°

so, let $q_0 = iq_4$, q^4 real

$\bar{q} = (\vec{q}, q_4)$

$$\text{So } I_n(a) = i \int \frac{\partial^3 \vec{q} \, dq^4}{(-q_4^2 - \vec{q}^2 + a)^n} = i \int \frac{\partial^4 \bar{q}}{(a - \bar{q}^2)^n}$$

Let S_4 be the area of a unit sphere in 4-dim

$$\int_0^\infty e^{-r^2} S_4 r^3 \, dr = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^4 = \pi^2$$

$$\tfrac{1}{2} S_4 \int e^{-y} y \, dy = \pi^2, \quad \text{So } S_4 = 2\pi^2$$

In general $S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$

So: $\partial^4 \bar{q} = 2\pi^2 \bar{q}^3 d\bar{q}$ $\quad I_n(a) = i\int_0^\infty \frac{2\pi^2 \bar{q}^3 d\bar{q}}{(a - \bar{q}^2)^n} = i\pi^2 \int_0^\infty \frac{y\,dy}{(a-y)^n}$

$$I_n(a) = \frac{i\pi^2}{(n-1)(n-2)\, a^{n-2}}$$

This goes to ∞ for $n=2$, so our integral diverges.

$$I_n(a) = i\int \frac{d^4\bar{q}}{(a-\bar{q}^2)^n} \sim \int \frac{\bar{q}^3 dq}{\bar{q}^4} \sim \int \frac{d\bar{q}}{\bar{q}} \to \infty$$

<u>Cut-off Procedures</u>: totally ad hoc.

① cut off integration at some momentum q_0, but this destroys covariance.

② <u>Feynman</u>: Replace $\frac{1}{p^2-m^2+i\epsilon}$ by $\frac{1}{p^2-m^2+i\epsilon} - \frac{1}{p^2-M^2+i\epsilon}$

For $p^2 \gg M^2, m^2$, then these terms go as $\sim 1/p^4$ then, take limit as $M \to \infty$

In effect this says that there is another field present with an imaginary coupling constant.

"Regulators"

③ <u>Pauli-Villars</u> "[illegible]"

Pauli thought there was entire family of higher mass mesons that would "conspire" to eliminate infinities, turned out to be nonsense

$$I(k^2, m^2) = \int_0^1 dx \int \frac{d^4q}{[q^2+k^2x(1-x)-m^2+i\epsilon]^2}$$

Interpret this as lim $M \to \infty$ of

$$I(k^2, m^2) - I(k^2, M^2) = \int_0^1 dx \int d^4q \left(\frac{1}{[q^2+k^2x(1-x)-m^2+i\epsilon]^2} - \frac{1}{[q^2+k^2x^2(1-x)-M^2+i\epsilon]^2}\right.$$

More generally:

$$I(k^2, m^2) \Rightarrow I(k^2, m^2) - \sum_j c_j' I(k^2 M_j^2)$$

$$I_R = \text{regulated } I = I(k^2, m^2) - I(k^2 M^2)$$

$$= -\int_{m^2}^{M^2} d\lambda \frac{\partial}{\partial\lambda} I(k^2, \lambda) = -\int_{m^2}^{M^2} d\lambda \int_0^1 dx \int \frac{\partial}{\partial\lambda} \frac{d^4q}{q^2+k^2x(1-x)-\lambda+i\epsilon}$$

$$= -\int_{m^2}^{M^2} d\lambda \int_0^1 dx \int \frac{2\,d^4q}{[q^2+\underbrace{k^2x(1-x)-\lambda}_{a}+i\epsilon]^3} \to \text{convergent.}$$

$$= -\int_{m^2}^{M^2} d\lambda \int_0^1 dx \frac{i\pi^2}{k^2x(1-x)-\lambda+i\epsilon}$$

$$= i\pi^2 \int_0^1 \log\left(\frac{M^2-k^2x(1-x)-i\epsilon}{m^2-k^2x(1-x)-i\epsilon}\right) dx \qquad m^2 \gg k^2$$

$$= \underbrace{i\pi^2 \log\frac{M^2}{m^2}}_{\downarrow \infty} - \int_0^1 i\pi^2 \log\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon'\right)$$

$$\langle k'|S^{(2)}|k\rangle = -\cancel{(2\pi)}^4 \delta^4(k-k') (-ig)^2 I(k^2 m^2)$$

$$= -i(2\pi)^4 \delta^4(k-k') \Pi^{(2)}(k^2 m^2) \rightarrow \text{in Q.E.D. a Tensor Polarization}$$

$$= i(2\pi)^4 \delta^4(k-k') \langle k'|J^{(2)}|k\rangle$$

$$I \to I_R;\ I = \lim_{M\to\infty} I_R = \lim_{M\to\infty} i\pi^2 \left\{ \log\frac{M^2}{m^2} - \int_0^1 \left(\log\left(1-\frac{k^2}{m^2}x(1-x)\right)-i\epsilon\right)dx \right\}$$

For a free meson:

$$\mathcal{H} = \tfrac{1}{2}\{\dot\varphi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\}$$

Altering the Mass; $\mu^2 \to \mu^2 + \delta(\mu^2)$ $\quad \mathcal{H} = \mathcal{H}_0 + \delta\left(\frac{\mu^2}{2}\right)\varphi^2$ so $\mathcal{H}_I = \frac{1}{2}\delta(\mu^2)\varphi^2$

$$S = T\left\{ e^{-i\int \mathcal{H}_I d^4x} \right\} \qquad \langle k'|S^{(1)}|k\rangle = \langle k'| -\tfrac{i}{2}\delta(\mu^2)\int \varphi^2 d^4x |k\rangle$$

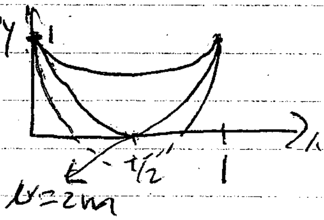
$$\langle k'|S^{(1)}|k\rangle = (2\pi)^4 \delta^4(k-k') \left(-\frac{2i}{2}\right) \delta(\mu^2)$$

So the second order calculation is equivalent to a first order change in mass.

For free meson $\delta(\mu^2) = \Pi'(\mu^2, m^2)$

$$= -\frac{g^2}{16\pi^2}\left\{ \log\frac{M^2}{m^2} - \int_0^1 \log\underbrace{\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right)}_{y} dx\right\}$$

The integrated term has form



So the integral takes on complex values after $\frac{\mu^2}{m^2} = 4$

So, if you increase meson mass past 2m, you go past particle production threshold.

So $\langle k|J^{(2)}|k\rangle$ real $\mu < 2m$, complex for $\mu > 2m$

So, for $\mu > 2m$, $\delta(\mu^2) = \text{Re}\,\Pi^{(2)}$

$-\mu\Gamma = \text{Im}\,\Pi^{(2)}$

So $\Pi^{(2)} = \delta(\mu^2) - i\mu\Gamma$ $\qquad$ so $\mu^2 \to \mu^2 + \delta(\mu^2) - i\mu\Gamma$

$\mu^2 \to \mu'^2 - i\mu'\Gamma$

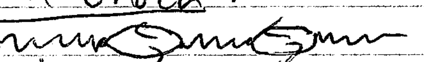
$\mu' \to \mu'\left(1 - \frac{i\Gamma}{2\mu'}\right)$

$$e^{-iNt} \rightarrow e^{-iN't - \frac{\Gamma}{2}t}$$

$$|e^{-iNt}|^2 \rightarrow e^{-\Gamma t}$$

Now, assuming $N < 2m$

<u>4th order</u>:



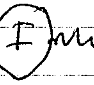
Just a repetition of the last one.

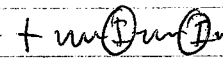
"Reducible" or "Improper": can be disconnected by removal of internal mesonline

New: "Proper" or "Irreducible"

These are "1-meson irreducible"

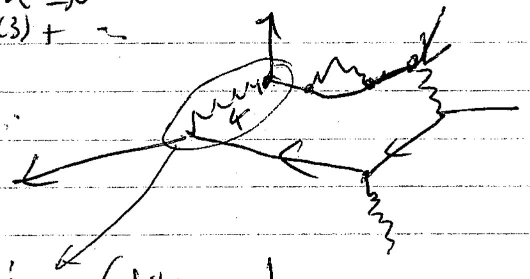
$$\langle K'| S_{\substack{1\text{ meson}\\ \text{irreducible}}} |k\rangle = -i(2\pi)^4 \delta(k-k') \, i\Pi(k^2, m^2) =$$ 

$$\langle k'| S-1 |k\rangle =$$  $+$  $+ \; - - - - =$ 

$$\text{but } \delta(N^2) = \Pi(k^2 m^2)\Big|_{k^2 = N^2}$$

$$= \Pi^{(2)} + \Pi^{(3)} + -$$

Ex:



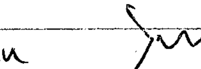
$$\frac{i}{(2\pi)^4}\int d^4k \, \frac{1}{k^2 - m^2 + i\epsilon}$$

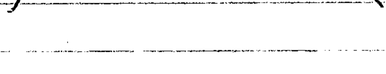
<u>Additional corrections</u>: But, can replace this line by  , then, ~~to lowest order~~

$$\frac{i}{(2\pi)^4}\int d^4k \, d^4k' (2\pi)^4 \delta(k-k') \frac{1}{k^2-m^2+i\epsilon}\left[-i\Pi(k^2)\right]$$
$$\times \frac{1}{k^2-m^2+i\epsilon} \frac{i}{(2\pi)^4}$$

$$\text{So: } = \frac{i}{(2\pi)^4}\int d^4k \, \frac{1}{k^2-\mu^2+i\epsilon} \Pi(k^2) \frac{1}{k^2-\mu^2+i\epsilon}$$

$$\text{So, } \frac{1}{k^2-\mu^2+i\epsilon} \rightarrow \frac{1}{k^2-\mu^2+i\epsilon} + \frac{1}{k^2-\mu^2+i\epsilon}\Pi(k^2)\frac{1}{k^2-\mu^2+i\epsilon}$$

But what we really want is all the iterations of .

we wan 

there $\frac{1}{k^2-\mu^2+i\epsilon} \to \frac{1}{k^2-\mu^2+i\epsilon} + \frac{1}{k^2-\mu^2+i\epsilon}\Pi(k^2)\frac{1}{k^2-\mu^2+i\epsilon}$

$$+ \frac{1}{k^2-\mu^2+i\epsilon}\Pi(k^2)\frac{1}{k^2-\mu^2+i\epsilon}\Pi(k^2)\frac{1}{k^2-\mu^2+i\epsilon} - \ldots$$

$$\to \frac{1}{k^2-\mu^2+i\epsilon}\sum_{n=0}^{\infty}\frac{\Pi(k^2)}{k^2-\mu^2+i\epsilon} = \frac{1}{k^2-\mu^2+i\epsilon}\frac{1}{1-\frac{\Pi(k^2)}{k^2-\mu^2+i\epsilon}}$$

so $\frac{1}{k^2-\mu^2+i\epsilon} \to \frac{1}{k^2-\mu^2-\Pi(k^2)+i\epsilon}$

Write $\Pi(k^2) = \Pi(\mu^2) + \tilde{\Pi}(k^2)$ $\qquad \tilde{\Pi}(k^2) = \Pi(k^2) - \Pi(\mu^2) = 0$ for $k^2=\mu^2$

$= \delta(\mu^2) + \tilde{\Pi}(k^2)$

We've shown that $\tilde{\Pi}(k^2)$ is non-divergent in second order.
(It's also true in other orders)

So $\frac{1}{k^2-\mu^2+i\epsilon} \to \frac{1}{k^2-[\mu^2+\delta(\mu^2)]-\tilde{\Pi}(k^2)+i\epsilon}$

So we have to replace the Feynman Propagator by this
the real problem is to compute $\tilde{\Pi}(k^2)$

$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - \mu_0^2\varphi^2)$ $\qquad \mu_0$ = bare mass

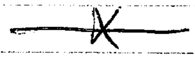
$= \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - \mu^2\varphi^2) + \frac{1}{2}(\mu^2-\mu_0)^2\varphi^2$ $\qquad \mu$ = interacting mass (observed)

$\mathcal{L}_0^{\text{new}} = \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - \mu^2\varphi^2)$

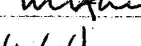
$\mathcal{L}_{\text{int}} = \frac{1}{2}(\mu^2-\mu_0)^2\varphi^2 = \frac{1}{2}\delta(\mu)\varphi^2$

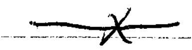
$\mathcal{H} = \mathcal{H}_0^{\text{new}} - \mathcal{H}_0^{\text{int}}$ $\qquad \boxed{\mathcal{H}_1^{\text{new}} = g:\psi^\dagger\psi\varphi: - \frac{1}{2}\delta(\mu)\varphi^2 - \delta(m^2)\psi^\dagger\psi}$

This introduces two new types of vertices:

 and 

Two new Feynman rules

Rule 5: For vertices such at , put in factor
$i(2\pi)^4\delta(\mu^2)\delta^4(k-k')$

Rule 6: For vertices such as 
put in factor
$i(2\pi)^4\delta(m^2)\delta^4(k-k')$

$$\langle k' | S_{\substack{1\text{ meson}\\ \text{irreducible}}} | k \rangle = (2\pi)^4 \delta^4(k-k')\, \Pi'(k^2)$$

$\Pi'(k^2) \neq \Pi$, but the meson mass does not change further:

So: $\Pi'(\mu^2) = 0$

Use this to determine $\delta(\mu^2)$

And $\frac{1}{k^2-\mu^2+i\epsilon} \longrightarrow \frac{1}{k^2-\mu^2-\Pi'(k^2)+i\epsilon}$

Physics 253 b

Quantization of Electromagnetic Field:

Cannot be done simply while maintaining covariance.

$\vec{E}, \vec{B}$

$$\nabla \times \vec{E} = -\tfrac{1}{c}\tfrac{\partial}{\partial t} B \qquad \vec{E} = -\nabla\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\vec{A}$$

$$\nabla \cdot \vec{B} = 0 \qquad \vec{B} = \nabla \times \vec{A}$$

Independent variables: $\varphi(r,t)$, $\vec{A}(r,t)$

Lagrangian Density: $\mathcal{L}(\varphi, \vec{A}, \nabla\varphi, \dot{\varphi}, \frac{\partial A^i}{\partial x_j}, \frac{\partial \vec{A}}{\partial t})$ ↗ 16 derivatives

Choose: $\mathcal{L} = \mathcal{L}(\varphi, \vec{A}, \vec{E}, \vec{B})$ ↗ 12 derivatives.

Missing derivatives: $\frac{\partial A_i}{\partial x_i}$, $\partial\varphi/\partial t$, the derivatives that make up a four-divergence.

$$\delta W = \delta \int_{t_0}^{t_1} L\, dt = \delta \int \mathcal{L}\, d^4x = 0$$

$$= \int \left\{ \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial A_j}\cdot \delta A_j + \frac{\partial \mathcal{L}}{\partial E_j}\delta E_j + \frac{\partial \mathcal{L}}{\partial B_j}\delta B_j \right\} d^4x = 0$$

$$= \int \left\{ \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial A}\delta A + \frac{\partial \mathcal{L}}{\partial E}\left(-\nabla\delta\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\delta A\right) + \frac{\partial \mathcal{L}}{\partial B}(\nabla \times \delta A) \right\} d^4x$$

since $\int \frac{\partial \mathcal{L}}{\partial E}\nabla\delta\varphi\, d\vec{r} = \cancelto{0}{\int \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial E}\delta\varphi\right) d\vec{r}} - \int \left(\nabla\cdot\frac{\partial \mathcal{L}}{\partial E}\right)\delta\varphi\, d\vec{r}$.

and $\int \frac{\partial \mathcal{L}}{\partial B}\cdot(\nabla\times\delta A)\, d\vec{r} = \cancelto{0}{-\int \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial B}\times\delta A\right) d\vec{r}} + \int \delta A\cdot\left(\nabla\times\frac{\partial \mathcal{L}}{\partial B}\right) d\vec{r}$

and $\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial E}\cdot \frac{\partial}{\partial t}\delta A\, dt = \cancelto{0}{\int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial E}\delta A\right) dt} - \int \delta A\cdot\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial E}\right) dt$

since variations vanish at end points.

$$\delta W = \int \left\{ \left[\frac{\partial \mathcal{L}}{\partial \varphi} + \nabla \frac{\partial \mathcal{L}}{\partial E} \right] \delta \varphi + \left[\frac{\partial \mathcal{L}}{\partial A} + \nabla \times \frac{\partial \mathcal{L}}{\partial B} + \frac{1}{c} \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial E} \right] \cdot \delta A \right\} d^4x = 0$$

Gives as eqns. of motion:

$$-\frac{\partial \mathcal{L}}{\partial \varphi} = \nabla \cdot \frac{\partial \mathcal{L}}{\partial E} \quad \text{and} \quad \nabla \times \frac{\partial \mathcal{L}}{\partial B} = -\frac{\partial \mathcal{L}}{\partial A} - \frac{1}{c}\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial E}$$

Guess for $\mathcal{L}$: $\mathcal{L} = \frac{1}{2}(E^2 - B^2) - \rho\varphi + \frac{1}{c}\vec{j} \cdot \vec{A}$

this is a relativistic scalar

gives: 1) $\nabla \cdot E = \rho$

2) $\nabla \times B = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$

Notice $\mathcal{L}$ is indep. of $\dot{\varphi}$

In point mechanics: if $\mathcal{L}(q_j, \dot{q}_j)$ is indep. of $\dot{q}_i$

the $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \quad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial}{\partial t} p_i = 0$

this fixes q_i in terms of $q_j, \dot{q}_j, j \neq i$.
You never should have used q_i as a separate coordinate.
So the scalar potential is defined in term of the other potentials, it is not independent.

In some sense, there are really only two indep. fields involved. Photons have not 4-polarizations, but two polarizations. this is as a result of the zero-rest mass of the photon.

<u>Gauge Invariance</u> Let $A' = A + \nabla\Lambda$ Λ arbitrary $f(r,t)$

Let $\varphi' = \varphi - \frac{1}{c}\partial_t\Lambda$, E, B are invariant.

$$\mathcal{L}' = \mathcal{L} + \frac{\rho}{c}\frac{\partial}{\partial t}\Lambda + \frac{1}{c}\vec{j} \cdot \nabla\Lambda$$

$$= \mathcal{L} + \frac{1}{c}\left(\frac{\partial}{\partial t}\Lambda\rho + \nabla \cdot jA\right) - \frac{1}{c}\Lambda\left(\frac{\partial \rho}{\partial t} + \nabla \cdot j\right)^{\nearrow 0 \text{ by current conservation}}$$

Gauge invariance $\Rightarrow$ current conservation.

So $\mathcal{L}'(E' B', \varphi', A')$ leads to same set of eqns. of motion.

$\nabla \cdot E = \rho$ or $\nabla(-\nabla\varphi - \frac{1}{c}\dot{A}) = \rho$

This will give $\nabla^2\varphi = \rho$ if $\underline{\nabla \cdot \dot{A} = 0}$

and φ will be determined as a functional of ρ by solving the poisson eq. This removes φ as a dynamical variable.

So you do not quantize φ: there are no scalar quanta in Q.E.D.

Write $\vec{E} = \overset{\text{longitudinal}}{E^{(L)}} + \overset{\text{transverse}}{E^{(T)}}$

defined by: $\nabla \times E^{(L)} = 0$ (irrotational)

$\nabla \cdot E^{(T)} = 0$ (solenoidal)

$\nabla \cdot E^{(L)} = \rho$

$\nabla \times E^{(T)} = -\frac{1}{c}\frac{\partial}{\partial t} B$

Since $\nabla \cdot B = 0$, B is transverse

Thm: If $\nabla \times E^{(L)} = 0$, i.e. E is irrotational, then there exists a scalar potential such that: $E^{(L)} = -\nabla\varphi$

$\nabla \cdot E^{(L)} = \rho \Rightarrow \nabla^2\varphi = -\rho$

An appropriate solution to this is $\varphi(\vec{r},t) = \int \frac{\rho(\vec{r}',t)}{4\pi|\vec{r}-\vec{r}'|} d\vec{r}'$

This is an instantaneous potential, not a retarded one, covariance is gone.

$$\nabla \times B = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}(E^{(L)} + E^{(T)}) = \frac{1}{c}\left(\vec{j} + \frac{\partial E^{(L)}}{\partial t}\right) + \frac{1}{c}\frac{\partial}{\partial t}E^{(T)}$$

Since $\nabla \cdot \nabla \times B = 0$ and $\nabla \cdot \frac{1}{c}\frac{\partial}{\partial t}E^{(T)} = 0$

$\nabla \cdot \frac{1}{c}\left(\vec{j} + \frac{\partial E^{(L)}}{\partial t}\right) = 0$, so $\frac{1}{c}\left(\vec{j} + \frac{\partial E^{(L)}}{\partial t}\right)$ must be a transverse current.

$$\nabla \cdot \frac{\partial}{\partial t}E^{(L)} = \nabla \cdot \frac{\partial}{\partial t}(-\nabla)\int \frac{\rho(\vec{r}',t)}{4\pi|\vec{r}-\vec{r}'|} d\vec{r}'$$

by continuity.

$$= -\frac{\partial}{\partial t}\int\{-\delta(\vec{r}-\vec{r}')\}\rho(\vec{r}',t)\,d\vec{r}' = \frac{\partial}{\partial t}\rho(\vec{r},t) = -\nabla \cdot \vec{j}$$

Maxwell eqs. for transverse field,

$\nabla \cdot E^{(T)} = 0$

$\nabla \cdot B = 0$

$\nabla \times E^{(T)} = -\frac{1}{c}\frac{\partial B}{\partial t}$

$\nabla \times B = \frac{1}{c} j^{(T)} + \frac{1}{c}\frac{\partial E^{(T)}}{\partial t}$

therefore: $\nabla \cdot (\vec{j} + \frac{\partial E^{(L)}}{\partial t}) = 0$

Define $j^{(T)} = j + \frac{\partial E^{(L)}}{\partial t}$

and $j^{(L)} = -\frac{\partial E^{(L)}}{\partial t}$

$\nabla \times B = \frac{1}{c} j^{(T)} + \frac{1}{c}\frac{\partial}{\partial t} E^{(T)}$

$E = -\nabla \varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = E^{(L)} - \frac{1}{c}\frac{\partial A}{\partial t}$, so $E^{(T)} = -\frac{1}{c}\partial_t \vec{A}$

Since $\nabla \cdot E^{(T)} = 0 = -\frac{1}{c}\nabla \cdot \dot{\vec{A}} = -\frac{1}{c}\partial_t(\nabla \cdot \vec{A})$

This is a gauge condition on $\vec{A}$, $\partial_t(\nabla \cdot A) = 0$

A field is determined if its curl and its divergence are determined

<u>Gauge condition</u>: $\nabla \cdot \vec{A} = 0$, or: $\vec{A}$ is transverse.

If we had $\nabla \cdot \dot{A} = 0$, $\nabla \cdot A \neq 0$, then let $A' = A + \nabla \Lambda$

$\nabla A' = \nabla \cdot A + \nabla^2 \Lambda = 0$

this is a gauge condition $\nabla^2 A = -\nabla \cdot A \rightarrow$ time-indep. source.

Since $\varphi' = \varphi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$, $\varphi' = \varphi$ since Λ is time-indep.

This is the radiation gauge or Coulomb gauge.
Universally used in low-energy problems.

Then, the only non-trivial Maxwell eq. is: $\nabla \times B = \frac{1}{c} j^{(T)} + \frac{1}{c}\frac{\partial E^{(T)}}{\partial t}$

$\nabla \times (\nabla \times A) = \frac{1}{c} j^{(T)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$

$\nabla(\cancel{\nabla \cdot A})^0 - \nabla^2 \vec{A} = \frac{1}{c} j^{(T)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} A$

or $\Box^2 \vec{A} = \frac{1}{c}\vec{j}$

$W = \int L\, dt = \int \{\frac{1}{2}(E^2 - B^2) - \rho\varphi + \frac{1}{c}\vec{j} \cdot \vec{A}\}\, d\vec{r}$

$\int E^2\, d\vec{r} = \int (E^{(L)} + E^{(T)})^2\, d\vec{r} = \int E^{(L)2} + 2E^{(L)} \cdot E^{(T)} + E^{(T)2}\, d\vec{r}$

$$= \int E^{(L)} \cdot E^{(T)} d\vec{r} = \int -\nabla\varphi \cdot E^{(T)} d\vec{r} = -\overbrace{\int \nabla \cdot (\varphi E^{(T)}) dr}^{0 \text{ at boundaries.}} + \int \varphi \underbrace{\nabla \cdot E^{(T)}}_{0} dr$$

$$\text{so } \int E^{(L)} \cdot E^{(T)} d\vec{r} = 0$$

$$j = j^{(L)} + j^{(T)} \qquad \text{since } j^{(L)} = -\frac{\partial}{\partial t} E^{(L)} = \nabla \frac{\partial}{\partial t}\varphi$$

$$\text{so } \int j^{(L)} \cdot A\, dr = \int \cdots\cdots \overset{0}{\cancel{(\nabla \cdot A)}}\, d\vec{r} = 0$$

So the longitudinal component is not even coupled to the vector potential.

$$L = L^{(T)} + L^{(L)}, \quad \mathcal{L} = \mathcal{L}^{(L)} + \mathcal{L}^{(T)}$$

$$L^{(T)} = \int \left\{ \frac{1}{2}\left(E^{(T)2} - B^2\right) + \frac{1}{c} \vec{j} \cdot \vec{A} \right\} d\vec{r}$$

$L^{(L)} = \int \left\{ \frac{1}{2} E^{(L)2} - \rho\varphi \right\} d\vec{r}$ this is completely determined given the charge distribution.

Independent variables: $\vec{A} = (A_1, A_2, A_3)$

Define momentum $\Pi_j = \frac{\partial \mathcal{L}^{(T)}}{\partial \dot{A}_j}$ or $\vec{\Pi} = \frac{\partial \mathcal{L}^{(T)}}{\partial \dot{A}} = \frac{\partial \mathcal{L}^{(T)}}{\partial E} \frac{\partial E}{\partial \dot{A}}$

$\vec{\Pi} = -\frac{1}{c} \vec{E}^{(T)} = \frac{1}{c^2} \dot{A}$ is the momentum conjugate to A.

$$H^{(T)} = \int \left\{ \Pi \cdot \dot{A} - \mathcal{L}^{(T)} \right\} d\vec{r} =$$

$$= \int \left\{ E^{(T)2} - \frac{1}{2}\left(E^{(T)2} - B^2\right) - \frac{1}{c} j \cdot A \right\} dr$$

$= \int \left\{ \frac{1}{2}\left(E^{(T)2} + B^2\right) - \frac{1}{c} j \cdot A \right\} d\vec{r}$ = energy of transverse field in E.M.

The total Hamiltonian: $\mathcal{H} = \int (\Pi \dot{A} - \mathcal{L})\, d\vec{r} = H^{(T)} - \int \mathcal{L}^{(L)} d\vec{r} = H^{(T)} + H^{(L)}$

$$H^{(T)} = \int \left\{ \frac{1}{2}\left(c^2 \Pi^2 + (\nabla \times A)^2\right) - \frac{1}{c} j \cdot A \right\} d\vec{r}$$

$$H^{(L)} = -\int \left(\frac{1}{2} E^{(L)2} - \rho\varphi \right) d\vec{r}$$

$$\text{Since } \int E^{(L)2} d\vec{r} = \int \nabla\varphi \cdot \nabla\varphi\, d\vec{r} = \int \nabla \cdot (\varphi \nabla \varphi)\, d\vec{r} - \int \varphi \underbrace{\nabla^2 \varphi}_{-\rho} dr$$

$$= -\int \rho\varphi\, d\vec{r}$$

so $H^{(L)} = \frac{1}{2}\int \rho \varphi \, d\vec{r}$: Electrostatic energy of the charge distribution.
Problem: must count self-interaction $\to \infty$

$$H^{(L)} = \frac{1}{2}\int \frac{\rho(r,t)\,\rho(r',t)}{4\pi|r-r'|}\, d\vec{r}\, d\vec{r}'$$

Only quantize transverse field; $\vec{A}$

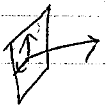
Canonical commutation would give: $[A_j(r,t), \Pi_\ell(r',t)] = i\hbar\, \delta(r-r')\, \delta_{j\ell}$
this is *incorrect*.
This violates the gauge conditions $\nabla\cdot\dot{A} = 0$, $\nabla\cdot A = 0$

since $\nabla\cdot\dot{A} = 0 \Rightarrow \nabla\cdot\Pi = 0$

and $[A_j(r,t), \Pi_\ell(r',t)] = i\hbar\, \delta(r-r')\, \delta_{j\ell} \Rightarrow [A_j, \nabla\cdot\Pi] \neq 0$

Helmholtz eq: $(\nabla^2 + \frac{\omega_k^2}{c^2})v = 0$, $v \sim e^{i\vec{k}\cdot\vec{r}}$ $\omega^2 = c^2k^2$
here $\omega \geq 0$

Vector solutions: $\vec{v} = \epsilon\, e^{i\vec{k}\cdot\vec{r}}$ require $\nabla\cdot\vec{v} = 0$, or, $\hat{\epsilon}\cdot k = 0$



take unit vectors perp. to $\vec{k}$ $\hat{\epsilon}_1^{(1)}, \hat{\epsilon}_2^{(2)}$

to represent circular polarizations
$\hat{\epsilon}^{(i)*}\cdot\hat{\epsilon}^{(j)} = \delta_{ij}$ form a complex orthonormal basis in plane $\perp$ to $\vec{k}$.

<u>Box Normalization</u> $\vec{v}_{\vec{k}\lambda}(r) = \hat{\epsilon}^{(\lambda)}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}/\sqrt{V}$

$$\int \vec{v}^*_{k\lambda}\cdot\vec{v}_{k'\lambda'}\, d\vec{r} = \delta_{\vec{k}\vec{k}'}\,\delta_{\lambda\lambda'}$$

$\overleftarrow{-k}$ $\overrightarrow{k}$ take $\hat{\epsilon}^{(\lambda)}(-k) = \hat{\epsilon}^{(\lambda)*}(k)$, a convention.

$\hat{\vec{A}}(r,t) = c\sum_{\vec{k},\lambda} Q_{k\lambda}(t)\,\vec{v}_{k\lambda}(\vec{r})$ Q's are complex coordinates

$A^* = A$, since $\vec{v}^*_{k\lambda}(r) = \vec{v}_{-k,\lambda}(r)$

so $Q_{k\lambda} = Q^*_{-k\lambda}$

$$A(r,t) = c\underset{\lambda}{\underset{\text{in every ⊥ plane of } k}{\sum}}\{Q_{k\lambda}U_{k\lambda} + Q^{*}_{k\lambda}U^{*}_{k\lambda}\}$$

<u>Transverse field</u>: $E = -\frac{1}{c}\dot{A} = -\sum \dot{Q}_{k\lambda}U_{k\lambda}(r)$

$$B = \nabla\times A = c\sum Q_{k\lambda}\nabla\times \frac{\epsilon^{(\lambda)}(k)\,e^{ikr}}{\sqrt{V}}$$

$$= -c\sum Q_{k\lambda}\left(\epsilon^{(\lambda)}\times\frac{\nabla e^{ikr}}{\sqrt{V}}\right)$$

$$= ic\sum Q_{k\lambda}(k\times\epsilon^{(\lambda)})\frac{e^{ikr}}{\sqrt{V}}$$

$$\int E^{(T)2}d\vec{r} = \sum \dot{Q}_{k\lambda}\dot{Q}_{k'\lambda'}\int \vec{U}_{k\lambda}\cdot\vec{U}_{k'\lambda'}\,d\vec{r}$$

$$= \sum \dot{Q}_{k\lambda}\dot{Q}_{k'\lambda'}\int U^{*}_{-k\lambda}\cdot U_{k'\lambda}\,d\vec{r} = \sum \dot{Q}_{k\lambda}\dot{Q}_{-k\lambda} = \sum \dot{Q}^{*}_{k\lambda}\dot{Q}_{k\lambda}$$

$\Rightarrow k=-k'$

$$\int B^{2}d\vec{r} = \int(\nabla\times A)^{2}d\vec{r}$$

$$= -c^{2}\sum Q_{k\lambda}Q_{k'\lambda'}\int\underbrace{(\vec{k}\times\epsilon^{(\lambda)}(k))(\vec{k}'\times\epsilon^{(\lambda')}(k'))}_{(k\cdot k')(\epsilon^{(\lambda)}\cdot\epsilon^{(\lambda')})-(\cancel{\vec{k}\cdot\epsilon^{(\lambda')}})(\cancel{k'\cdot\epsilon^{(\lambda)}})}\frac{e^{i(k+k')r}}{V}\,d\vec{r}$$

$$k^{2}\,\delta_{k,-k'}\,\delta_{\lambda\lambda'}$$

so $\int B^{2}d\vec{r} = \sum \omega_k^{2}Q_{k\lambda}Q_{-k\lambda} = \sum\omega_k^{2}Q^{*}_{k\lambda}Q_{k\lambda}$

so $L = \int\mathcal{L}^{(T)}d\vec{r} = \frac{1}{2}\sum_{\text{all }k\lambda}(\dot{Q}^{*}_{k\lambda}\dot{Q}_{k\lambda} - \omega^{2}Q^{*}_{k\lambda}Q_{k\lambda})$

Define $P_{k\lambda} = \frac{\partial L}{\partial\dot{Q}_{k\lambda}} = \dot{Q}^{*}_{k\lambda} = \dot{Q}_{-k\lambda}$ note factor of 2.

$$H = \sum P_{k\lambda}\dot{Q}_{k\lambda} - L$$

$$= \sum\dot{Q}^{*}_{k\lambda}\dot{Q}_{k\lambda} - L = \frac{1}{2}\sum\{\dot{Q}^{*}_{k\lambda}\dot{Q}_{k\lambda} + \overset{c^2k^2}{\omega_k^{2}}Q^{*}_{k\lambda}Q_{k\lambda}\}$$

$$= \frac{1}{2}\sum_{k\lambda}\{P^{*}_{k\lambda}P_{k\lambda} + \omega_k^{2}Q^{*}_{k\lambda}Q_{k\lambda}\}$$

Two constraints: specify φ #, and $\nabla \cdot \vec{A} = 0$

We lose covariance here.

$$A(r,t) = c \sum_{k\lambda} Q_{k\lambda}^{(t)} e^{(\lambda)}(\vec{k}) \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \qquad \begin{matrix} \lambda = 1,2 \\ e^{(\lambda)} \cdot \vec{k} = 0 \end{matrix} \qquad w_k = c|\vec{k}|$$

$$P_{k\lambda} = \frac{\partial L}{\partial Q_{k\lambda}} = \dot{Q}_{k\lambda}^{*} = \dot{Q}_{-k\lambda}$$

$$H = \frac{1}{2} \sum_{k\lambda} \{ P_{k\lambda}^{*} P_{k\lambda} + w_k^2 Q_{k\lambda}^{*} Q_{k\lambda} \}$$

Sum of Harmonic oscillators with complex amplitudes.

Reality Constraint $Q_{k\lambda}^{*} = Q_{-k\lambda}$ and $P_{k\lambda}^{*} = P_{-k\lambda}$

$$Q_{k\lambda} = \sqrt{\frac{\hbar}{2w_k}} (a_{k\lambda} + a_{-k\lambda}^{+})$$

$$P_{k\lambda} = i \sqrt{\frac{\hbar w_k}{2}} (a_{k\lambda}^{+} - a_{k\lambda})$$ satisfy the reality constraints.

$$a_{k\lambda} = \sqrt{\frac{w_k}{2\hbar}} Q_{k\lambda} + \frac{i}{\sqrt{2\hbar w_k}} P_{-k\lambda}$$

Commutation Relations: $[Q_{k\lambda}, Q_{k'\lambda'}] = 0$, $[P_{k\lambda}, P_{k'\lambda'}] = 0$

$$[Q_{k\lambda}, P_{k'\lambda'}] = i\hbar \delta_{\lambda\lambda'} \delta_{\vec{k}\vec{k}'}$$

or $[a_{k\lambda}, a_{k'\lambda'}] = [a_{k\lambda}^{+}, a_{k'\lambda'}^{+}] = 0$

$$[a_{k\lambda}, a_{k'\lambda'}^{+}] = \delta_{\lambda\lambda'} \delta_{\vec{k}\vec{k}'}$$

$$H = \frac{1}{2} \sum \{ P_{k\lambda}^{+} P_{k\lambda} + w_k^2 Q_{k\lambda}^{+} Q_{k\lambda} \} = \frac{1}{2} \sum \{ \hbar w_k (a_{k\lambda}^{+} a_{k\lambda} + a_{k\lambda} a_{k\lambda}^{+})$$

$$= \text{[illegible]} = \sum_{k\lambda} \hbar w_k (a_{k\lambda}^{+} a_{k\lambda} + 1/2)$$

Poynting vector: $\vec{P} = \vec{E} \times \vec{B} \, dr$

Field Momentum $\vec{P} = \frac{1}{c} \int \vec{E} \times \vec{B} \, d\vec{r}$ classically.

Measureability of the E.M. fields: Bohr and Rosenfeld.

Most accurate measurements possible are those using large charges.

Define (to make it Hermitian)

$$\vec{P} = \frac{1}{2c} \int (E \times B - B \times E) \, d\vec{r}$$

$$\vec{P} = \sum_{k\lambda} \hbar\vec{k}\, a^{\dagger}_{k\lambda} a_{k\lambda}$$

$$\vec{A}(r,t) = c\sum \sqrt{\frac{\hbar}{2\omega_k}}\,(a_{k\lambda} + a^{\dagger}_{-k\lambda})\frac{e^{ikr}}{\sqrt{V}}\cdot e^{(\lambda)}$$

$$= c\sum_{k\lambda}\sqrt{\frac{\hbar}{2\omega_k V}}\left\{a_{k\lambda}(t)e^{(\lambda)}(k)\,e^{i\vec{k}\cdot\vec{r}} + a^{\dagger}_{k\lambda}(t)e^{(\lambda)*}(k)\,e^{-i\vec{k}\cdot\vec{r}}\right\}$$

This is implicitly in the Heisenberg picture,

$$\dot{a}_{k\lambda} = \frac{1}{i\hbar}[a_{k\lambda}, H] = -i\omega_k a_{k\lambda} \quad , \quad a_{k\lambda}(t) = a_{k\lambda}(0)\,e^{-i\omega_k t}$$
$$= a_{k\lambda}$$

$$A(r,t) = c\sum\sqrt{\frac{\hbar}{2\omega_k V}}\left\{a_{k\lambda}\,e^{(\lambda)}(k)e^{-ikx} + a^{\dagger}_{k\lambda}e^{(\lambda)*}(k)\,e^{ikx}\right\}$$

$\omega_k = ck_0$

$kx = k_0ct - \vec{k}\cdot\vec{r} = \omega_k t - \vec{k}\cdot\vec{r}$

Klein Gordon eq. since $\mu = 0$ $\quad \Box^2 A = 0$

This has solutions $\vec{f}^{(\pm)} = (\text{const})(\text{vector})\cdot e^{\mp ikx}$ for $k^2 = 0$
$k_0^2 = \vec{k}^2$

Complete set of transverse waves: $(\text{vector})\cdot\vec{k} = 0$

$$\vec{f}^{(+)}_{\vec{k}\lambda} = \text{const}\cdot e^{(\lambda)}e^{-ikx} \quad , \quad \vec{e}^{(\lambda)}\cdot\vec{k} = 0$$

$$e^{(\lambda)*}\cdot e^{(\lambda')} = \delta_{\lambda\lambda'} \qquad \text{convention } e^{(\lambda)}(-k) = e^{(\lambda)*}(k)$$

$$\vec{f}^{(-)}_{k\lambda} = \text{const}\cdot e^{(\lambda)*}e^{ikx}$$

$$\text{So } (f^{(+)}_{k\lambda}, f^{(+)}_{k'\lambda'}) = \frac{i}{\hbar c^2}\int \vec{f}^{(+)*}_{k\lambda}\cdot \overleftrightarrow{\partial_0}\,\vec{f}^{(+)}_{k'\lambda'}\,d\vec{r} = \delta_{kk'}\delta_{\lambda\lambda'}$$

$$(f^{(-)}_{k\lambda}, f^{(-)}_{k'\lambda'}) = -\delta_{kk'}\delta_{\lambda\lambda'}$$

$$\text{So } \vec{f}^{(+)}_{k\lambda} = c\sqrt{\frac{\hbar}{2\omega_k V}}\,e^{(\lambda)}(k)\,e^{-ikx}$$

Commutation relations for $\vec{A}$ field.

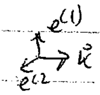
$$\vec{\Pi} = \frac{1}{c}\vec{E} = \frac{1}{c^2}\dot{\vec{A}} \quad , \quad \vec{\Pi}(r,t) = \frac{-i}{c^2}\sum\sqrt{\frac{\hbar\omega_k}{2V}}\left\{a_{k\lambda}e^{(\lambda)}e^{-ikx} - a^{\dagger}_{k\lambda}e^{(\lambda)*}e^{ikx}\right\}$$

$$[A_j(r,t), \Pi_l(r',t)] = \frac{i}{\cancel{\hbar}}\sum_{k\lambda}\frac{\hbar}{2V}\,2e^{(\lambda)}_j e^{(\lambda)*}_l e^{-ik(x-x')}$$

$$= i\hbar\frac{1}{(2\pi)^3}\int d\vec{k}\sum_{\lambda} e^{(\lambda)}_j(k)e^{(\lambda)}_l(k)\,e^{+i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\sum_{\lambda=1}^{3} e^{(\lambda)} e^{(\lambda)*} = \mathbb{1} = \text{unit dyadic,}$$

a completeness relation for $e^{(\lambda)}$



Define $\hat{e}^{(3)} = \hat{k} = \frac{\vec{k}}{|k|}$

$$\sum_{\lambda=1}^{3} e_j^{(\lambda)} e_\ell^{(\lambda)*} = \delta_{j\ell}$$

$$\text{so } \sum_{\lambda=1,2} e_j^{(\lambda)} e_\ell^{(\lambda)*} = \delta_{j\ell} - \hat{k}_j \hat{k}_\ell = \delta_{j\ell} - \frac{k_j k_\ell}{|k|^2}$$

$$[A_j(r,t), \Pi_\ell(r' t)] = i\hbar \frac{1}{(2\pi)^3} \int d\vec{k} \left(\delta_{j\ell} - \frac{k_j k_\ell}{|k|^2}\right) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

$$= i\hbar\, \delta_{j\ell}^{(T)}(\vec{r}-\vec{r}')$$ the transverse delta function.

free fields $$[A_j(r,t), A_\ell(r',t')] = c^2 \sum \frac{\hbar}{2\omega_k V} e^{(\lambda)}(k) e^{(\lambda)*}(k) \left\{ e^{-ik(x-x')} - e^{ik(x-x')} \right\}$$

$$= i\hbar c \frac{1}{(2\pi)^3 i} \int \frac{d\vec{k}}{k} \left(\delta_{j\ell} - \frac{k_j k_\ell}{|k|^2}\right) \left\{ e^{-ik(x-x')} - e^{ik(x-x')} \right\}$$

$$= i\hbar c\, D_{j\ell}^{(T)}(x-x')$$

<u>Vacuum Expectation</u>: $\langle 0| A_j(x) A_\ell(y) |0\rangle = c^2 \sum \frac{\hbar}{2\omega_k V} e^{(\lambda)} e^{(\lambda)*} e^{-ik(x-y)}$

$$\langle 0| A_j(x), A_\ell(y) |0\rangle = i\hbar c \frac{1}{(2\pi)^3 i} \int \frac{d\vec{k}}{2k} \left(\delta_{j\ell} - \frac{k_j k_\ell}{|k|^2}\right) e^{-ik(x-y)}$$

$$= i\hbar c\, D_{j\ell}^{(+)(T)}(x-y)$$

$$\langle 0| T(A_j(x), A_\ell(y)) |0\rangle = i\hbar c \left\{ D_{j\ell}^{+(T)}(x-y)\, \theta(x^0-y^0) + D_{\ell j}^{(+)(T)}(y-x)\, \theta(y_0-x_0) \right\}$$

$$= i\hbar c \frac{1}{(2\pi)^4} \int d^4k \left(\delta_{j\ell} - \frac{k_j k_\ell}{k^2}\right) \frac{e^{-ik(x-y)}}{k^2+i\epsilon}$$

$$= i\hbar c\, D_{F j\ell}^{(T)}(x-y)$$

$$\langle 0| A_\ell(x) A_m(y) + A_m(y) A_\ell(x) |0\rangle = i\hbar c \left\{ D_{\ell m}^{(+)T}(x-y) + D_{m\ell}^{(+)T}(y-x) \right\}$$

$$\equiv \hbar c\, D_{\ell m}^{(1)(T)}(x-y)$$

$$= \frac{\hbar c}{(2\pi)^3} \int \frac{d\vec{k}}{2|k|} \left(\delta_{\ell m} - \frac{k_\ell k_m}{|k|^2}\right) \left\{ e^{-ik(x-y)} + e^{ik(x-y)} \right\}$$

$$= \hbar c \frac{1}{(2\pi)^3} \int d^4k \left(\delta_{\ell m} - \frac{k_\ell k_m}{|k|^2}\right) \delta(\underset{\uparrow\, k_\mu k^\mu}{k^2})\, e^{-ik(x-\cancel{y})}$$

Sample Calculation:

$$H_0 = \frac{1}{2}\int (E^{(T)^2} + B^2)\, d\vec{r}$$

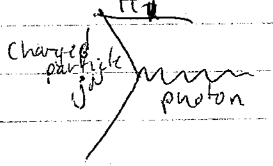
$$H_1 = -\frac{1}{c}\int \vec{j}\cdot\vec{A}(r,t)\, d\vec{r} = -\frac{1}{c}\sum_{\ell=1}^{3}\int j^{\ell}(r,t)\, A_\ell(r,t)\, d\vec{r}$$

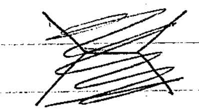
and $H_2 = H_{coulomb} = \frac{1}{2}\int \rho(r,t)\, \varphi(r,t)\, d\vec{r} = \frac{1}{2}\int \frac{\rho(r,t)\,\rho(r',t)}{4\pi|r-r'|}\, d\vec{r}\, d\vec{r}\,'$

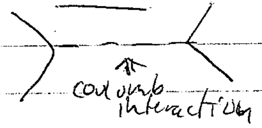
$\mathcal{H}_{I}(t) = H_1(t) + H_2(t)$ $\quad j^{\nu} = (c\rho, \vec{j})$ density current 4-vector,

$j_\nu = (c\rho, -\vec{j})$

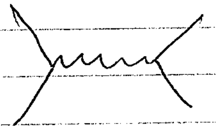
$\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$: $\partial_\nu j^\nu = 0$: Charge conservation.







Look at Second order contribution to S-matrix



$$S^{(2)} = \frac{1}{2}\left(\frac{i}{\hbar c}\right)^2 \sum_{\ell m}\int j^{\ell}(x)\, \langle 0|T(A_\ell(x)A_m(y))|0\rangle\, j^{m}(y)\, d^4x\, d^4y$$

$$-\frac{i}{\hbar}\int H_c(t)\, dt$$

$$= \frac{1}{2}\left(\frac{i}{\hbar c}\right)^2 \sum_{\ell m}\int j^{\ell}(x)\, \overset{i\hbar c}{D}{}^{(T)}_{F\ell m}(x-y)\, j^{m}(y)\, d^4x\, d^4y - \frac{i}{\hbar}\int H_c(t)\, dt$$

$$D^{(T)}_{F\ell m}(x-y) = \frac{1}{(2\pi)^4}\int d^4k \left(\delta_{\ell m} - \frac{k_\ell k_m}{|\vec{k}|^2}\right) \frac{e^{-ik(x-y)}}{k^2 + i\epsilon}$$

$$= \delta_{\ell m} D_F(x-y) - \partial_{x_\ell}\partial_{y_m} D(x-y)$$

where $D(x-y) = \frac{1}{(2\pi)^4}\int \frac{d^4k}{|\vec{k}|^2}\, \frac{e^{-ik(x-y)}}{k^2 + i\epsilon}$

$$S^{(2)} = -\frac{i}{2\hbar c}\sum_{\ell m}\int j^{\ell}(x)\left\{\delta_{\ell m} D_F(x-y) - \partial_{x_\ell}\partial_{y_m} D(x-y)\right\} j^{m}(y)\, d^4x\, d^4y - \frac{i}{\hbar}\int H_c(t)\, dt$$

Let $I_\ell = \frac{i}{2\hbar c}\sum_{\ell m}\int j^{\ell}(x)\, \partial_{x_\ell}\partial_{y_m} D(x-y)\, j^{m}(y)\, d^4x\, d^4y$

Integrate by parts: $I = \frac{i}{2\hbar c}\int \nabla_x\cdot\vec{j}(x)\, D(x-y)\, \nabla_y\cdot\vec{j}(y)\, d^4x\, d^4y$

Since $\vec{j} = \vec{j}^{(T)} + \vec{j}^{(L)}$, $\nabla\cdot\vec{j}^{(T)} = 0$, $\vec{j}^{(L)}$ defined as a gradient.

$\int \vec{j}^{(L)} \cdot A \, d\vec{r} = 0$ since $\nabla \cdot A = 0$

So the $\nabla_x \cdot \vec{j}(x)$, $\nabla_y \cdot \vec{j}_y$ vanish if the current is purely transverse.

$\therefore$ Kekn terms contribute nothing if $\nabla \cdot \vec{j}^{(L)} = 0$, i.e. $\vec{j} = \vec{j}^{(T)}$

Continuing for arbitrary $\vec{j}$: $\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$ $\qquad x^0 = ct$

$$I = \frac{i}{2\hbar c} \int \left(-c \frac{\partial \rho}{\partial x^0}\right) D(x-y) \left(-c \frac{\partial \rho(y)}{\partial y^0}\right) d^4x \, d^4y$$

$$= \frac{i}{2\hbar c} \int c^2 \rho(x) \frac{\partial^2}{\partial x^0 \partial y^0} D(x-y) \rho(y) \, d^4x \, d^4y$$

$$= -\frac{i}{2\hbar c} \int c^2 \rho(x) \frac{\partial^2}{\partial x_0^2} D(x-y) \rho(y) \, d^4x \, d^4y \qquad \text{by parts}$$

$$\frac{\partial^2}{\partial x^{0\,2}} D(x-y) = -\frac{}{(2\pi)^4} \int \frac{k^{0\,2}}{|\vec{k}|^2} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \, d^4k$$

$$= -\frac{}{(2\pi)^4} \int \frac{|\vec{k}|^2 + k^{0\,2} - |\vec{k}|^2}{|\vec{k}|^2} \frac{d^4k}{} \; 1$$

$$= - \left\{ \frac{1}{(2\pi)^4} \int \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} d^4k + \frac{1}{(2\pi)^4} \int \frac{k^2}{k^2 + i\epsilon} \frac{e^{-i\{k^0(x^0-y^0) - \vec{k}(\vec{x}-\vec{y})}}{|\vec{k}|^2} \right.$$

$$= - \left\{ D_F(x-y) + \frac{1}{(2\pi)^4} \int \frac{e^{-i\{k^0(x^0-y^0) - \vec{k}(\vec{x}-\vec{y})\}}}{|\vec{k}|^2} \, dk^0 \, d^3\vec{k} \right.$$

$$= - \left\{ D_F(x-y) + \frac{1}{(2\pi)^4} \cdot 2\pi (\delta(x^0 - y^0)) \int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{|\vec{k}|^2} \, d^3\vec{k} \right.$$

$$= - \left\{ D_F(x-y) + \frac{\delta(x^0 - y^0)}{4\pi |\vec{x} - \vec{y}|} \right\}$$

→ Instantaneous Coulomb Interaction.

$$S^{(2)} = -\frac{i}{2\hbar c} \int \left\{ \sum_{\ell m} j^\ell(x) \, \delta_{\ell m} D_F(x-y) j^m(y) - c^2 \rho(x) D_F(x-y) \rho(y) \right\} d^4x \, d^4y$$
$$+ \cancel{\frac{ic^2}{2\hbar c} \int \rho(x) \frac{\delta(x^0-y^0)}{4\pi |\vec{x}-\vec{y}|} \rho(y) \, d^4x \, d^4y}$$
$$- \cancel{\frac{i}{\hbar} \frac{1}{2} \int \frac{\rho(x,t) \rho(y,t)}{4\pi |\vec{x}-\vec{y}|} d\vec{x} \, d\vec{y} \, dt}$$

$x^0 = ct_x$
$y^0 = ct_y$

$$\boxed{S^{(2)} = \frac{i}{2\hbar c} \int j^\mu(x) \, g_{\mu\nu} D_F(x-y) j^\nu(y) \, d^4x \, d^4y}$$

This reduction takes place to all orders in Q.E.D.

Photon propagator in 4-dim, is in effect: $-i\hbar c \, g_{\mu\nu} D_F(x-y)$

So, for an internal photon line, insert $-\frac{i}{(2\pi)^4}\frac{g_{\mu\nu}}{k^2+i\epsilon}$

If we could quantize all 4 components,
$[A_\mu(x), A_\nu(y)] = -i\hbar c\, g_{\mu\nu} D(x-y)$ D(x-y) a scalar,

We would have 4-varieties of photons, would have to work it so they are unobservable. they would turn out to have neg. energies. (because $g_{00} = -\delta_{00}$

Scalar photons: neg. energy
Long. photon: pos. energy.
You would have to compute these in pairs.

Solution: Use a pseudo-Hilbert space: indefinite metric.

In this theory: $\langle 0|T\{A_\mu(x), A_\nu(y)\}|0\rangle = -i\hbar c\, g_{\mu\nu} D_F(x-y)$
Calculations here are done in the Lorentz gauge:
$$\Box A_\mu = 0$$
This introduces neg. probabilities into the theory
(Gupta-Bleuler procedure).

c=1 **Covariant Formulation**: Must use for any extended calculation

$$\underset{\text{contravariant}}{A^\mu = (\varphi, \vec{A})} \qquad A_\mu = (\varphi, -\vec{A})^{\text{covariant}}$$

$$F_{\mu\nu} = \frac{\partial}{\partial x^\nu} A_\mu - \frac{\partial A_\nu}{\partial x^\mu} = -F_{\nu\mu}$$

$$j=1,2,3. \quad F_{j0} = \frac{\partial A_j}{\partial x_0} - \frac{\partial A_0}{\partial x_j} = -\frac{\partial \vec{A}_j}{\partial t} - \nabla_j \varphi = E_j$$

$$F_{12} = \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} = +(\nabla \times A)_3 = +B_3$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & +B_3 & -B_2 \\ -E_2 & -B_3 & 0 & +B_1 \\ -E_3 & +B_2 & -B_1 & 0 \end{pmatrix}$$

$$E = -\nabla\varphi - \dot{A} \qquad B = \nabla \times A$$
together imply
Homogeneous Maxwell eqs: $\nabla \times E = -\dot{B}$, $\nabla \cdot B = 0$

Inhomogeneous Maxwell eqs: $\frac{\partial F_{\mu\nu}}{\partial x^\nu}$

$\mu=0$: $\frac{\partial F^{0j}}{\partial x_j} = \nabla\cdot E = \rho$

$\mu=1$ $\frac{\partial F^{1\nu}}{\partial x^\nu} = -\frac{\partial E_1}{\partial t} + \frac{\partial B_3}{\partial x_2} - \frac{\partial B}{\partial x_3} = -\frac{\partial E_1}{\partial t} + (\nabla\times B)_1 = j_1$

So: $\boxed{\frac{\partial F_{\mu\nu}}{\partial x^\mu} = j^\mu}$

$\frac{\partial^2 F^{\mu\nu}}{\partial x^\mu \partial x^\nu} = 0$ since This must be antisymmetric.

$= \frac{\partial j^\mu}{\partial x^\mu}$ which gives current conservation

Gauge Transformations: $A \to A' = A + \nabla\Lambda$

$\varphi \to \varphi' = \varphi - \dot{\Lambda}$

where Λ is any function.

so $A^\mu \to A^\mu - \frac{\partial \Lambda}{\partial x_\mu}$ or $A_\mu \to A_\mu - \frac{\partial \Lambda}{\partial x^\mu}$

Under this transformation:

$F^{\mu\nu} \to F^{\mu\nu}$

Inhom. eq. $\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\nu} A^\mu - \frac{\partial}{\partial x_\mu} A^\nu\right) j^\mu$

$\Box^2 A^\mu - \frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) = j^\mu$

To treat the A^μ comps. separately: impose $\frac{\partial A^\nu}{\partial x^\nu} = 0$
by Gauge transformation:

Suppose $\frac{\partial A^\nu}{\partial x^\nu} \neq 0$ $\frac{\partial A^\nu}{\partial x^\nu} \to \frac{\partial A^\nu}{\partial x^\nu} - \frac{\partial \Lambda}{\partial x^\nu \partial x_\nu} = 0$

Solve $\Box^2 \Lambda = \frac{\partial A^\nu}{\partial x^\nu} \neq 0$, an inhom. wave eq.

Lorentz gauge: $\boxed{\frac{\partial A^\nu}{\partial x^\nu} = 0}$

Once we have done this, we can still do $A^\mu \to A^\mu - \frac{\partial \Lambda}{\partial x_\mu}$
where $\Box^2 \Lambda = 0$
So, the Lorentz gauge is not unique.

In the Lorentz gauge $\boxed{\Box^2 A^\mu = j^\mu}$

So: Maxwell eqs + Lorentz condition $\Rightarrow$ wave eq.

Wave eqn + Lorentz condn ⇒ Maxwell eqn.
This is what we will use in Q.E.D.

Lorentz invariants: $A^\mu A_\mu$ rel. invariant, not gauge invariant.

$$F^{\mu\nu}F_{\mu\nu} = -2(\vec{E})^2 + 2(\vec{B})^2$$
$$= 2(E^2 - B^2)$$

$$\mathcal{L}^{(0)} = \frac{1}{2}(E^2 - B^2) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$\underline{\mathcal{L} = \frac{1}{2}(E^2 - B^2) - \rho\varphi + \vec{j}\vec{A} = \boxed{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu}}$$

This doesn't involve $\dot{\varphi}$, or in general $\frac{\partial A^\mu}{\partial x^\mu}$ (not summed)

Fermi: 1932: Reviews of Modern Physics

Try $\boxed{\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\frac{\partial A^\mu}{\partial x^\mu}\right)^2 - j_\mu A^\mu}$

This makes the eqn. of motion the wave eqn.
This has restricted gauge invariance: $\square^2\Lambda = 0$
Does not have general gauge invariance.

$$\mathcal{L} = -\frac{1}{4}(\partial^\nu A^\mu - \partial^\mu A^\nu)(\partial_\nu A_\mu - \partial_\mu A_\nu) - \frac{1}{2}(\partial_\mu A^\mu)^2 - j_\mu A^\mu$$
$$\to = (\partial_\mu g^{\mu\nu} A_\nu)^2$$

Eqns of motion: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\nu}$ for $\nu = 0, 1 \ldots 3$

$$= \partial_\mu\{F^{\mu\nu} - g^{\mu\nu}(\partial_\lambda A^\lambda)\} = -j^\nu$$

$$= \partial_\mu\{\partial^\nu A^\mu - \partial^\mu A^\nu - g^{\mu\nu}(\partial_\lambda A^\lambda)\} = -j^\nu$$

$$= -\square^2 A^\nu + \cancel{\partial_\mu\partial^\nu A^\mu} - \cancel{\partial^\nu(\partial_\lambda A^\lambda)} = -j^\nu$$

$$\square^2 A^\nu = j^\nu$$

Alternatively: Add a Fourdivergence: $\partial_\nu \frac{1}{2}\{A^\nu \partial_\lambda A^\lambda - A^\lambda \partial_\lambda A^\nu\}$

So this has no effect on the Action integral, so no effect on the eqns. of motion.

$$\mathcal{L}' = \mathcal{L} + \partial_\nu \frac{1}{2}\{A^\nu \partial_\lambda A^\lambda - A^\lambda \partial_\lambda A^\nu\}$$

Then: $\boxed{\mathcal{L}' = -\frac{1}{2}\frac{\partial A^\nu}{\partial x^\mu}\frac{\partial A_\nu}{\partial x_\mu} - j_\nu A^\nu}$

Eqns. of Motion: $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial \mathcal{L}}{\partial A_\nu}$

$$\partial_\mu(-\partial^\mu A^\nu) = -j^\nu$$

$$\underline{\Box^2 A^\nu = j^\nu}$$

Canonical Momenta: $= \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\nu)} = -\partial^0 A^\nu = \Pi^\nu = -\frac{\partial A^\nu}{\partial t}$

Π^ν is not a four-vector since $\partial/\partial t$ is not an invariant notion.

Canonical Quantization: $[q_j, p_k] = i\hbar\delta_{jk}$ ~~t~~ $= t'$

$$[A_\mu(\vec{r},t), \Pi^\nu(\vec{r}',t)] = i\hbar\delta_\mu{}^\nu\delta(\vec{r}-\vec{r}')$$

$$[A_\mu(\vec{r},t), \tfrac{\partial}{\partial t}A^\nu(\vec{r}',t)] = -i\hbar\delta_\mu{}^\nu\delta(\vec{r}-\vec{r}')$$

$$[A^\lambda(\vec{r},t), \tfrac{\partial}{\partial t}A^\nu(\vec{r}',t)] = -i\hbar g^{\lambda\nu}\delta(\vec{r}-\vec{r}')$$

for spatial components sign is same as in scalar case.

$$[A^\mu(\vec{r},t), A^\nu(\vec{r}',t')] = M^{\mu\nu}(\vec{r}-\vec{r}', t-t')$$

$$[A^\mu(x), A^\nu(y)] = M^{\mu\nu}(x,y)$$

$$\Box^2[A^\mu(x), A^\nu(y)] = \Box^2 M^{\mu\nu}(x,y) = 0$$

This is wave eq. with Initial conditions: ① $[A^\mu(\vec{r},t), A^\nu(\vec{r}',t)] = 0$

and $[A^\lambda(\vec{r},t), \frac{\partial}{\partial t}A^\nu(\vec{r}',t)] = -i\hbar g^{\lambda\nu}\delta(\vec{r}-\vec{r})$

so ② $\frac{\partial}{\partial y_0}M^{\mu\nu}(x-y)\Big|_{y_0=x_0} = -i\hbar g^{\mu\nu}\delta(\vec{r}-\vec{r}')$

For the scaler field: $[\varphi(\vec{r},t), \dot{\varphi}(\vec{r}',t)] = i\hbar\,\delta(\vec{r}-\vec{r}')$

which led to $[\varphi(r,t), \varphi(r',t')] = i\hbar\,\Delta(\vec{r}-\vec{r}', t-t')$

$i\hbar \to -i\hbar g^{\mu\nu}$

So, would expect $[A^\mu(x), A^\nu(y)] = -i\hbar g^{\mu\nu} D(x-y)$

where $D(x) = \lim_{\mu \to 0} \Delta(x) = \frac{-1}{(2\pi)^3}\int \frac{d\vec{k}}{\omega_k} e^{i\vec{k}\vec{r}} \sin\omega_k t\ , \quad \omega_k = |\vec{k}|$

Doing ang. integrations. $= -\frac{4\pi}{(2\pi)^3}\int \frac{k^2 dk}{\cancel{k}} \frac{\sin kr}{\cancel{k}r} \sin kt$

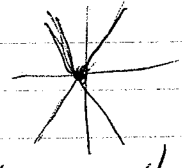
$$= -\frac{4\pi}{(2\pi)^3 r}\int_0^\infty \frac{1}{2}\{\cos k(r-t) - \cos k(r+t)\}\,dk$$

$$= \frac{-4\pi}{4(2\pi)^3 r}\int_{-\infty}^\infty \left(e^{ik(r-t)} - e^{ik(r+t)}\right) dk$$

$$= \frac{-\pi(2\pi)}{(2\pi)^3 r}\{\delta(r-t) - \delta(r+t)\}$$

This is zero everywhere except on the light cone.

$$= -\frac{1}{2\pi}\,\delta(r^2-t^2)\,\epsilon(t) = -\frac{1}{2\pi}\,\delta(x^2)\,\epsilon(x^0)$$



where $\epsilon(x^0) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

For interacting charged fields, this would not necessarily vanish inside the light cone.

$[A^\mu(x), A^\nu(y)] = -i\hbar g^{\mu\nu} D(x-y)$

Scalar case $\langle 0|\varphi(x)\varphi(y)|0\rangle = i\hbar\,\Delta^{(+)}(x-y)$

$\langle 0|\varphi(x)\varphi(y) + \varphi(y)\varphi(x)|0\rangle = \hbar\,\Delta^{(1)}(x-y) - i\hbar(\Delta^{(+)}(x-y) + \Delta^{(+)}(y-x))$

$\langle 0|T(\varphi(x)\varphi(y))|0\rangle = i\Delta_F(x-y)$

N=0 : QED:

$$\langle 0|A^\mu(x)A^\nu(y)|0\rangle = -i\hbar g^{\mu\nu} D^{(+)}(x-y)$$

$$D^{(+)}(x) = \frac{1}{(2\pi)^3}\int \frac{d\vec{k}}{2\omega_k} e^{-ikx} \qquad \omega_k = |\vec{k}|$$

$$\langle 0|A^\mu(x)A^\nu(y) + A^\nu(y)A^\mu(x)|0\rangle = -\hbar g^{\mu\nu} D^{(1)}(x-y)$$

$$D^{(1)}(x) = \frac{1}{(2\pi)^3}\int \frac{d\vec{k}}{\omega_k} \cos kx$$

$$\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle = -i\hbar g^{\mu\nu} D_F(x-y)$$

$$D_F(x) = \frac{1}{(2\pi)^4}\int d^4k \, \frac{e^{-ikx}}{k^2+i\epsilon}$$

Eqns. of motion: $\Box^2 A^\mu = j^\mu$

How do we get back to Maxwell eqs:

Wave eqn. + Lorentz gauge condition ⇒ Maxwell eqs.

So, somehow must impose $\frac{\partial A^\mu}{\partial x^\mu} = 0$

but this is not consistent with the free field commutator:

$$\left[\frac{\partial A^\mu(x)}{\partial x^\mu}, A^\nu(y)\right] = -i\hbar \frac{\partial}{\partial x^\mu} g^{\mu\nu} D(x-y)$$

$$= -i\hbar \frac{\partial}{\partial x_\nu} D(x-y) \neq 0$$

Fermi: Only admissible states $|\Psi\rangle$ are ones for which

$$\frac{\partial A^\mu}{\partial x^\mu}|\Psi\rangle = 0$$

The Maxwell eqs. then hold as expectation values.

$$\frac{\partial}{\partial x^\mu}\langle\Psi|A^\mu(x)|\Psi\rangle = 0$$

this gives the Maxwell eqs. for $\langle\Psi|F^{\mu\nu}|\Psi\rangle$

But the condition $\frac{\partial A^\mu}{\partial x^\mu}|\Psi\rangle = 0$ is nonsense, cannot be satisfied.

$$A = \underset{\substack{\uparrow \\ \text{annihilation}}}{A^{(+)}} + A^{(-)} \leftarrow \text{creation}$$

How can you apply creation operators and get 0 - impossible. Point is that you never use this supplementary condition in calculations.

This statement implies $\langle \psi | \frac{\partial A^\nu}{\partial x} = 0$

Examine $\langle \psi | [\frac{\partial A^\nu(x)}{\partial x^\nu}, A^\nu(y)] | \psi \rangle$

$$= -i\hbar \frac{\partial}{\partial x_\nu} D(x-y) \langle \psi | \psi \rangle \neq 0$$

$= 0$ if Fermi condition holds so nonsensical.

Weakened Fermi Condition:

$$\frac{\partial A^{\nu(+)}}{\partial x^\nu} | \psi \rangle = 0 \quad , \quad \frac{\partial A^\nu(x)}{\partial x^\nu} = \left(\frac{\partial A^\nu}{\partial x^\nu}(x) \right)^{(+)}$$

$$\Box^2 A^\nu = j^\nu \Rightarrow \Box^2 \frac{\partial A^\nu}{\partial x^\nu} = \frac{\partial j^\nu}{\partial x^\nu} = 0$$

So $\frac{\partial A^\nu}{\partial x^\nu}$ obeys a homogeneous wave eq. so there is unambiguous pos-neg. energy parts for separation.

$$\langle \psi | \frac{\partial A^{\nu(-)}}{\partial x^\nu} = 0 \Rightarrow \langle \psi | \frac{\partial A^{\nu(-)}}{\partial x^\nu} + \frac{\partial A^{\nu(+)}}{\partial x^\nu} | \psi \rangle = 0$$ implies Max. eq for $\langle \psi | F^{\mu\nu} | \psi \rangle$

Take $\vec{e}^{(1)}(k), \vec{e}^{(2)}(k)$, Transverse polarizations

Relativistic notation: $k^\nu = (k^0, \vec{k}) \quad k^0 = |\vec{k}| = k$

Unit "scalar" polarization vector: $e^{(0)\nu} = (1,0,0,0) = e^{(0)}_\nu$

Unit "longitudinal" polarization vector $e^{(3)\nu} = (0, \frac{\vec{k}}{|k|}), e^3_\nu = (0, -\frac{\vec{k}}{|k|})$

Spatial orthogonality rule: $\lambda = 1, 2$,

$$\vec{e}^{(\lambda)} \cdot \vec{e}^{(\lambda')*} = \delta^{\lambda\lambda'} \quad \text{or} \quad e^{(\lambda)}_\nu e^{(\lambda')\nu} = -\delta^{\lambda\lambda'}$$

for $\lambda = 0, 1..3$, $\quad e^{(\lambda)}_\nu e^{(\lambda')\nu *} = g^{\lambda\lambda'}$

Complete set of vector solns of wave eq.

$$f^{(+)(\lambda)}_k(x) = \sqrt{\frac{\hbar}{2\omega_k V}}\, e^{(\lambda)}(k)\, e^{-ikx}$$

$$f^{(-)(\lambda)}_k(x) = \sqrt{\frac{\hbar}{2\omega_k V}}\, e^{(\lambda)*}(k)\, e^{ikx} = [f^{(+)(\lambda)}_k(x)]^*$$

$$((f_k^{(+)(\lambda)}, f_{k'}^{(+)(\lambda')})) \equiv \frac{i}{\hbar}\int f_{k\nu}^{(+)(\lambda)}(x)\, \overleftrightarrow{\partial_0}\, f_{k'}^{(+)(\lambda')\nu}(x)\, d\vec{r}$$ (summed over ν)

$$= g^{\lambda\lambda'}\delta_{kk'}$$

$$((f_k^{(-)(\lambda)}, f_{k'}^{(-)(\lambda')})) = -g^{\lambda\lambda'}\delta_{kk'}$$

$$A_\nu(x) = \sum_{k\lambda}\{a_{k\lambda} f_{k\nu}^{(+)(\lambda)}(x) + a^\dagger_{k\lambda} f_{k\nu}^{(-)(\lambda)}(x)\}$$

multiply by $f_{k'}^{(+)(\lambda')}$, take scalar product:

$$((f_k^{(+)(\lambda')}, A_\nu)) = \sum_{k'\lambda'} a_{k'\lambda'}\, g^{\lambda\lambda'}\delta_{kk'} = \sum_{\lambda'} g^{\lambda\lambda'} a_{k\lambda'}$$

$$a_{k\lambda} = g_{\lambda\lambda'}\, ((f_k^{(+)(\lambda')}, A))$$
$$a^\dagger_{k\lambda} = -g_{\lambda\lambda'}\, ((f_k^{(-)(\lambda')}, A))$$

Equal-time Commutation Relations $[A_\nu(r,t), A_\nu(r',t)] = 0$
$[\pi^\nu(r,t), \pi^\nu(r',t)] = 0$

$$[A_\nu(r,t), \frac{\partial A^\nu}{\partial t}(r',t)] = -i\hbar g^{\nu\nu}\delta(\vec{r}-\vec{r}')$$

Calculate: $[a_{k\lambda}, a_{k'\lambda'}] = 0$ $\quad [a^\dagger_{k\lambda}, a^\dagger_{k'\lambda'}] = 0$

$$[a_{k\lambda}, a^\dagger_{k'\lambda'}] = -g_{\lambda\lambda'}\delta_{kk'}$$ $\lambda\lambda' = 1,2,3$ sign is familiar

but $[a_{k0}, a^\dagger_{k0}] = \cancel{\neq} -1$

$$A_\nu(x) = \sum_{k\lambda=0..3}\sqrt{\frac{\hbar}{2\omega_k V}}\{a_{k\lambda}e_\nu^{(\lambda)}(k)e^{-ikx} + a^\dagger_{k\lambda}e_\nu^{(\lambda)*}e^{ikx}\}$$

$$\mathcal{H} = \sum_{\nu=0}^{3}\pi^\nu\frac{\partial A_\nu}{\partial t} - \mathcal{L} \qquad \pi^\nu = -\partial^0 A^\nu$$

$$= -\sum_\nu \overset{p}{\partial^0 A^\nu}\,\overset{q}{\partial_0 A_\nu} + \frac{1}{2}\partial^\nu A^\nu\partial_\nu A_\nu + j_\nu A^\nu$$

$$= -\frac{1}{2}\partial_0^0 A^\nu\partial_0 A_\nu + \sum_{j=1}^{3}\frac{1}{2}\partial^j A^\nu\partial_j A_\nu + j_\nu A^\nu$$

sum over ν

$$\mathcal{H} = \underbrace{\frac{1}{2}\frac{\partial\vec{A}}{\partial t}\cdot\frac{\partial\vec{A}}{\partial t} + \frac{1}{2}\nabla\vec{A}:\nabla\vec{A}}_{\text{pos. definite}} \underbrace{-\frac{1}{2}\frac{\partial A^0}{\partial t}\frac{\partial A^0}{\partial t} - \frac{1}{2}\nabla A^0\cdot\nabla A^0}_{\text{neg. definite}} + j_\nu A^\nu$$

In terms of a's, a†'s, same calculation ~~as~~ in scalar case

$$H = \int\mathcal{H}\,d\vec{r} = -\frac{1}{2}\sum_{k\lambda}\hbar\omega_k g^{\lambda\lambda'}(a^\dagger_{k\lambda}a_{k\lambda'} + a_{k\lambda}a^\dagger_{k\lambda'})$$ (not normal-ordered)

not pos. defi

$$H = \sum \hbar\omega_k \{a^\dagger_{k1} a_{k1} + a^\dagger_{k2} a_{k2} + a^\dagger_{k3} a_{k3} - a^\dagger_{k0} a_{k0}\}$$

+ zero-point term which doesn't occur when normal ordered

Using the supplementary condition $\frac{\partial A^{\nu(+)}}{\partial x^\nu} |\Psi\rangle = 0$

$$-i \sum \sqrt{\frac{\hbar}{2\omega_k V}}\, a_{k\lambda} k_\nu e^{(\lambda)\nu}(k) e^{-ikx} |\Psi\rangle = 0$$ for all x / at least for all $\vec{r}$

For each $\vec{k}$ $\sum_{\lambda=0}^{3} a_{k\lambda} k_\nu e^{(\lambda)\nu} |\Psi\rangle = 0$

for $\lambda = 1, 2$, $k_\nu e^{(\lambda)\nu} = 0$ so only $\lambda = 0, 3$ contribute

$k_\nu e^{0(\nu)} = k_0 = k$, $k_\nu e^{(3)\nu} = -k$

So $(a_{k0} k - a_{k3} k) |\Psi\rangle = 0$

$\boxed{a_{k0} |\Psi\rangle = a_{k3} |\Psi\rangle}$ and $\langle\Psi| a^\dagger_{k0} = \langle\Psi| a^\dagger_{k3}$

$$\langle\Psi| a^\dagger_{k3} a_{k3} - a^\dagger_{k0} a_{k0} |\Psi\rangle = 0$$

So, of $H = \sum \hbar\omega_k \{a^\dagger_{k1} a_{k1} + a^\dagger_{k2} a_{k2} + \underbrace{\boxed{a^\dagger_{k3} a_{k3} - a^\dagger_{k0} a_{k0}}}_{\text{contribute nothing to expectation value.}}\}$

$$[a_{k\lambda}, a^\dagger_{k'\lambda'}] = -g_{\lambda\lambda'} \delta_{\vec{k}\vec{k}'}$$

Def. of vacuum (Ground state of universe)

Gauge invariant def: $E^{(+)}(x) |0\rangle = 0$

$B^{(+)}(x) |0\rangle = 0$

or $F^{(+)}_{\mu\nu}(x) |0\rangle = 0$

$$\left(\frac{\partial A_\nu^{(+)}}{\partial x_\mu} - \frac{\partial A_\mu^{(+)}}{\partial x_\nu}\right) |0\rangle = 0$$

$$A^{(+)}_\mu(x) = \sum \sqrt{\frac{\hbar}{2\omega_k V}}\, a_{k\lambda} e^{(\lambda)}_\mu e^{-ikx}$$

So $-i \sum \sqrt{\frac{\hbar}{2\omega_k V}}\, a_{k\lambda} (k_\nu e^{(\lambda)}_\mu - k_\mu e^{(\lambda)}_\nu) e^{-ikx} |0\rangle = 0$

So: For each k; $\sum_\lambda a_{k\lambda}(k_\nu e^{(\lambda)}_\mu - k_\mu e^{(\lambda)}_\nu) |0\rangle = 0$

multiply by $e^{(j)\mu}$ $j = 1, 2$

Then: $\sum_\lambda a_{k\lambda} (k_\nu g^{j\lambda} - 0)|0\rangle = 0$ so for $j=1,2$ $a_{kj}|0\rangle = 0$

Take the product of above by k^ν

Then $\sum_\lambda a_{k\lambda}(k_\nu (k^\nu e^{(\lambda)}_\nu) - 0)|0\rangle = 0$.

$k_\nu (a_{k0} k_0 - a_{k3} \underset{k_0}{k}) |0\rangle = 0$ so $(a_{k0} - a_{k3})|0\rangle = 0$

So, the vacuum state obeys the supplementary condition.

There are no states that satisfy this condition in the Hillbert space, no normalizable states.

$[a_{k0}, a^\dagger_{k0}] = -1 \Rightarrow$ a_{k0} - creation op.
$a^\dagger_{k0}$ - annihilation operator.

$a_{k3} = a$ $a^\dagger_{k0} = b$

solve: $a|\Psi\rangle = b|\Psi\rangle \Rightarrow |\Psi\rangle = c e^{b^\dagger a^\dagger} |\Psi_0\rangle$

since $a|\,\rangle = \alpha|\,\rangle \Rightarrow |\,\rangle = c e^{\alpha a^\dagger}|0\rangle$ But this is not normalizable.

$\langle\Psi|\Psi\rangle = |c|^2 \langle\Psi_0| e^{ba} e^{b^\dagger a^\dagger}|\Psi_0\rangle$

Ex: If $|\Psi_0\rangle$ has no b, a quanta. $\langle\Psi|\Psi\rangle = |c|^2 \sum \frac{1}{(n!)^2}$
$\times \langle 0|b^n b^{\dagger n}|0\rangle \langle 0|a^n a^{\dagger n}|0\rangle$
$= |c|^2 \sum \frac{(n!)^2}{(n!)^2} = |c|^2 \sum_n 1 \to \infty$

Answer: Eliminate basic Q.M. Axiom: go to pseudo-Hilbert space

Scalar Products: $\langle\Psi|\varphi\rangle = (\Psi^\dagger, \varphi)$ $\Psi^\dagger$ the hermitian conjugate of Ψ.

Define new adjoint: $\Psi^{\#} = \Psi^\dagger \eta$ η = metric operator.

New Scalar Product: $(\Psi^{\#}, \varphi) = \langle\Psi|\eta|\varphi\rangle$

$(\Psi^{\#}, \Psi) = \langle\Psi|\eta|\Psi\rangle$ Require that this be real

so $\eta^\dagger = \eta$, won't in general be positive.

Require: $\eta^\dagger \eta = I$, η unitary

O, Operator $O^\dagger$ = Hermitian adjoint

New Adjoint $O^* = \eta^{-1} O^\dagger \eta$ (assume η^{-1} exists)

$$(\psi^*, O^*\varphi) = \langle\psi|\eta O^*|\varphi\rangle = \langle\psi|O^\dagger \eta|\varphi\rangle$$

$$= \langle\varphi|\eta^\dagger O|\psi\rangle^* = (\varphi^*, O\psi)^* \qquad \text{since } \eta^\dagger = \eta$$

So this scalar product acts like before.

Expectation value of O in ψ: $(\psi^*, O\psi)$

Gupta

$$(\psi^*, O^*\psi) = (\psi^*, O\psi)^*$$

If $O^* = O$, (self-adjoint), then expectation value is real.

$O = \eta^{-1} O^\dagger \eta$, so ηO is Hermitian by this def.

Fields A^ν contained $a_{k\lambda}$, $a^\dagger_{k\lambda}$

but now, with the "screwed-up" metric, want A selfadjoint not hermitian,

$$A_\nu(x) = \sum \sqrt{\frac{\hbar}{2\omega_k V}} \left\{ a_{k\lambda} e_\nu^{(\lambda)} e^{-ikx} + a^*_{k\lambda} e_\nu^{(\lambda)*} e^{ikx} \right\}$$

Use equal-time commutation relations to derive commutation relations for a, a^*'s

$$[a_{k\lambda}, a_{k'\lambda'}] = [a^*_{k\lambda}, a^*_{k'\lambda'}] = 0 \qquad [a_{k\lambda}, a^*_{k'\lambda'}] = -g_{\lambda\lambda'}\delta_{kk'}$$

Supplementary condition: $\frac{\partial A^{\nu(+)}}{\partial x^\nu}|\psi\rangle = 0$

or $\langle\psi|\left(\frac{\partial A^{\nu(+)}}{\partial x^\nu}\right)^\dagger = 0$

multiply by η from the right.

$$\langle\psi|\eta\frac{\partial}{\partial x^\nu}\eta^{-1}A^{\nu(+)\dagger}\eta = 0$$

$$\langle\psi|\eta\frac{\partial}{\partial x^\nu}A^{\nu(+)*} = 0$$

So ~~taking product~~ operating on ~~by~~ $|\psi\rangle$:

$$\langle\psi|\eta\left(\frac{\partial A^{\nu(+)}}{\partial x^\nu} + \frac{\partial A^{\nu(+)*}}{\partial x^\nu}\right)|\psi\rangle = \langle\psi|\eta\frac{\partial A^\nu}{\partial x^\nu}|\psi\rangle = 0$$

Choose η such that $[\eta, a_{kj}] = 0 \quad j = 1,2,3.$
and $[\eta, a^{+}_{kj}] = 0$

Want $\eta a_{k0} = -a_{k0}\eta$
and $\eta a^{+}_{k0} = -a^{+}_{k0}\eta \qquad a^{*}_{k0} = \eta^{-1} a^{+}_{k0}\eta = -\eta^{-1}\eta a^{+}_{k0} = -a^{+}_{k0}$

$[a_{k0}, a^{*}_{k0}] = -1 \Rightarrow [a_{k0}, a^{+}_{k0}] = 1$ $\qquad$ a_{k0} : annihilation
a^{+}_{k0} : creation.

For each $\vec{k}$ $\quad |n\rangle_0 = \frac{(a^{+}_{k0})^{n}}{\sqrt{n!}}|0\rangle_0$ as in harmonic oscillator.

Choose η as a function of the a_{k0}, a^{+}_{k0}
Let $N = (-1)^{N_0}, \qquad N_0 = \sum_k a^{+}_{k0} a_{k0}$

Norms : $\underset{0}{\langle} n|\eta|n\rangle_0 = \underline{\underline{(-1)^{n}}}$

<u>Vacuum state</u>: $\left.\begin{array}{l} a_{kj}|0\rangle = 0 \text{ for } j = 1,2 \\ (a_{k0} - a_{k3})|0\rangle \equiv 0 \end{array}\right\}$ gauge invariant characterization of the vacuum.

All states created by a_{k0}, a_{k3} have zero norm except for one, the one with no scalar or long. states.

<u>Non gauge invariant def</u>: $A_{\nu}^{(+)}|0\rangle = 0 \Rightarrow a_{k\lambda}|0\rangle = 0$ all $\vec{k}\lambda$.
But gauge invariant vacua can be reached by a gauge transformation from this one.

Coupling quantized E.M. field to an external source

$H_0 = \frac{1}{2}\int (E^{(+)2} + B^{2})\, d\vec{r}$ $\qquad$ Radiation Gauge, $c=1$

$H_1 = -\int \vec{j}\cdot\vec{A}\, dr + \underbrace{\frac{1}{2}\int \rho\varphi\, d\vec{r}}_{H_c \text{ : Coulomb Hamiltonian}}$ $\qquad$ $\vec{J}$ is a c-number
$\vec{A}$ is a quantized field.

$H_c = \frac{1}{2}\int \frac{\rho(r)\rho(r')}{4\pi|r-r'|}\, dr\, dr'$

Interaction picture: $i\hbar\,\partial_t|t\rangle = \left(-\int j\cdot A\, dr + H_c\right)|t\rangle$

Let $|t\rangle = e^{i/\hbar \int_{-\infty}^{t} j\cdot A\, dt'\, d\vec{r}}\,|t\rangle' \rightarrow$ a new state vector

Let $a(t) \equiv \frac{i}{\hbar}\int j\cdot A(\vec{r},t)\, d\vec{r}$

$$|t\rangle = \cancel{e^{\frac{i}{\hbar}\int_{-\infty}^{t} \vec{j}\cdot\vec{A}\, dt'\, d\vec{r}}}\; e^{\int_{-\infty}^{t} a(t')\, dt'} |t\rangle'$$

New state $|t\rangle'$ obeys Schrod. eq. with H_{eff} of order e^2

Prescribed sources $\vec{j}(r,t)$, $\rho(r,t)$ in the Radiation Gauge:

$$i\hbar \frac{\partial}{\partial t}|t\rangle = \left(-\int \vec{j}\cdot A\, d\vec{r} + \mathcal{H}_c\right)|t\rangle$$

$$|t\rangle = e^{i/\hbar \int_{-\infty}^{t} \vec{j}\cdot\vec{A}\, d\vec{r}\, dt} |t\rangle'$$

Let $a(t) = \frac{i}{\hbar}\int \vec{j}\cdot A\, d\vec{r}$ this will be a sum of creation, annihilation operators.

$$i\hbar\, e^{-\int_{-\infty}^{t} a(t')dt'}\left(\frac{\partial}{\partial t} e^{\int_{-\infty}^{t} a(t')\,dt'}\right)|t\rangle' + i\hbar\frac{\partial}{\partial t}|t'\rangle = -e^{-\int_{-\infty}^{t} a(t')dt'} \int \vec{j}\cdot A\, d\vec{r}\, e^{\int_{-\infty}^{t} a(t')dt'} + \mathcal{H}_c\} |t\rangle'$$

this is the Schrodinger eq. for $|t'\rangle$

$$e^{-\lambda B} A e^{+\lambda B} = A + \lambda[A,B] + \frac{\lambda^2}{2}[[A,B],B] + \frac{\lambda^3}{3!}[[[\quad]]]$$

Since, in this case $[A,B] = c$-number: all the other terms go to zero.

$$F(\lambda) = e^{-\lambda S(t)} \frac{\partial}{\partial t} e^{\lambda S(t)} \qquad \frac{\partial F}{\partial \lambda} = e^{-\lambda S}\frac{\partial}{\partial t}(S e^{\lambda S}) - e^{-\lambda S} S \frac{\partial}{\partial t} e^{\lambda S}$$

$$= e^{-\lambda S}\frac{\partial S}{\partial t} e^{\lambda S}$$

$$F(1) = \int_0^1 \underbrace{e^{-\lambda S}\left(\frac{\partial S}{\partial t}\right) e^{\lambda S}}_{\text{Use multiple commutator again:}} d\lambda \qquad \text{since } F(0) = 0$$

So:
$$\int_0^1 e^{-\lambda\int a(t')dt'} a(t)\, e^{\lambda \int a(t')dt'} d\lambda\, |t'\rangle + \frac{\partial}{\partial t}|t\rangle'$$

$$= \{e^{-\int a\, dt'} a(t)\, e^{\int a(t')dt'} + \frac{1}{i\hbar}\mathcal{H}_c\}|t\rangle'$$

$$\{\cancel{a(t)} + \tfrac{1}{2}\int [a(t), a(t')]\, dt'\}|t\rangle' + \frac{\partial}{\partial t}|t\rangle'$$

$\int_0^1 \lambda\, d\lambda$

$$= \{a(t) + \int [a(t), a(t')]\, dt' + \frac{1}{i\hbar}\mathcal{H}_c\}\, |t\rangle'$$

Now: $\boxed{\frac{\partial}{\partial t}|t\rangle' = \{\frac{1}{2}\int [a(t), a(t')]\, dt' - \frac{i}{\hbar}\mathcal{H}_c\}\, |t\rangle'}$ new Schrodinger eq.

This doesn't work if $\vec{j}$ and ρ are not c-numbers.

$$\frac{d}{dt}|t\rangle' = \mathcal{H}^{(2)}_{eff}\,|t\rangle' \qquad \mathcal{H}^{(2)}_{eff} = \text{c-number}$$

$$|t\rangle' = e^{-i/\hbar \int_{-\infty}^{t} \mathcal{H}^{(2)}_{eff}(t')dt'}\,|-\infty\rangle' \qquad \text{Assuming } |-\infty\rangle = |-\infty\rangle'$$

$$\doteq e^{-i\theta(t)}\,|-\infty\rangle \qquad \theta(t) \text{ a real-valued c-number function.}$$

$$= i/\hbar \int_{-\infty}^{t} \mathcal{H}^{(2)}_{eff}(t')dt'$$

So: $$|t\rangle = e^{i/\hbar \int_{-\infty}^{t} j(r,t')A(r,t')\,dt'dr}\, e^{-i\theta(t)}\,|-\infty\rangle$$

$$= U(t,-\infty)|-\infty\rangle$$

$$S = U(\infty,-\infty)$$

Let $S' = e^{i\theta(t)}\,U(\infty,-\infty) = e^{i/\hbar \int_{-\infty}^{\infty} j\cdot A\,d\vec r\,dt}$ $\qquad a = a^{(+)} + a^{(-)}$

$$S' = e^{\int_{-\infty}^{\infty} a(t')dt'} = e^{\int a^{(-)}(t')dt'}\, e^{\int a^{(+)}(t')dt'}\, e^{\frac{1}{2}\iint [a^{(+)}(t'),a^{(-)}(t'')]dt}$$

since $e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]}$ if $[A,B]$ a c-number.

$$S' = e^{\int a^{(-)}dt'}\, e^{\int a^{(+)}dt'}\, e^{-\frac{1}{2}W}$$

$$W = -\iint [a^{(+)}(t'),a^{(-)}(t'')]\,dt'dt'' = -\iint \langle 0|[a^{(+)}(t'),a^{(-)}(t'')]|0\rangle\,dt'dt''$$

$$= -\iint \langle 0|a^{(+)}(t')a^{(-)}(t'')|0\rangle\,dt'dt'' = -\tfrac{1}{2}\iint \langle 0|a^{(+)}(t')a^{(-)}(t'') + a^{(+)}(t'')a^{(-)}(t')|0\rangle\,dt'dt''$$

$$W = \frac{1}{2\hbar^2}\iint j^{l}(r',t')\,\langle 0|A^{(+)}_{l}(r',t')A^{(-)}_{m}(r'',t'') + A^{(+)}_{m}(r'',t'')A^{(-)}_{l}(r',t')|0\rangle\, j^{m}(r'',t'')\,dr'dt'dr''dt''$$

$l,m = 1,2,3.$

since $\langle 0|A^{(+)}(x)A^{(-)}(y)|0\rangle = \langle 0|A(x)A(y)|0\rangle$, can remove ±'s

$$W = \frac{1}{2\hbar^2}\iint j^{l}(r',t')\,\hbar D^{(1)T}_{lm}(r',t',r'',t'')\,j_m(r'',t'')\,dr'dt'dr''dt''$$

Coulomb field is produced by exchange of ~~longitudinal~~ + scalar particles, but these are unreal.

In the Lorentz Gauge: No $\mathcal{H}_c$ $\quad H_I = \int j^{\nu}(x)A_{\nu}(x)\,d\vec r$

$$= \int(\rho A_0 - j\cdot A)\,dr$$

So $\hbar D^{(1)(T)}_{lm} \rightarrow -\hbar g_{\mu\nu} D^{(1)}(x-y)$

This gives identically the same answer.

$$D^{(1)(T)}_{\ell m}(x-y) = \frac{1}{(2\pi)^3}\int\left(\delta_{\ell m} - \frac{k_\ell k_m}{|k|^2}\right)\delta(k^2)\, e^{-ik(x-y)}\, d^4k$$

$$= \delta_{\ell m} D^{(1)}(x-y) - \frac{\partial}{\partial x_\ell}\frac{\partial}{\partial y_m}\mathcal{D}^{(1)}(x-y)$$

Where we Define $\mathcal{D}^{(1)}(x-y) = \frac{1}{(2\pi)^3}\int \frac{1}{|k|^2}\delta(k^2)\, e^{-ik(x-y)}\, d^4k$

$$\int j^\ell(x)\frac{\partial}{\partial x_\ell}\frac{\partial}{\partial y^m}\mathcal{D}^{(1)}(x-y)\, j^m(y)\, d^4x\, d^4y$$

$$= -\int \nabla\cdot\vec{j}(x)\,\mathcal{D}^{(1)}(x-y)\,\nabla\cdot\vec{j}(y)\, d^4x\, d^4y$$ Integrate by parts

$$= -\int \frac{\partial\rho}{\partial x_0}\,\mathcal{D}^{(1)}(x-y)\,\frac{\partial\rho}{\partial y_0}\, d^4x\, d^4y$$ Using current condemnation

Integrate by parts

$$= +\int \rho(x)\frac{\partial^2}{\partial x_0^2}\mathcal{D}^{(1)}(x-y)\,\rho(y)\, d^4x\, d^4y.$$

but $\frac{\partial^2}{\partial x_0^2}\mathcal{D}^{(1)} = -D^{(1)}$ $\rangle = -\int \rho(x)\, D^{(1)}(x-y)\, \rho(y)\, d^4x\, d^4y$

$$W = \frac{1}{2\hbar}\iint\{j^\ell(x)\, j^m(y)\,\delta_{\ell m} - \rho(x)\rho(y)\}\, D^{(1)}(x-y)\, d^4x\, d^4y$$

$$= \frac{1}{2\hbar}\iint j^\nu(x)\, D^{(1)}(x-y)\, j_\nu(y)\, d^4x\, d^4y$$: Lorentz Result.

$$S' = e^{\int_{-\infty}^{\infty} a(t)\, dt} \qquad a(t) \equiv \frac{i}{\hbar}\int \vec{j}(r,t)\cdot\vec{A}(r,t)\, d\vec{r}$$

Normally ordered: $S' = e^{\int a^{(-)}dt}\, e^{\int a^{(+)}dt}\, e^{-\frac{1}{2}W}$

$$W = \frac{1}{2\hbar}\int j^\ell(r,t)\, D^{(1)(T)}_{\ell m}(r-r', t-t')\, j^m(r',t')\, d\vec{r}\, dt\, d\vec{r}'\, dt' \geq 0$$

$$= -\frac{1}{2\hbar}\int j^\nu(x)\, D^{(1)}(x-y)\, j_\nu(y)\, d^4x\, d^4y$$

Last term: $\mathcal{H} = -g\,\rho(x)\,\varphi(x)$ ⇒ by diagram Summation found $S = :e^{-i\int g\rho\varphi\, dx}: e^{-i\frac{g^2}{2}\int \rho(x)\Delta_F(x-y)\rho(y)\, d^4x\, d^4y}$

Here $\mathcal{H} = -\vec{j}\cdot\vec{A}$

which is basically the same problem.

Start with $|t=-\infty\rangle = |0\rangle$

$$S'|0\rangle = e^{\int a^{(+)}(t)\, dt}|0\rangle\, e^{-\frac{1}{2}W}$$

n-photon component: $S_n'|0\rangle = \frac{1}{n!}\left(\int a^{(-)}(t)dt\right)^n|0\rangle e^{-\frac{1}{2}W}$
only n-photons

n-photon probability: $w_n = \langle 0|S_n'^{\dagger}S_n'|0\rangle =$

$$= \frac{1}{(n!)^2}\langle 0|\left(\int a^{(+)}(t)dt\right)^n\left(\int a^{(-)}(t)dt\right)^n|0\rangle e^{-W}$$

since $\langle 0|a^n a^{\dagger n}|0\rangle =$ $n!(\langle 0|a a^{\dagger}|0\rangle)^n$

so $w_n = \frac{n!}{(n!)^2}\left(\langle 0|\int a^{(+)}dt\int a^{(-)}dt|0\rangle\right)^n e^{-W}$

$w_n = \frac{W^n}{n!}e^{-W}$ ⇒ a Poisson distribution

$\sum w_n = 1$ $\langle n\rangle = \sum n w_n = W$

Lowest order Perturbation theory:
$w_1 = W$, $w_0 = 1-W$ $w_2, w_3 = \dots = 0$

But W is not the single photon emission probability, it is the mean number of photons emitted.
W may be >1, even infinite.

Coherent states

1-mode: $D(\alpha) = e^{\alpha a^{\dagger} - \alpha^* a}$ α = arbitrary complex #
$D^{\dagger}(\alpha) = D^{-1} = D(-\alpha)$

$$\left.\begin{array}{l} D(\alpha)aD^{\dagger}(\alpha) = a-\alpha \\ D(\alpha)a^{\dagger}D^{\dagger}(\alpha) = a^{\dagger}-\alpha^* \end{array}\right\}$$ so D's are displacement operators

$D(\alpha)|0\rangle$ = displaced ground state
$aD(\alpha)|0\rangle = D(\alpha)(a+\alpha)|0\rangle = \alpha D(\alpha)|0\rangle$

So $D(\alpha)|0\rangle \equiv |\alpha\rangle$ is a coherent state, amplitude α.

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{\alpha a^{\dagger}-\alpha^* a}|0\rangle = e^{\alpha a^{\dagger}-\alpha^* a}|0\rangle e^{-\frac{1}{2}|\alpha|^2}$$
$$= \sum_{n=0}^{\infty}\frac{(\alpha a^{\dagger})^n}{n!}|0\rangle e^{-\frac{1}{2}|\alpha|^2}$$
$$= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$ alt. def. for coherent state

Many Modes: $D(\{\alpha_k\}) = \prod_k D_k(\alpha_k) = e^{\sum_k(\alpha_k a_k^\dagger - \alpha_k^* a_k)}$

Many mode coherent state: $|\{\alpha_k\}\rangle = D(\{\alpha_k\})|0\rangle$

$$S' = e^{\int_{-\infty}^{\infty} a(t)\,dt} = e^{\frac{i}{\hbar}\int \vec{j}(r,t)\cdot\sum_{k\lambda}\sqrt{\frac{\hbar}{2\omega_k V}}\left(\rho^{(\lambda)}(k)\, a_{k\lambda}\, e^{-ikx} + \rho^{(\lambda)*}(k)\, a_{k\lambda}^\dagger e^{ikx}\right) d\vec{r}\,dt}$$

$$= D(\{\alpha_{k\lambda}\})$$

$$\text{where } \alpha_{k\lambda} = \frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega_k V}}\int \vec{J}(r,t)\cdot\vec{\rho}^{(\lambda)*} e^{-i(k\cdot r - \omega_k t)}\, d\vec{r}\,dt$$

So there is a Poisson distribution in each mode, so the sum is a poisson distribution.

Particle physics: Each mode is in 3-dim continuum never more than one photon/mode.
Resonant Cavities: Prescribed modes, here the Poisson distribution is relevant.

Example: Moving charged particle:
predetermined trajectory $\vec{r}(t)$

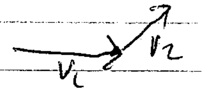
$\rho(r',t) = \delta(\vec{r}' - \vec{r}(t))$

$\vec{j}(r',t) = e\vec{v}(t)\,\delta(\vec{r}' - \vec{r}(t)) \qquad \vec{v}(t) = \frac{\partial\vec{r}}{\partial t}$

$$W = -\frac{1}{2\hbar}\int(\rho(x)\rho(y) - \vec{j}(x)\cdot\vec{j}(y))\,D^{(1)}(x-y)\,d^4x\,d^4y$$

$$= -\frac{e^2}{2\hbar}\int(1 - \vec{V}(t)\cdot\vec{v}(t'))\,D^{(1)}(\vec{r}(t) - \vec{r}(t'), t-t')\,dt\,dt'$$

$$= -\frac{1}{(2\pi)^3}\frac{e^2}{2\hbar}\int(1 - \vec{v}(t)\cdot\vec{v}(t'))\,\delta(k^2)\,e^{i[k(r(t)-r(t')) - k_0(t-t')]}\,d^4k\,dt\,dt'$$

Suppose 

$\vec{r}(t) = \vec{v}_1(t) \quad t \leq 0$

$\vec{r}(t) = \vec{v}_2(t) \quad t \geq 0$

Since by adiabatic coupling integrand is 0 for $t = -\infty$

$$\int_{-\infty}^{0} e^{i(k\cdot v_1 - k_0)t}\,dt = \frac{1}{i(\vec{k}\cdot\vec{v}_1 - k_0)}$$

$$\int_0^{\infty} e^{i(k\cdot v_2 - k_0)t} = -\frac{1}{i(k\cdot v_2 - k_0)}$$

$$W = -\frac{1}{(2\pi)^3}\frac{e^2}{2\hbar}\int \left\{\left(\frac{1}{(k\cdot v_1 - k_0)} - \frac{1}{k\cdot v_2 - k_0}\right)^2 - \left(\frac{v_1}{k\cdot v_1 - k_0} - \frac{v_2}{k\cdot v_2 - v_0}\right)^2\right\}\delta(k^2)\, d^4k$$

since $\frac{m(k\cdot v_1 - k_0)}{\sqrt{1-v_1^2}} = \vec{k}\cdot\vec{p}_1 - k_0 p_1^0 = -(kp_1)$

so $k\cdot v_1 - k_0 = -\frac{\sqrt{1-v_1^2}}{m}(kp_1) = -\frac{(kp_1)}{p_1^0}$

$$\frac{1}{k\cdot v_1 - k_0} = -\frac{p_1^0}{(kp_1)} \qquad \frac{\vec{v}_1}{k\cdot v_1 - k_0} = -\frac{\vec{p}_1}{(kp)}$$

So: $W = -\frac{1}{(2\pi)^3}\frac{e^2}{2\hbar}\int \left(\frac{p_1^0}{(kp_1)} - \frac{p_2^0}{(kp_2)}\right)^2 - \left(\frac{\vec{p}_1}{(kp_1)} - \frac{\vec{p}_2}{(kp_2)}\right)^2 \delta(k^2)\, d^4k$

$$= \frac{-1}{(2\pi)^3}\, e^2/2\hbar \int \left(\frac{p_1^\nu}{kp_1} - \frac{p_2^\nu}{kp_2}\right)\left(\frac{p_{1\nu}}{kp_1} - \frac{p_{2\nu}}{kp_2}\right)\delta(k^2)\, d^4k \geq 0$$

Non-relativistic expansion : 2^{nd} order in the velocities v_1, v_2

$$W = \frac{e^2}{2\hbar}\frac{1}{(2\pi)^3}\int\left\{\frac{(\vec{v}_1 - \vec{v}_2)^2}{k_0^2} - \frac{[k\cdot(v_1 - v_2)]^2}{k_0^4}\right\}\delta(k_0^2 - k^2)\, dk_0\, d\vec{k}$$

$$= \frac{e^2}{2\hbar}\frac{1}{(2\pi)^3}\int \frac{2\, d\vec{k}}{2k\cdot k^2}\frac{[k\times(v_1 - v_2)]^2}{k^2}$$

$k_0 = \pm k$ so a factor of 2 + a factor of $\frac{1}{2k}$

$$= \frac{e^2}{2\hbar}\frac{2}{3}(v_1 - v_2)^2 \int \frac{d\vec{k}}{k^3}$$

Since $\int \frac{dk}{k^3} = \int \frac{4\pi k^2 dk}{k^3} = 4\pi\int_0^\infty \frac{dk}{k} = 4\pi \lim \log \frac{k_{max}}{k_{min}}$

This has logaritmic sings at. 0, ∞.

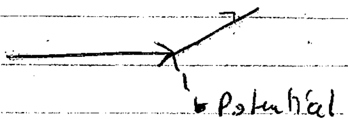
So the exp. value of number of photons emitted is inf.
their spectrum is the integrand $4\pi/k$

High-energy end Problem: the velocity was made discontinuous, so you have an inf. acceleration. Once you make the vel. continuous, the high freq. problem goes aw

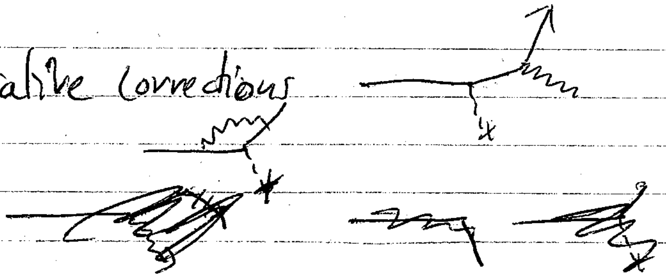
Low frequency: Infrared catastrophe.
The Energy spectrum doesn't diverge here $\propto \int^\infty 4\pi\, dk$

Since $w_n = \frac{w^n}{n!}e^{-W} = 0$ for all n.
$\rightarrow$ Prob. of observing any number of photons is 0.

If $\vec{J}$ is quantized, get same low-energy results.



Radiative corrections



If your photons have low enough freq., no recoil you will get ~~that~~ inf. number of photon exp.

Answer: Sensitivity of your detector; What is its threshold?

Probability for radiation less - scattering: 0

$w_0 = e^{-W} = 0$

Any finite - number of photons: still 0.

If you take into account threshold, set those below threshold as radiationless, than the infinities cancel, to each order

Alternate method: give photon a mass.

<u>Coupling to charges + currents</u>

Single spin-zero partic:

$H_{part} = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A}(r,t))^2 + e\varphi(r,t) + V(r)$ $\vec{r}, \vec{p}$ operators

$H_{rad} = \frac{1}{2}\int (E^{(T)2} + B^2)\, d\vec{r}$ $H = H_{part} + H_{rad}$?

Electrostatic term: $e\varphi(r,t)$ $\rho(\vec{r}) = e\,\delta(\vec{r}' - \vec{r})$

$$e\varphi(r,t) \approxeq \int \rho(r')\,\varphi(r',t)\,d\vec{r}'$$
$$\neq \frac{1}{2}\int \rho(r')\,\varphi(r',t)\,d\vec{r}'$$

$\varphi(r,t) = \int \frac{\rho(r',t)}{4\pi|r-r'|}\,dr'$ so $\frac{1}{2}\int \rho(r')\,\varphi(r')\,dr' = \frac{1}{2}\int \frac{\rho(r')\,\rho(r'')}{4\pi|r'-r''|}\,dr'\,dr''$

$\rho(r') = e\,\delta(\vec{r}' - \vec{r}) + \rho'(r')$

$\rightarrow$ density of <u>other</u> charges.

$$\frac{1}{2}\int \varrho(r') \varphi(r')\, dr = \frac{1}{2}\int \frac{(e\delta(r'-r)+\rho'(r'))(e\delta(r''-r)+\rho'(r''))}{4\pi|r'-r''|}\, dr' dr''$$

$$= \int \frac{e\delta(r'-r)e'(r'')dr'dr''}{4\pi|r-r'|} + \boxed{\frac{1}{2}\int \frac{e^2\delta(r'-r)\,\delta(r''-r)}{4\pi|r'-r''|}\, dr'dr'' + \frac{1}{2}\int e^2 \frac{\rho'(r')\rho'(r'')}{4\pi|r'-r''|}\, dr'dr''}$$

inf. constant self-inter.

constant, may be inf.

the two last term are usually ommitted.

Define $\varphi'(r) = \int \frac{\rho'(r')}{4\pi|r-r'|}\, dr'$ ≗ Potential due to other charges.

So the term $e\varphi(r t)$ is really $e\varphi'(r)$

So $H = \frac{1}{2m}\{ p^2 - \frac{e}{c}(p\cdot A + A\cdot p) + \frac{e^2}{c^2}A^2 \} + e\varphi'(r,t) + V(r) + H_{rad}$.

Coulomb (radiation) gauge: $\nabla\cdot A = 0$

$\sum_j [p_j, A_j] = \frac{\hbar}{i}\nabla\cdot A = 0$ so $p\cdot A = A\cdot p$

Does the coupling term have form $-\frac{1}{c}\int \vec{j}\cdot\vec{A}\, d\vec{r}$?

Gauge-invariant current: Let $\vec{j}(r',t) = \frac{e}{2m}\{ (p - \frac{e}{c}A)\delta(\vec{r}'-\vec{r}) + \delta(\vec{r}'-\vec{r})(p-\frac{e}{c}A) \}$

then $H_{coupling} = \frac{-1}{2m}\{\frac{e}{c}(p\cdot A + A\cdot p) + \frac{e^2}{c^2}A^2\} = -\frac{1}{c}\int \vec{j}\cdot\vec{A}\, d\vec{r}$

When we varied Lagrangian to get eqs. of motion, varied rel. to $\vec{A}$ not taking into account variation in $\vec{j}$, $\vec{j}$ was treated as a current.

So there must be an extra term in the Lagrangian to account for this. (What is it?)

Many-body problem: Many spin-zero charges.

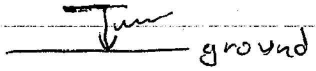
$\varrho(\vec{r}) = e\psi^\dagger(r)\psi(r)$ $\quad$ $j(\vec{r}) = \frac{e}{2m}\{ \psi^\dagger(r)(\frac{\hbar}{i}\nabla - \frac{e}{c}A)\psi(r) + (-\frac{\hbar}{i}\nabla - \frac{e}{c}A)\psi^\dagger(r)\psi(r) \}$

H_0 part $= \int \psi^\dagger\{-\frac{\hbar^2}{2m}\nabla^2 + V(r)\}\psi(r)\, dr$

$$H = H_{0\,particles} - \frac{1}{c}\int j(r')\cdot A(r')\, dr' + \frac{1}{2}\int \frac{\psi^\dagger(r)\psi^\dagger(r')\psi(r')\psi(r)}{4\pi|r-r'|}\, dr dr' + H_{rad},$$

This omits all of the electrostatic self-interactions. You need at least two particles for it to annihilate, no single particle term.

Radiation damping:

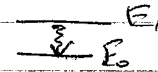


$\mathcal{H}_I = -\vec{j}\cdot\vec{A}$ $\quad\vec{j}$ confined to a <u>small</u> volume. (electric dipole approx.)

$\vec{j}$ of 0-order in A, so drop A^2 terms. (typically this is small) (emission, absorption of two photons scattering of 1-photon)

Supposing both states have 0-ang. momentum than you have to go to A^2 to get radiation.

Assume closed system has only two states, initial and final lying at energies E_1 and E_0



This is formally equivalent to a spin 1/2 particle.

E_1 corresponds to $\sigma_z = 1$, (up) E_0 to $\sigma_z = -1$, (down):

$$H_{atom} = \tfrac{1}{2}(E_1 + E_0) + \tfrac{1}{2}(E_1 - E_0)\sigma_z \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

no spatial significance

$$H_{rad} = \Sigma \hbar\omega_k a_k^\dagger a_k = \Sigma \hbar\omega_k N_k$$ after integrating the field Hamiltonian

State-changing interaction must be a linear combination of σ_x, σ_y

Rotation $e^{i\theta\sigma_y}$ make any lin-comb into e.g. σ_x

So: Choose interaction $\propto \sigma_x$ $\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H_{int} = M\sigma_x E(\overset{\vec{r}=0}{0}, t)$$

quantized E field

In the Heisenberg picture $H = \tfrac{1}{2}(E_1 + E_0) + \tfrac{1}{2}(E_1 - E_0)\sigma_z + M\sigma_x E(0,t) + \Sigma \hbar\omega_k N_k$

M = matrix element, of coupling constants.
This formulation ignores the fact that $\vec{A}$ is a vector field.
No worrying about photon polarization/
Assume that E_0 is the ground state.

$$E = E^{(+)} + E^{(-)}$$

$$\sigma_+ = \tfrac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ raising}$$

$$\sigma_- = \tfrac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ lowering}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x = \sigma_+ + \sigma_-$$

$$\sigma_x E = \sigma_+ E^+ + \sigma_- E^- + \cancel{\underbrace{\sigma_+ E^{-}}_{\substack{\text{create photon}\\ 0\to 1\text{ state}}}} + \cancel{\underbrace{\sigma_- E^{(+)}}_{\substack{\text{annihilate photon}\\ 1\to 0\text{ state}}}}$$

These violate Energy conservation, ignore them. If t-dependence taken into account, these would be the rapidly-oscillating terms.

$$H = \tfrac{1}{2}(E_1+E_0) + \tfrac{1}{2}(E_1 - E_0)\sigma_z + M\{\sigma_+ E^{(+)} + \sigma_- E^{(-)}\} + \sum \hbar\omega_k N_k.$$

Atomic excitation number $\frac{1}{2}(1+\sigma_z) + \sum N_k$ is conserved.
So $\frac{1}{2}\sigma_z + N$ commutes with H, $N = \sum N_k$

$[\sigma_z, \sigma_+] = 2\sigma_+ \quad [\sigma_z, \sigma_-] = -2\sigma_-$

$[N, E^{(+)}] = -E^{(+)} \quad [N, E^{(-)}] = E^{(-)}$

Initial state $|\underset{\sigma_z=1}{1}, \underset{vac}{0}\rangle$ so $(\frac{1}{2}\sigma_z + N)|1,0\rangle = \boxed{\tfrac{1}{2}}|1,0\rangle$ constant of motion

So $|t\rangle$ can be expanded in terms of $|1,0\rangle$ and $\{\underset{\frac{1}{2}(1+\sigma_z)=0}{|0,k\rangle}\}$

$\langle 1,0|t\rangle = u(t)$
$\langle 0,k|t\rangle = v_k(t)$ then $|t\rangle = u(t)|1,0\rangle + \sum_k v_k(t)|0k\rangle$

Let $(E_1 - E_0) = \hbar\omega$ $\quad i\hbar\frac{\partial}{\partial t}|t\rangle = \left[\frac{1}{2}(E+E_0) + \frac{1}{2}\hbar\omega\sigma_z + M\{\sigma_+ E^{(+)} + \sigma_- E^{(-)}\} + \sum_k \hbar\omega_k N_k\right]|t\rangle$

of "units of excitation" is conserved.

$$H = \tfrac{1}{2}(E_0 + E_1) + \tfrac{1}{2}\hbar\omega\sigma_z + M\{\sigma_+ \mathcal{E}^{(+)} + \sigma_- \mathcal{E}^{(-)}\} + \sum_k \hbar\omega_k a_k^\dagger a_k$$

$\frac{1}{2}\sigma_z + \sum_k a_k^\dagger a_k = \text{const.}$

$|t=0\rangle = |\underset{\frac{1}{2}(1+\sigma_z)}{1}, \underset{vac}{0}\rangle$

$|t\rangle = \alpha(t)|1,0\rangle + \sum \beta_k(t)|0,k\rangle$ $\quad u \to \alpha \quad v_k \to \beta_k$

$i\hbar\frac{\partial}{\partial t}|t\rangle = H|t\rangle$

Change phase of $|t\rangle$ if necessary to make $E_0 \to 0$

then $E_0 + E_1 = \hbar\omega$

$$\mathcal{H} = \tfrac{1}{2}\hbar\omega(1+\sigma_z) + M\{\sigma_+ \mathcal{E}^{(+)} + \sigma_- \mathcal{E}^{(-)}\} + \sum_k \hbar\omega_k a_k^\dagger a_k$$

Eqs. become: $\underline{\cancel{\hbar}\ i\hbar \frac{\partial\alpha(t)}{\partial t} = \hbar\omega\alpha(t) + iM\sum_k \sqrt{\frac{\hbar\omega_k}{2V}}\,\beta_k(t)}$

left multiplying by $\langle 1,0|$

$$\text{since}\quad \begin{aligned} \mathcal{E}^{(+)}(0,0) &= -\frac{\partial}{\partial t}A^{(+)}(0,0) = i\sum_k\sqrt{\frac{\hbar\omega_k}{2V}}\,a_k \\ \mathcal{E}^{(-)}(0,0) &= [\mathcal{E}^{(+)}]^\dagger \end{aligned}$$

Left multiplying by $\langle 0, k|$:

$$i\hbar\frac{\partial\beta(t)}{\partial t} = \hbar\omega_k\beta_k(t) - iM\sqrt{\frac{\hbar\omega_k}{2V}}\,\alpha(t)$$

$$\boxed{\begin{aligned} \dot{\alpha} &= -i\omega\alpha + M\sum_k\sqrt{\frac{\omega_k}{2\hbar V}}\,\beta_k \\ \dot{\beta}_k &= -i\omega_k\beta_k - M\cancel{\sum_k}\sqrt{\frac{\omega_k}{2\hbar V}}\,\alpha \end{aligned}}$$

Unitarity requires $\langle t|t\rangle = |\alpha(t)|^2 + \sum|\beta_k(t)|^2 = 1$
these equations satisfy this.

Ex: Phonon in crystal at finite T
In elementary particles T = 0 so don't have this problem.
Finite temp: $|t=0\rangle \neq |1,0\rangle$
Lorentz: natural line width due to radiation damping
Radiating system was described as a harmonic oscillator

damped harmonic oscillator: $m\ddot{x} = -m\omega^2 X \cancel{+} m\gamma\dot{x}$

$$\ddot{x} + \gamma\dot{x} + \omega^2 X = 0 \Rightarrow \text{damped solutions}$$

If $p = m\dot{x}$ $\quad\ddot{p} + \gamma\dot{p} + \omega^2 p = 0 \Rightarrow$ damped solutions.
start with $[X,P]_{t=0} = i\hbar \Rightarrow [X(t), P(t)] \to 0$ (whether operators or c-numb
commutators damped.

Classical context: $T \neq 0$ Direct relation between amount of dissipation and fluctuations imposed by surroundings on system

Q.M.: Even at $T=0$: Vacuum fluctuations
Dissipative coupling $\Rightarrow$ random driving force fromsurroundings.

So: Must take into effect zero-point fluctuations.

One oscillator $a, a^\dagger$ coupled to many $\{b_k, b_k^\dagger\}$ $[b_k, b_{k'}^\dagger] = \delta_{kk'}$
b_k are "heat bath" operators. (radiation field)

$H = \hbar\omega a^\dagger a + \sum \hbar\omega_k b_k^\dagger b_k + \hbar\sum_k(\lambda_k^* a b_k^\dagger + \lambda_k b_k a^\dagger)$ λ_k = coupling coeff.
(Assume that many of the ω_k are near ω.) dense in neighborhood of ω.

this Hamiltonian conserves quanta, since interaction both creates and destroys:

$$a^\dagger a + \sum b_k^\dagger b_k = \text{const}$$

$$\boxed{\begin{aligned} \dot a &= \frac{1}{i\hbar}[a, H] = -i\omega a - i\sum \lambda_k b_k \\ \dot b_k &= \frac{1}{i\hbar}[b_k, H] = -i\omega_k b_k - i\lambda_k^* a \end{aligned}}$$

Solutions take the form: $a(t) = a(0)u(t) + \sum b_k(0) v_k(t)$
$b_k(t) = \sum b_{k'}(0) X_{k'k}(t) + a(0) y_k(t)$

$u(0) = 1$ $v_k(0) = 0$
$X_{k'k}(0) = \delta_{kk'}$ $y_k(0) = 0$

$a(t) = F(a(0), \{b_k(0)\}, t)$
$b_k(t) = G_k(a(0), \{b_k(0)\}, t)$

Thm: If you start out with coherent state, state will stay coherent forever.
Because operators and their adjoints are not coupled, this gives conservation.

Heisenberg picture: assume state is coherent
label it $|\alpha, \{\beta_k\}\rangle$, α, β_k are complex numbers
this is direct product of coherent states.

i.e.: $a(0)|\alpha \{\beta_k\}\rangle = \alpha|\alpha\{\beta_k\}\rangle$ etc.

Then $a(t)|\alpha, \{\beta_k\}\rangle = F(\alpha\{\beta_k\}, t)|\alpha, \{\beta_k\}\rangle$

$b_k(t)|\alpha, \{\beta_k\}\rangle = G(\alpha\{\beta_k\}, t)|\alpha, \{\beta_k\}\rangle$

Schrodingen state $|t\rangle = U(t)|\alpha, \{\beta_k\}\rangle \rightarrow$ Hecsenberg state

$$a(t) = U^{-1}(t) a(0) U(t)$$
$$b_k(t) = U^{-1}(t) b_k(0) U(t)$$

Schro = Heis. at $t=0$

So $a(0)U(t)|\alpha\{\beta_k\}\rangle = a(0)|t\rangle = \underbrace{F(\alpha\{\beta_k\}t)}_{\alpha(t)}|t\rangle$ likewise for β_k

So $a(0)|t\rangle = \alpha(t)|t\rangle$
$b_k(0)|t\rangle = \beta_k(t)|t\rangle$ for all k.

$\alpha(t)$ and $\beta_k(t)$ must obey same eqs. as operators:

$$\boxed{\begin{aligned}\dot{\alpha}(t) &= -i\omega\alpha - i\sum \lambda_k \beta_k \\ \dot{\beta}_k(t) &= i\omega_k \beta_k - i\lambda_k^{+}\alpha\end{aligned}}$$

Initial conditions:
$\left.\begin{aligned}\alpha(0) &= \alpha \\ \beta(0) &= \beta\end{aligned}\right\}$ arbitrary c-numbers.

this is just the radiation damping eqs.
with $-i\lambda = M\sqrt{\frac{\omega_k}{2\hbar V}}$ $\quad |\alpha(t)|^2 + \sum_k |\beta_k(t)|^2 =$ cons

In a coherent state $|\alpha\rangle$ α means a displacement of a groundstate of harmonic oscillator.
$\alpha, \{\beta_k\}$ are the centers in complex plane of these oscillators.

For $T \neq 0$ $\quad |t=0\rangle = |\alpha\{\beta_k\}\rangle$
$|t\rangle =$ phase factor $\times |\alpha(t), \beta_k(t)$

$\left.\begin{aligned}\rho(0) &= |\alpha, \{\beta_k\}\rangle\langle\alpha\{\beta_k\}| \quad \text{pure coherent state.} \\ \rho(t) &= |\alpha(t)\{\beta_k(t)\}\rangle\langle\alpha(t),\{\beta_k(t)\}|\end{aligned}\right\}$ Solves the Liouville eq.

Chaotic state $\langle n\rangle$ fixed

$$\rho = \frac{1}{\pi\langle n\rangle}\int e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha\rangle\langle\alpha| d^2\alpha = \frac{1}{1+\langle n\rangle}\sum_j \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^j |j\rangle\langle j|$$

j-quantum states

This includes Thermal excitations: $\langle n_k\rangle = \frac{1}{e^{\frac{\hbar\omega_k}{kT}} - 1}$
(black body radiation)

Heat bath : Chaotic excitations

$$\rho(0) = \int |\alpha \{\beta_k\}\rangle\langle\alpha\{\beta_k\}| \prod_k \frac{e^{-\frac{|\beta_k|^2}{\langle n_k\rangle}}}{\pi\langle n_k\rangle} d^2\beta_k$$

α is in coherent state
β_k are in chaotic state, know only averag quantum numbers

$$\rho(t) = \int |\alpha(t)\{\beta_k(t)\}\rangle\langle\alpha(t),\{\beta_k(t)\}| \prod_k \frac{e^{-\frac{|\beta_k|^2}{\langle n_k\rangle}}}{\pi\langle n_k\rangle} d^2\beta_k$$

Look only at a-oscillator.
Partial or reduced density op. for a:

$$\rho_A(t) = \mathrm{Trace}_\beta\, \rho(t) = \int |\alpha(t)\rangle\langle\alpha(t)| \prod_k \frac{e^{-\frac{|\beta_k|^2}{\langle n_k\rangle}}}{\pi\langle n_k\rangle} d^2\beta_k$$

depend linearly on β_K

$$\alpha(t) = \underset{\equiv \alpha}{\alpha_0}\, U(t) + \overbrace{\sum \underset{\equiv}{\beta_k(0)}\, V_k(t)}^{\Gamma}$$

$$\rho_A(t) = \int |\alpha U(t) + \Gamma\rangle\langle\alpha U(t) + \Gamma|\, d^2\Gamma \int \delta^2\left(\Gamma - \sum \beta_k V_k(t)\right) \times \prod_k \frac{e^{-\frac{|\beta_k|^2}{\langle n_k\rangle}}}{\pi\langle n_k\rangle} d^2\beta_k$$

Put in Fourier integral rep of δ-fn, complete square,

$$\rho_A(t) = \int |\alpha U(t) + \Gamma\rangle\langle\alpha U(t) + \Gamma| \frac{1}{\pi\sum_k \langle n_k\rangle |V_k(t)|^2} e^{-\frac{|\Gamma|^2}{\sum_k \langle n_k\rangle |V_k(t)|^2}} d^2\Gamma$$

$$= \int P(\alpha, \overset{t=0}{0} | \gamma, t)\, |\gamma\rangle\langle\gamma|\, d^2\gamma$$

conditional prob. weight fn

where $P(\alpha, 0 | \gamma, t) = \frac{1}{\pi\sum_k \langle n_k\rangle |V_k(t)|^2} e^{-\frac{|\gamma - \alpha U(t)|^2}{\sum \langle n_k\rangle |V_k(t)|^2}}$

for $T = 0$ $\langle n_k\rangle \to 0$ so then this is a δ-function
in any case it is a δ-fn at $t = 0$ since $V_k(0) = 0$

$P(\alpha, 0 | \gamma t) \to \delta^2(\gamma - \alpha)$ for $t = 0$

Chaotic Heat bath: mean occupation #'s $\langle n_k \rangle$

$\rho_A(0) = |\alpha(0)\rangle\langle\alpha(0)|$ pure coherent state reduced density operator

But this pure state is coupled to the chaotic state

$$\rho_A(t) = \int P(\alpha 0|\gamma, t)\,|\gamma\rangle\langle\gamma|\, d^2\gamma$$

$$P(\alpha(0)\,0|\gamma, t) = \frac{1}{\pi \sum \langle n_k\rangle |V_k(t)|^2}\, e^{-\frac{|\gamma-\alpha(0)\,U(t)|^2}{\sum \langle n_k\rangle |V_k(t)|^2}}$$

mean value at time t: $\langle\gamma\rangle = \alpha(0)\,U(t)$

$V_k(0) = 0$

so $P(\alpha(0), 0|\gamma, 0) = \delta^{(2)}(\gamma - \alpha(0))$

At $T=0$ $\langle n_k\rangle = 0$ $P(\alpha(0)\,0|\gamma\, t) = \delta^{(2)}(\gamma - \alpha(0)\,U(t))$

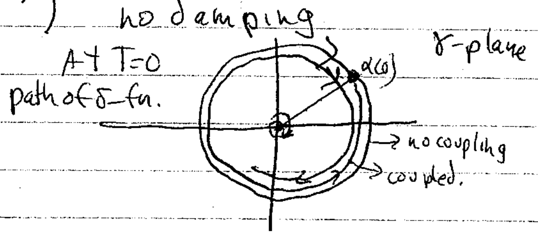
↑ this will contain the damping

Suppose coupling $= 0$, all $\lambda_k = 0$ then $V_k(t) = 0$ for all t

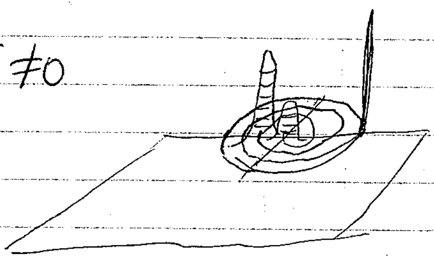
then $\dot\alpha = -i\omega\alpha$

$\alpha(t) = \alpha(0)\, e^{-i\omega t}$ $U(t) = e^{-i\omega t}$

$P(\alpha(0)\,0|\gamma\, t) = \delta^2(\gamma - \alpha(0)\, e^{-i\omega t})$ no damping



for $T \neq 0$



As $t \to \infty$, Gaussian dist. at origin, width characteristic of the $\langle n_k\rangle$ (equipartition.)

$$\boxed{\begin{aligned}\dot\alpha &= -i\omega\alpha - i\sum \lambda_k \beta_k \\ \dot\beta_k &= -i\omega_k\beta_k - i\lambda_k^*\alpha\end{aligned}}$$

find $\alpha(t)$, $\beta(t)$

given for $t=0$ $\alpha(0)$, $\beta(0)$

Solve by Laplace Transform, not Fourier transform like most people.

$$\bar\alpha(s) = \int_0^\infty e^{-st}\alpha(t)\,dt \qquad \bar\beta_k(s) = \int_0^\infty e^{-st}\beta_k(t)\,dt$$

Analytic for $\mathrm{Re}(s) > 0$

$$\int_0^\infty \frac{d\alpha}{dt} e^{-st}\,dt = \alpha(t)e^{-st}\Big]_0^\infty + s\int_0^\infty \alpha e^{-st}\,dt = -\alpha(0) + s\bar\alpha(s)$$

Taking the ~~exponential~~ Transform of these eqs.

$$s\,\bar{\alpha}(s) = \alpha(0) - i\omega\bar{\alpha}(s) - i\sum \lambda_k \bar{\beta}_k(s)$$

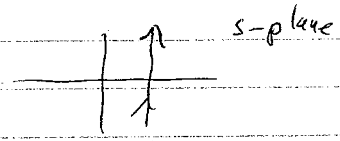
$$s\,\bar{\beta}_k(s) = \beta_k(0) - i\omega_k \beta_k(s) - i\lambda_k^* \bar{\alpha}(s)$$

$$(s+i\omega_k)\,\bar{\beta}_k(s) = \beta_k(0) - i\lambda_k^* \bar{\alpha}(s)$$

$$\bar{\beta}_k(s) = \frac{\beta_k(0) - i\lambda_k^*\bar{\alpha}(s)}{s+i\omega_k}$$

Then $s\,\bar{\alpha}(s) = \alpha(0) - i\omega\bar{\alpha}(s) - \frac{i\sum \lambda_k \beta_k(0)}{s+i\omega_k} - \sum \frac{|\lambda_k|^2}{s+i\omega_k}\bar{\alpha}(s)$

$$\bar{\alpha}(s) = \frac{\alpha(0) - i\sum \frac{\lambda_k \beta_k(0)}{s+i\omega_k}}{s+i\omega+\sum\frac{|\lambda_k|^2}{s+i\omega_k}}$$



Fourier-Mellin Inversion thm.

$$\alpha(t) = \frac{1}{2\pi i}\int_C e^{st}\,\bar{\alpha}(s)\,ds$$

C chosen to go from $c-i\infty \rightarrow c+i\infty$ so that $\bar{\alpha}(s)$ is always analytic to the right

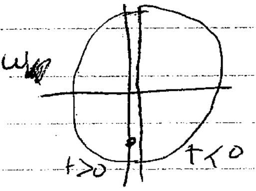
If $\alpha(t) = \alpha(0)\,U(t) + \sum \beta_k(0)\,V_k(t)$

$$\text{so } U(t) = \frac{1}{2\pi i}\int_C \frac{e^{st}}{s+i\omega+\sum_k\frac{|\lambda_k|^2}{s+i\omega_k}}\,ds$$

$$V_k(t) = -\frac{i}{2\pi i}\lambda_k\int_C \frac{e^{st}}{(s+i\omega_k)\left(s+i\omega+\sum_{k'}\frac{|\lambda_{k'}|^2}{s+i\omega_{k'}}\right)}\,ds$$

I If $\lambda_k = 0$: No coupling. $\frac{1}{s+i\omega}$: pole at $s=-i\omega$

$$U(t) = \begin{cases} 0 & t<0 \\ e^{-i\omega t} & t>0 \end{cases}$$



II Only one $\lambda_k \neq 0 \Rightarrow$ 2 poles. on neg. imaginary axis.
for λ small, one near $-i\omega$
If N λ_k's $\neq 0 \Rightarrow$ N+1 poles on neg im. axis., one near $-i\omega$.

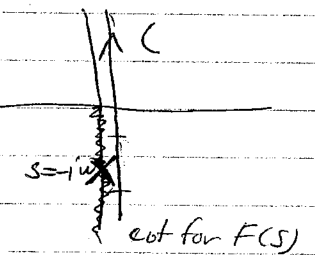
Poincaré recurrence: for N-finite, start with all energy in one mode: there is no true irreversibility. Energy will sooner or later all come back into that one mode.

Take $\lim_{N\to\infty}$, assume $\lambda_k \sim 1/\sqrt{N}$ so the integrals work.

$$F(s) \equiv \sum_k \frac{|\lambda_k|^2}{s+i\omega_k} \rightarrow \int \frac{|\lambda(k)|^2}{s+i\omega_k} g(\omega_k)\, d\omega_k$$

$g(\omega_k) \equiv$ spectral density of heat bath oscillator system.

$$U(t) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{st}}{s+i\omega+F(s)}\, ds$$



Most contribution to U(t) will come from near $s=-i\omega$

On the contour $s=\epsilon-i\omega'$

Weisskopf-Wigner Approxima
i

Approximate integral by: $F(s) \rightarrow F(\epsilon-i\omega)$

$$U(t) \approx \frac{1}{2\pi i}\int \frac{e^{st}}{s+i\omega+F(\epsilon-i\omega)} ds$$

write $F(\epsilon-i\omega)=i\delta\omega+k$

$\lim_{\epsilon\to 0} F(\epsilon-i\omega) = i\delta\omega+k$

$$= \frac{1}{2\pi i}\int \frac{e^{st}}{s+i(\omega+\delta\omega)+k} ds$$

pole at $s=-i(\omega+\delta\omega)-k$

$$= e^{-i(\omega+\delta\omega)t-kt}$$

for $t>0$ $\quad$ 0 for $t<0$

so freq. shift $\delta\omega$, exp. decrease e^{-kt}

In the same way; $V_k(t) = \frac{-i\lambda_k}{2\pi i}\int \frac{e^{st}}{(s+i\omega_k)\{s+i\omega+F(\epsilon-i\omega)\}}$

$$= \frac{-i\lambda_k}{k+i(\omega+\delta\omega-\omega_k)}\left\{e^{-i\omega_k t} - e^{-\{k+i(\omega+\delta\omega)t\}}\right\}$$ fo

$$F(\epsilon-i\omega) = \sum \frac{|\lambda_k|^2}{\epsilon-i(\omega-\omega_k)} \rightarrow \int \frac{|\lambda_k|^2}{\epsilon-i(\omega-\omega_k)} g(\omega_k)\, d\omega_k$$

$$= \sum \frac{|\lambda_k|^2\{\epsilon+i(\omega-\omega_k)\}}{\epsilon^2+(\omega-\omega_k)^2} \rightarrow \int \frac{|\lambda_k|^2 g(\omega_k)(\epsilon-i(\omega-\omega_k))}{\epsilon^2+(\omega-\omega_k)^2} d\omega_k$$

Since $\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^2+x^2} = \pi\delta(x)$ $\quad$ $\lim_{\epsilon\to 0}\frac{x}{\epsilon^2+x^2} = P(1/x)$

So $\delta\omega = P\sum_k \frac{|\lambda_k|^2}{\omega-\omega_k} \rightarrow P\int \frac{|\lambda_k|^2 g(\omega_k) d\omega_k}{\omega-\omega_k} \rightarrow \infty$ must be renormalized

$$k = \pi\sum_k |\lambda_k|^2\delta(\omega-\omega_k) \rightarrow \int |\lambda_k|^2 g(\omega_k)\pi\delta(\omega-\omega_k)\, d\omega_k$$

$$\boxed{k = \pi|\lambda_{k_\omega}|^2 g(\omega)}$$

After renormalization of $\delta\omega$, you get the Lamb shift.

The expression for k is just the transition probability evaluated to lowest order in perturbation theory:

$$\alpha(t) = \alpha(0)\, u(t) + \sum_k \beta_k(0)\, V_k(t)$$

$$u(t) = \frac{1}{2\pi i}\int_c \frac{e^{st}}{s+i\omega+F(s)}\, ds \quad \text{where } F(s) = \sum \frac{|\lambda_k|^2}{s+i\omega_k}$$

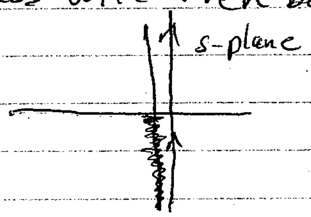
$$\rightarrow \int_0^\infty \frac{|\lambda(k)|^2 g(\omega_k)}{s+i\omega_k}\, d\omega_k$$

$$u(t) = e^{-i\omega' t - kt} \qquad \omega' = \omega + \delta\omega$$

$$V_k(t) = \frac{-i\lambda_k}{k+i(\omega-\omega_k)}\left\{ e^{-i\omega_k t} - e^{-(k+i\omega')t} \right\}$$

Problem: the integral $\int |\lambda_k|^2 g(\omega_k)\, d\omega_k$ is divergent.
But, assume that g is restricted so that it converges.

$F(s)$ will then be analytic except on the cut



the integrand of $u(t)$: $\frac{e^{st}}{s+i\omega+F(s)}$

$G(s) = \frac{1}{s+i\omega+F(s)}$

$Re(s)>0$ is analytic in the right half plane (since it is a Laplace transform)

$$F(-s^*) = \int \frac{|\lambda|^2 g}{-s^*+i\omega_k}\, d\omega_k = -F^*(s), \qquad G(-s)^* = -G^*(s)$$

So $G(s)$ is analytic for $Re(s)<0$ as well as $Re(s)>0$

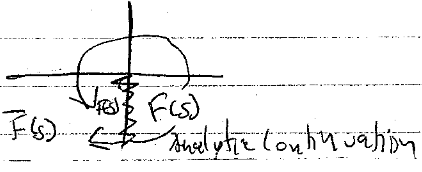
Discontinuity across the cut: $F(\epsilon - i\omega) - F(-\epsilon - i\omega)$

$$= \lim_{\epsilon\to 0} \int_0^\infty |\lambda(k)|^2 g(\omega_k) \left\{ \frac{1}{\epsilon - i(\omega_k-\omega)} + \frac{1}{\epsilon + i(\omega_k-\omega)} \right\} d\omega_k$$

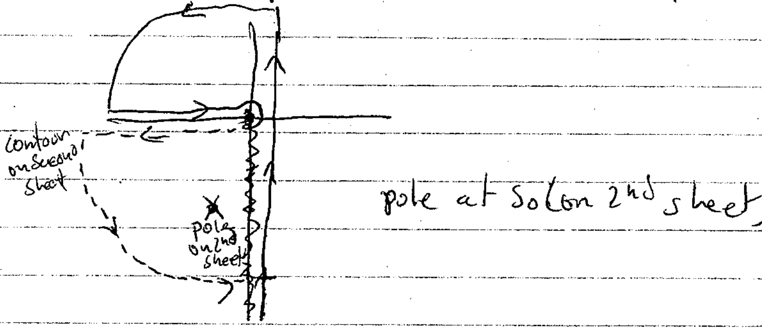
$$= \lim_{\epsilon\to 0} \int_0^\infty |\lambda|^2 g \frac{2\epsilon}{\epsilon^2+(\omega_k-\omega)^2}\, d\omega_k$$

$$= \int 2\pi |\lambda|^2 g\, \delta(\omega_k-\omega)\, d\omega_k = 2\pi |\lambda(k_\omega)|^2 g(\omega)$$

Use analytic continuation to continue $F(s)$ across the cut



$\bar{F}(s)$: Function continued onto secondsheet
$\bar{G}(s)$: " " " " "
$\bar{G}(s)$ has a pole on 2nd sheet,



$$\int_{\epsilon-i\infty}^{\epsilon+i\infty} + \int_{\swarrow} + \int_{\rightleftarrows} + \int_{\searrow} = 2\pi i \cdot \text{Residue at } s_0$$

$$u(t) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} ds = \text{Residue at } s_0 - \int_{\rightleftarrows} ds$$

$$u(t) = \underset{\text{at } s_0}{\text{Res}}\left(\frac{e^{st}}{s+i\omega+\bar{F}(s)}\right) \int_{\rightleftarrows} \frac{e^{st}}{s+i\omega+F(s)}\,ds$$

$\bar{F} \to F$ on 2nd sheet

$$s_0 + i\omega + \bar{F}(s_0) \equiv 0$$

$$s + i\omega + \bar{F}(s) = \cancel{s_0 + i\omega + \bar{F}(s_0)}^{0} + s - s_0 + (s-s_0)\,\bar{F}'(s_0) + \cdots$$

$$= s - s_0\{1 + \bar{F}'(s_0)\}$$

$$\text{so } \underset{s=s_0}{\text{Res}}\left\{\frac{e^{st}}{s+i\omega+\bar{F}(s)}\right\} = \frac{e^{s_0 t}}{1+\bar{F}'(s_0)}$$

$$\text{so } u(t) = \frac{e^{s_0 t}}{1+\bar{F}'(s_0)} - \int_{\rightleftarrows} \frac{e^{st}}{s+i\omega+F(s)}\,ds$$

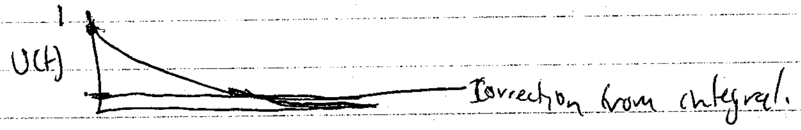
Approximation for s_0: $s_0 = -K - i(\omega + \delta\omega)$
there will be higher order corrections to this.

Correction to Weisskopf-Wigner: $(1+\bar{F}'(s_0))$ $\bar{F}'(s_0) \simeq O\left(\frac{K}{\omega}, \text{ or } \frac{\delta\omega}{\omega}\right)$

$$-\frac{1}{2\pi i}\int_{\rightleftarrows} \frac{e^{st}}{s+i\omega+F(s)}\,ds = -\frac{1}{2\pi i}\int_{\infty}^{0} e^{st}\left\{\frac{1}{s+i\omega+F(s)} - \frac{1}{s+i\omega+\bar{F}(s)}\right\}ds$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{0} \frac{e^{st}(\bar{F}(s)-F(s))}{(s+i\omega+F(s))(s+i\omega+\bar{F}(s))}\,ds$$

for $t=0$
this integral is $O\left(\frac{K}{\omega}, \text{ or } \frac{\delta\omega}{\omega}\right)$
At long times, this can dominate the exponential decay, giving another law (no experimental verification for this)



$$-i\lambda_k = M\sqrt{\frac{\omega_k}{2\hbar V}} \qquad K = \pi|\lambda(k_\omega)|^2 g(\omega)$$

$$2K = \frac{\Gamma}{\hbar} = \frac{2\pi}{\hbar}\left|M\sqrt{\frac{\hbar\omega_k}{2V}}\right|^2 \frac{1}{\hbar} g(\omega) = \frac{2\pi}{h}\,|\text{Matrix element}|^2 \cdot \rho(E)$$

Fermi's "Golden rule": Time-dependent transition probabilities.

$$\delta\omega = P\int \frac{|\lambda_k|^2 g(\omega)}{\omega - \omega_k}\,d\omega = P\int \frac{|M|^2 \frac{\omega_k}{2\hbar V} g(\omega)\,d\omega}{\omega - \omega_k}$$

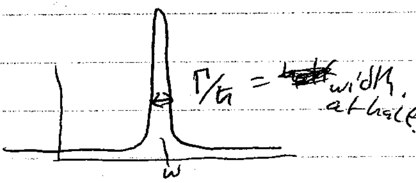
$c=1 \qquad dN = \frac{V\,d\vec{k}^3}{(2\pi)^3} = \frac{V 4\pi k^2}{(2\pi)^3}\,dk \quad , \quad \frac{dN}{d\omega} = g = \frac{V 4\pi k^2}{(2\pi)^3}$

so $\delta\omega = P\int \frac{|M|^2 \frac{\omega_k}{2\hbar}}{\omega - \omega_k}\frac{4\pi k^2}{(2\pi)^3}\,d\omega$ diverges as $\int k^2 d\omega$.

Remember $V_k(t) = \frac{-i\lambda_k}{K + i(\omega' - \omega_k)}\left\{e^{-i\omega_k t} - e^{-(K + i\omega' t)}\right\}$

for $t \gg 1/K \qquad |V_k(t)|^2 \to \frac{|\lambda_k|^2}{K^2 + (\omega' - \omega_k)^2}$

a typical resonance curve



$$= \frac{|\lambda_k|^2}{(\omega_k - \omega')^2 + \left(\frac{\Gamma}{2\hbar}\right)^2}$$

$a(t) = a(0)u(t) + \Sigma\, b_k(0) V_k(t)$

$$[a(t), a^\dagger(t)] = 1 = [a(0)u(t) + \Sigma b_k(0) V_k(t),\ a^\dagger(0)u^*(t) + \Sigma b_k^\dagger(0) V_k^*(t)]$$

$$1 = |u(t)|^2 + \Sigma |V_k(t)|^2$$

Weisskopf-Wigner Approximation: $|u(t)|^2 = e^{-2Kt}$, $\Sigma|V_k(t)|^2 = 1 - e^{-2Kt}$

Chaotic heat bath; $\sum_k \langle n_k\rangle |V_k(t)|^2$ assume $\langle n_k\rangle$ varies slowly with freq.

$$\Sigma\langle n_k\rangle|V_k(t)|^2 \approx \langle n_{\omega'}\rangle \Sigma|V_k(t)|^2$$
$$= \langle n_{\omega'}\rangle(1 - |u(t)|^2) = \langle n_{\omega'}\rangle(1 - e^{-2KT})$$

Recall: $P(\alpha 0|\gamma t) = \frac{1}{\pi \Sigma \langle n_k\rangle |V_k(t)|^2} e^{\frac{-|\gamma - \alpha u(t)|^2}{\Sigma \langle n_k\rangle |V_k(t)|^2}}$

In the Weisskopf-Wigner approx:

$$P(\alpha 0|\gamma t) = \frac{1}{\pi\langle n_{\omega'}\rangle(1-e^{-2\kappa t})} e^{-\frac{|\gamma - \alpha e^{-(\kappa + i\omega)t}|^2}{\langle n_{\omega'}\rangle(1-e^{-2\kappa t})}}$$

Width of gaussian increases monotonically, goes to $\langle n_{\omega'}\rangle$ eq. with oscillators of the renormalized frequency.

<u>Electrodynamics of spin-zero particles or Scalar Electrodynamics</u>

Lagrangian for charged $\underset{\text{scalar}}{}$ particles interacting with A^ν field (external)

$$\mathcal{L} = :(\partial^\nu - \tfrac{ie}{\hbar}A^\nu)\varphi^\dagger(\partial_\nu + \tfrac{ie}{\hbar}A_\nu)\varphi: - \mu^2 :\varphi^\dagger\varphi:$$

$$\mathcal{L}_0 = :\partial^\nu\varphi^\dagger\partial_\nu\varphi: - \mu^2 :\varphi^\dagger\varphi:$$

Normal order everything.

$$\mathcal{L} = \mathcal{L}_{0_{\text{scalar}}} - \frac{ie}{\hbar}(\varphi^\dagger\partial^\nu\varphi - \partial^\nu\varphi^\dagger\varphi)A_\nu + \frac{e^2}{\hbar^2}\varphi^\dagger\varphi A^\nu A_\nu$$

$$= \mathcal{L}_{0_{\text{scalar}}} - \frac{ie}{\hbar}(\varphi^\dagger\overleftrightarrow{\partial^\nu}\varphi)A_\nu + \frac{e^2}{\hbar^2}\varphi^\dagger\varphi A^\nu A_\nu$$

$\vec{a}_j$ is implicit in A^ν

$$\pi = (\partial^0 - \tfrac{ie}{\hbar}A^0)\varphi^\dagger \qquad \pi^\dagger = (\partial^0 + \tfrac{ie}{\hbar}A^0)\varphi$$

Add $L_{0_{EM}}$

The $\mathcal{H} = \pi\dot{\varphi} + \pi^\dagger\dot{\varphi}^\dagger - \mathcal{L}_{0\text{scalar}} + \mathcal{H}_{0_{EM}}$

$$= \pi(\pi^\dagger - \tfrac{ie}{\hbar}A^0\varphi) + \pi^\dagger(\pi + \tfrac{ie}{\hbar}A^0\varphi^\dagger) - \{\pi\pi^\dagger - (\nabla + \tfrac{ie}{\hbar}\vec{A})\varphi^\dagger \cdot (\vec{\nabla} - \tfrac{ie}{\hbar}\vec{A})\varphi$$
$$- \mu^2\varphi^\dagger\varphi\} + \mathcal{H}_{0_{EM}}.$$

(Normal ordered)

$$\mathcal{H} = \pi^\dagger\pi + (\nabla + \tfrac{ie}{\hbar}\vec{A})\varphi^\dagger(\nabla - \tfrac{ie}{\hbar}\vec{A})\varphi + \frac{ie}{\hbar}\{\pi^\dagger\varphi^\dagger - \pi\varphi\}A^0$$
$$\underline{(+\mu^2\varphi^\dagger\varphi +)\mathcal{H}_{0_{EM}}}$$

$$\mathcal{H} = \underbrace{(\pi^\dagger\pi + \nabla\varphi^\dagger\cdot\nabla\varphi + \mu^2\varphi^\dagger\varphi)}_{\mathcal{H}_{0\text{scalar}}} + \frac{ie}{\hbar}\varphi^\dagger\overleftrightarrow{\nabla}\varphi\cdot\vec{A} + \frac{ie}{\hbar}(\pi^\dagger\varphi^\dagger - \pi\varphi)A^0$$
$$+ \frac{e^2}{\hbar^2}(\vec{A})^2\varphi^\dagger\varphi + \mathcal{H}_{0_{EM}}.$$

$\mathcal{H} = \mathcal{H}_{0\,scalar} + \mathcal{H}_{0\,EM} + \mathcal{H}_1$

Interaction picture: $|t\rangle_{Schrod} = e^{-\frac{i}{\hbar}H_0 t}|t\rangle_I$

$H_0 = \int (\mathcal{H}_{0\,scalar} + \mathcal{H}_{0\,EM})\, d^3x$

$i\hbar \frac{d}{dt}|t\rangle_I = H_I(t)|t\rangle_I$ where $H_I(t) = \int \mathcal{H}_I(\vec{r},t)\, d\vec{r}$

$\mathcal{H}_I(r,t) = e^{\frac{i}{\hbar}H_0 t}\mathcal{H}_1(r,t)\, e^{-\frac{i}{\hbar}H_0 t}$

$\equiv \mathcal{H}_1(e^{\frac{i}{\hbar}H_0 t}\varphi\, e^{-\frac{i}{\hbar}H_0 t}, ---)$

$= \mathcal{H}_1(\varphi_I(r,t)\ldots$

$\varphi_I(r,t) = e^{\frac{i}{\hbar}H_0 t}\varphi_{Schrod}\, e^{-\frac{i}{\hbar}H_0 t}$

so φ_I is a free field

$\Pi_I^\dagger(r,) = e^{\frac{i}{\hbar}H_0 t}\Pi_{Schrod}\, e^{-\frac{i}{\hbar}H_0 t} = \dot{\varphi}_I$ (Since these are free fields now)

$\dot{\varphi}_I = \frac{1}{i\hbar}[\varphi_I, H_0] = \pi^\dagger$

$\mathcal{H}_I(r,t) = \frac{ie}{\hbar}(\varphi^\dagger \overleftrightarrow{\nabla}\varphi)\cdot\vec{A} + \frac{ie}{\hbar}(\varphi^\dagger\dot{\varphi} - \dot{\varphi}^\dagger\varphi)A^0 + \frac{e^2}{\hbar^2}(\vec{A})^2\varphi^\dagger\varphi$

$= \frac{ie}{\hbar}(\varphi^\dagger\partial^\nu\varphi - \partial^\nu\varphi^\dagger\varphi)A_\nu + \frac{e^2}{\hbar^2}\varphi^\dagger\varphi(\vec{A})^2$

$\mathcal{H}_I(r,t) = \frac{ie}{\hbar}(\varphi^\dagger\overleftrightarrow{\partial^\nu}\varphi)A_\nu + \frac{e^2}{\hbar^2}\varphi^\dagger\varphi(\vec{A})^2$

this is not a relativistic scalar because of 2nd term

$\mathcal{H}_I(r,t) = \frac{ie}{\hbar}(\varphi^\dagger\overleftrightarrow{\partial^\nu}\varphi)A_\nu - \frac{e^2}{\hbar^2}\varphi^\dagger\varphi\{A^\nu A_\nu - A^0 A_0\}$

Tomonaga: General $\mathcal{H}_I$ depends on spacelike surface it is evaluated at: $\mathcal{H}_I = \mathcal{H}_I(x,\sigma)$

If n is a unit normal to space-like surface,
$n = (1,0,0,0)$ for $\sigma \neq t = \text{const}$

Here $\mathcal{H}_I = \frac{ie}{\hbar}(\varphi^\dagger \overleftrightarrow{\partial^\nu} \varphi) A_\nu - \frac{e^2}{\hbar^2}\varphi^\dagger\varphi\{A^\nu A_\nu - (n\cdot A)^2\}$

this is covariant, but arbitrary n.

so, we'll just do all calculations with

$$\mathcal{H}_I(r,t) = \frac{ie}{\hbar}(\varphi^\dagger \overleftrightarrow{\partial^\nu} \varphi) A_\nu + \frac{e^2}{\hbar^2}\varphi^\dagger\varphi(\vec{A})^2$$

When we evaluate S-matrix, factors caused by vac. exp. of time derivatives will cancel the non-covariance, giving an invariant S-matrix.

Bjorken + Drell avoid this, work with interaction Lagrangian which is an invariant.

$$T(A(x),B(y)) = \frac{1}{2}\{A(x),B(y)\} + \frac{1}{2}[A(x),B(y)]\,\epsilon(x^0-y^0) \qquad \epsilon(x^0) = \begin{cases} 1 & x^0 \\ -1 & x^0 \end{cases}$$

$$\frac{\partial}{\partial x^\mu} T(A(x),B(y)) = \frac{1}{2}\left\{\frac{\partial A}{\partial x^\mu}, B\right\} + \frac{1}{2}\left[\frac{\partial A}{\partial x^\mu}, B\right]\epsilon(x^0-y^0)$$
$$+ \underset{\text{so } 0}{[A(x),B(y)]}\,\delta_{\mu 0}\,\delta(x^0-y^0)$$

Assume $[A(x),B(y)] = 0$ for $x^0=y^0$, a spacelike surface,

$$\frac{\partial^2}{\partial x^\mu \partial y^\nu} T(A(x),B(y)) = \frac{1}{2}\left\{\frac{\partial A}{\partial x^\mu}, \frac{\partial B}{\partial y^\nu}\right\} + \frac{1}{2}\left[\frac{\partial A}{\partial x^\mu}, \frac{\partial B}{\partial y^\nu}\right] - \left[\frac{\partial A}{\partial x^\mu}, B\right]\delta_{\nu 0}\,\delta(x^0-y^0)$$
$$= T\left(\frac{\partial A}{\partial x^\mu}, \frac{\partial B}{\partial y^\nu}\right) - \left[\frac{\partial A}{\partial x^\mu}, B\right]\delta_{\nu 0}\,\delta(x^0-y^0)$$

Let $A = \varphi^\dagger$ $B = \varphi$

$$T\left(\frac{\partial \varphi^\dagger}{\partial x^\mu}, \frac{\partial \varphi}{\partial x^\nu}\right) = \frac{\partial^2}{\partial x^\mu \partial y^\nu} T(\varphi^\dagger(x),\varphi(y)) + \underbrace{\left[\frac{\partial \varphi^\dagger}{\partial x^\mu}, \varphi(y)\right]\delta_{\mu 0}\,\delta(x^0-y^0)}_{\delta_{\mu 0}[\pi(x),\varphi(y)]\,\delta(x^0-y^0)}$$

for $\mu = 0$, $\dot{\varphi}^\dagger = \pi$

$$= \delta_{\mu 0}(-i\hbar)\,\delta(\vec{x}-\vec{y})\,\delta(x^0-y^0)$$

$$\text{so } T\left(\frac{\partial \varphi^\dagger}{\partial x^\mu}, \frac{\partial \varphi}{\partial y^\nu}\right) = \frac{\partial^2}{\partial x^\mu \partial y^\nu} T(\varphi^\dagger(x),\varphi(y)) - i\hbar\,\delta_{\mu 0}\delta_{\nu 0}\,\delta^4(x-y)$$

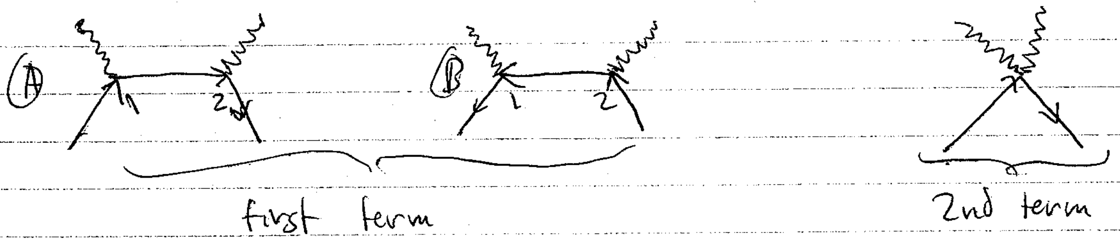
$$\langle 0|\,T\left(\frac{\partial \varphi^\dagger}{\partial x^\mu}, \frac{\partial \varphi}{\partial y^\nu}\right)|0\rangle = \frac{\partial^2}{\partial x^\mu \partial x^\nu}(i\Delta_F(x-y)) - i\hbar\,\delta_{\mu 0}\delta_{\nu 0}\,\delta^4(x-y)$$

$$S = T\left\{ e^{-i/\hbar \int \mathcal{H}_I d^4x} \right\}$$

$$\mathcal{H}_I = \mathcal{H}_I^{(1)} + \mathcal{H}_I^{(2)} = \frac{ie}{\hbar} (\varphi^+ \overleftrightarrow{\partial^\mu} \varphi) A_\mu + \frac{e^2}{\hbar^2} (\varphi^+ \varphi)(\vec{A})^2$$

Let $\hbar \to 0$

$$S^{(2)} = \frac{(-i)^2}{2!} \iint T\{\mathcal{H}_I^{(1)}(x_1), \mathcal{H}_I^{(1)}(x_2)\} d^4x_1 d^4x_2 - i \int \mathcal{H}_I^{(2)} d^4x$$



$$S^{(2)} = \frac{e^2}{2} \iint T\{\varphi^+(x_1)\, \partial_1^\mu \varphi(x_1)\, \varphi^+(x_2)\, \partial_2^\nu (x_2)$$
$$- \partial_1^\mu \varphi^+(x_1)\, \varphi(x_1)\, \varphi^+(x_2)\, \partial_2^\nu \varphi(x_2)$$
$$- \varphi^+(x_1)\, \partial_1^\mu \varphi(x_1)\, \partial_2^\nu \varphi^+(x_2)\, \varphi(x_2)$$
$$+ \partial_1^\mu \varphi^+(x_1)\, \varphi(x_1)\, \partial_2^\nu \varphi^+(x_2)\, \varphi(x_2)\} A_\mu(x_1) A_\nu(x_2)\, d^4x_1 d^4x_2$$

$$- ie^2 \int \varphi^+ \varphi (\vec{A})^2 d^4x$$

Doing the contractions (each term can be done two ways, corresponding to the two diagrams)

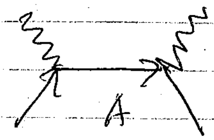
Ex: second term : $- \partial_1^\mu \varphi^+(x_1)\, \varphi(x_1)\, \varphi^+(x_2)\, \partial_2^\nu \varphi(x_2)$ or

~~First contraction~~, the extra term which appears in these all derivative contraction will, turn the 2nd term into a relativistic invariant.

the 2n term $\to + ie^2 \int \varphi^+ \varphi A^\mu A_\mu d^4x$ a scalar invariant.

$$\mathcal{H}_I = \frac{ie}{\hbar}\varphi^+ \overleftrightarrow{\partial_\nu}\varphi A^\nu + \frac{e^2}{\hbar^2}\varphi^+\varphi \vec{A}^2$$

Scattering of Photon by charged particle.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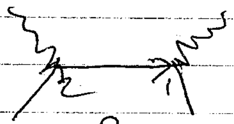


B

$$\hbar=1 \quad S_A^{(2)} = \frac{(-i)^2}{2}(ie)^2 \iint \Big\{ \partial_1^\nu \varphi_1 \langle 0|T(\varphi_1^+ \partial_2^\nu \varphi_2)|0\rangle \varphi_2^+$$
$$- \boxed{\varphi_1 \langle 0|T(\partial_1^\nu \varphi_1^+ \partial_2^\nu \varphi_2)|0\rangle \varphi_2^+}$$
$$- \partial_1^\nu \varphi_1 \langle 0|T(\varphi_1^+ \varphi_2)|0\rangle \partial_2^\nu \varphi_2^+$$
$$+ \varphi_1 \langle 0|T(\partial_1^\nu \varphi_1^+ \varphi_2 |0\rangle \partial_2^\nu \varphi_2^+ \Big\} A_\nu(x) A_\nu(x_2)\, d^4x_1\, d^4x_2$$

$$\cancel{S_A^{(2)}+} S_B^{(2)} = \frac{(-i)^2}{2}(ie)^2 \iint \varphi_1^+ \overleftrightarrow{\partial_1^\nu}\, i\Delta_F(x_1-x_2)\, \partial_2^\nu \varphi_2 A_\nu(x_1) A_\nu(x_2)\, d^4x_1\, d^4x_2$$
$$+ \frac{ie^2}{2}\int \varphi_1^+\varphi_1 A_0(x_1) A_0(x_1)\, d^4x_1$$

$$S^{(2)} = S_A^{(2)} + S_B^{(2)} + S_C^{(2)} \qquad S_C^{(2)} = -ie^2 \int \varphi^+\varphi \vec{A}^2\, d^4x$$

$$\to \frac{(-i)^2}{2}(ie)^2 \iint \Big\{ \varphi_1 \overleftrightarrow{\partial_1^\nu}\, i\Delta_F(x_1-x_2) \overleftrightarrow{\partial_2^\nu} \varphi_2$$
$$+ \varphi_1^+ \overleftrightarrow{\partial_1^\nu}\, i\Delta_F(x_1-x_2) \overleftrightarrow{\partial_2^\nu} \varphi_2 \Big\} A_\nu(x_1) A_\nu(x_2)\, d^4x\, d^4x_2$$

$$+ ie^2 \int \varphi_1^+\varphi_1 A^\nu(x_1) A_\nu(x_1)\, d^4x_1$$
$\to$ scalar invariant.

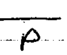
These results would be the same with

$$\mathcal{H}_{I\cancel{\ }}' = \left(\frac{ie}{\hbar}\varphi^+ \overleftrightarrow{\partial_\nu}\varphi\right) A^\nu - \frac{e^2}{\hbar^2}\varphi^+\varphi A^\nu A_\nu$$

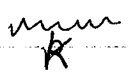
(if you ignore the second term in the $T(\partial_1^\nu \varphi_1^+ \partial_2^\nu \varphi_2)$)

Feynman rules:

① Propagators :- scalar: $\frac{i}{(2\pi)^4}\frac{1}{p^2-m^2+i\epsilon}$



(for A^μ, contravariant) photon : $-\frac{i}{(2\pi)^4}\frac{g^{\mu\nu}}{k^2+i\epsilon}$



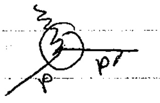
② External lines:

Invariant normalization:

$$\langle 0|\varphi(x)|p\rangle = e^{-ipx} \Rightarrow \text{factor } 1$$

$$\langle 0|A^\mu(x)|k\lambda\rangle = \rho^{(\lambda)\mu}(k)e^{-ikx} \Rightarrow \text{factor } \rho^{(\lambda)\mu}(k)$$

③ Vertices: "Meson Nucleon" theory: $(-ig)(2\pi)^4\delta^{(4)}(\Sigma p_{out}-\Sigma p_{in})$

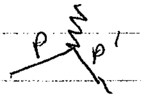


$$\varphi_1^\dagger \overleftrightarrow{\partial}_1^\mu\, i\Delta_F(x_1-x_2)\overleftrightarrow{\partial}_2^\nu\, \varphi_2$$

$$e^{ipx_1}\frac{\overleftrightarrow{\partial}}{\partial x_{1\mu}}\overset{\downarrow}{\frac{i}{(2\pi)^4}}\int\frac{e^{-ip'(x_1-x_2)}}{p'^2-m^2+i\epsilon}d^4p'$$

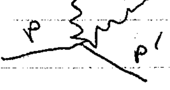
$$= -i(p^\mu+p'^\mu)$$

So, for one photon vertices: $(-i)(ie)(-i)(p+p')_\mu$



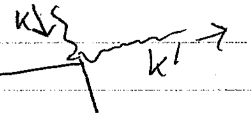
$$= -ie(p+p')_\mu\cdot(2\pi)^4\delta^{(4)}(\Sigma p_{out}-\Sigma p_{in})$$

Two-photon vertices:



take eff. H: $-\frac{e^2}{\hbar^2}\varphi^\dagger\varphi A^\mu A_\mu = \frac{-e^2}{\hbar^2}\varphi^\dagger\varphi\, g_{\mu\nu}A^\mu A^\nu$

If



k : absorbed
k' : emitted.

then $\langle k'|A^\mu(x)A^\nu(x)|k\rangle = \langle k'_{\lambda'}|A^\mu(x)|0\rangle\langle 0|A^\nu(x)|k\lambda\rangle$
$+\langle k'_{\lambda'}|A^\nu(x)|0\rangle\langle 0|A^\mu(x)|k\lambda\rangle$

since either A^μ or A^ν can act on k or k'

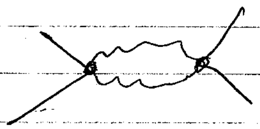
$$= e^{(\lambda')\mu *}_{(k')}e^{(\lambda)\nu}_{(k)} + e^{(\lambda')\nu *}_{(k')}e^{(\lambda)\mu}_{(k)}$$

Since $g_{\mu\nu}$ is symmetric.

For real e's $g_{\mu\nu}\langle k'\lambda'|A^\mu A^\nu|k\lambda\rangle = 2g_{\mu\nu}e^{(\lambda)\mu}_{(k)}e^{(\lambda')\nu}_{(k')}$

So, for the vertex: $2ie^2g_{\mu\nu}(2\pi)^4\delta^{(4)}(\Sigma p_{in}-\Sigma p_{out})$

Higher orders:



Closed loop of two photon lines.

Using the 2-factor here would cause double counting
To avoid double counting, insert factor of 1/2

Spin 1/2 particles

3-dim space: Ang. moment. ops. generators of rotations.
$\downarrow$
Unitary rotations in spin space

Pauli spin matrices σ^j $j = 1, 2, 3$ — 4 linearly indep. 2x2 matrices, $\sigma_j + I$

$$Q = -\vec{r} \cdot \vec{\sigma} = -\Sigma x^j \sigma^j$$

U unitary: $U^\dagger = U^{-1}$

$$\text{Let } Q' = U Q U^{-1} = -\Sigma x^j U \sigma^j U^{-1}$$

$$U \sigma^j U^{-1} = \Sigma \sigma^l a_l^{\ j} + a_0^{\ j}$$

Take trace of both sides:, since $\text{Tr}\, \sigma^j = 0$
then $a_0^{\ j} = 0$ for all j

$a_j^{\ l}$ are a real 3X3 matrix.

$$\text{So } U \sigma^j U^{-1} = \Sigma \sigma^l a_l^{\ j}$$

$$Q' = -\Sigma x'^j \sigma^j = -\Sigma \sigma^l a_l^{\ j} x^j$$

$$\text{Then } x'^j = \sum_l a_j^{\ l} x^l = a_j^{\ l} x^l \quad \text{a linear trans.}$$

$$Q'^2 = (\Sigma x^j \sigma^j)^2 = \Sigma x^{j2} = U Q^2 U^{-1} = U(\Sigma x^{j2}) U^{-1} = \Sigma x_j'^2$$

due to anticomm of pauli matrices

so U is an orthogonal transformation, preserves the sum of the squares

$$\Sigma x'^{j2} = \Sigma x^{j2}$$

$$Q = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$$

$$\det Q = -x^{3\,2} - (x_1^1 + ix^2)(x^1 - ix^2) = -\Sigma x^{j\,2}$$

$$\det Q' = \det U Q U^{-1} = \det U \det Q \det U^{-1}$$
$$= \det(UU^{-1}) \det Q = \det Q$$ So $\Sigma x_j'^2$ is preserved.

Inversions omitted

Can we generate the transformation $a_j^{\ell} = -\delta_j^{\ell}$?

i.e. $\sigma'^j = U\sigma^j U^{-1} = -\sigma^j$?

No! Since $\vec{\sigma} \times \vec{\sigma} = 2i\vec{\sigma}$
and $\vec{\sigma}' \times \vec{\sigma}' = 2i\vec{\sigma}'$ which is inconsistent with $\sigma' = -\sigma$

So we can only generate proper rotations. $\det a = 1$

Let $Q = t - (\vec{\sigma} \cdot \vec{r}) = - \begin{pmatrix} x^3 - x_0 & x^1 - ix_2 \\ x^1 + ix_2 & -x^3 - x_0 \end{pmatrix}$

$x^0 = t \quad x^j$

$$\det Q = -(x^3 - x^0)(x^3 + x^0) - (x^1 + ix^2)(x^1 - ix^2)$$
$$= x^{0\,2} - x^{3\,2} - x_1^2 - x_2^2 = x^{\mu}x_{\mu} = t^2 - r^2$$

Look for transformations that leave det Q invariant.
Unitary transformation

eg. $Q' = UQU^{-1} \quad U^{\dagger} = U^{-1}$

This corresponds to a spatial rotation, t is unchanged.
Want to preserve reality properties, hermiticity of Q!

~~To keep~~ Let $Q' = TQT^{\dagger}$ will preserve hermiticity

$x^{\mu}x_{\mu}$ preserved $\Rightarrow$

$$\det Q' = \det T \det Q \det T^{\dagger}$$
$$= (\det T)(\det T)^{*} \det Q$$

Require $|\det T|^2 = 1$ T contains arbitrary phase factor set it 0

so $\det T = 1$

The group of 2x2 'Unimodular transformations
Lorentz transformations.

$Q = t - \vec{\sigma}\cdot\vec{r}$ $\det Q = x^\nu x_\nu = t^2 - r^2$

Write $\sigma^\nu = (1, \vec{\sigma})$ $\sigma^0 = 1$

$\sigma_\nu = (1, -\vec{\sigma})$ σ^ν are a complete set.

$Q = \sigma^\nu x_\nu = \sigma_\nu x^\nu$

Any transformation of Q induces a linear trans on x^ν

Define $Q' = TQT^\dagger$ $\det T = 1$

$= \sigma_\nu x^{\nu\prime} = \sigma_\nu' x^\nu$ $x^{\nu\prime} x_{\nu\prime} = x^\nu x_\nu$

where $\sigma_\nu' = T\sigma_\nu T^\dagger = \sum_\lambda \sigma_\lambda a^\lambda{}_\nu$ — inverse transformations.

Use of $TQT^\dagger$ preserves the reality properties of Q.

$T\sigma_\nu T^\dagger = \sum_\lambda \sigma_\lambda a^\lambda{}_\nu$

Adjoint eq. $T\sigma_\nu T^\dagger = \sum_\lambda \sigma_\lambda a^{\lambda *}{}_\nu$

Subtracting: $\sum \sigma_\lambda (a^\lambda{}_\nu - a^\lambda{}_\nu{}^*) = 0 \Rightarrow a^\lambda{}_\nu{}^* = a^\lambda{}_\nu$

$\sigma_\nu x^{\nu\prime} = \sigma_\nu a^\nu{}_\lambda x^\lambda \Rightarrow x^{\nu\prime} = a^\nu{}_\lambda x^\lambda$

for $\nu = 0$ $TT^\dagger = \sum \sigma_j a^j{}_0 + a^0{}_0$ trace $\sigma^j = 0$ $j = 1,2,3$.

So, taking traces $a^0{}_0 = \frac{1}{2}\,\mathrm{Trace}\, TT^\dagger \geq 0$ also ≥ 1

$a^0{}_0 \geq 0 \Rightarrow$ transformations cannot reverse time.

$\sigma_\lambda a^\lambda{}_\nu = T\sigma_\nu T^\dagger$

$\langle i|\sigma_\lambda|j\rangle a^{*}_\nu = \sum_{k,l,\lambda} \langle i|T|k\rangle\langle k|\sigma_\nu|l\rangle\langle l|T^\dagger|j\rangle$

Let $(ij|\lambda) \equiv \langle i|\sigma_\lambda|j\rangle$ a 4x4 matrix.

$\sum_\lambda (ij|\lambda)\, a^\lambda{}_\nu = \sum_{kl} \langle i|T|k\rangle\langle j|T^*|l\rangle (kl|\nu)$

Outer product notation:

$$(A \otimes B)_{ij,kl} = A_{ik} B_{jl}$$ (2×2) matrix × (2×2) matrix = 4×4

$$\sum_{\lambda} (ij|\lambda)\, a^{\lambda}{}_{\nu} = \sum_{kl} \langle ij | T \otimes T^{*} | kl \rangle (kl|\nu)$$

$$\det(ij|\lambda)\, \det a = \det(T \times T^{*})\, \det(kl|\nu)$$

$\det(ij|\lambda) \neq 0$ since σ_λ are lin. indep.

so $$\det a = \det(T \otimes T^{*}) = \det(T \otimes 1)(1 \otimes T^{*})$$
$$= \det(T \otimes 1) \cdot \det(1 \otimes T^{*})$$
$$= \det(T)^2 \det(T^{*})^2 = |\det T|^4 = 1$$

You cannot have $\det a = -1$, (as shown before that spatial and time inversions were impossible)

So: $a^{\lambda}{}_{\nu}$: proper orthochronous Lorentz group.

$SO(3) \longleftrightarrow SU(2)$ for rotations
$1 \longleftrightarrow 2$

$SO(3,1) \longleftrightarrow SL(2,C)$ for Lorentz transformations
$1 \longleftrightarrow 2$ since T or $-T$ is same,

Infinitesimal transformations

Rotations: $\delta\vec{\omega}$ ∥ to axis of rotation
$|\delta\omega|$ = angle of rotation.

$$\delta\vec{A} = \delta\vec{\omega} \times \vec{A}$$

Passive transformations: rotations of coordinate axes.

$\hat{i}^j$ unit vectors along the coordinate axes.

$$\hat{i}^{j\prime} = \hat{i}^j + \delta\vec{\omega} \times \hat{i}^j \qquad x^{j\prime} = r \cdot \hat{i}^j = x^j + r \cdot (\delta\vec{\omega} \times \hat{i}^j)$$
$$= x^j + (r \times \delta\omega)^j$$

Lorentz transformations : $\delta\vec{v} \parallel x$-axis

$$x' = x - \delta v\, t \qquad t' = t - \delta v\, x$$

Arbitrary $\delta\vec{v}$: $$x^{j\prime} = x^j - \delta v^j x_0$$
$$x^{0\prime} = x^0 - \vec{r}\cdot\delta\vec{v}$$

combined : $$x^{j\prime} - x^j = -(\delta\vec{\omega} \times \vec{r})^j - \delta v^j x^0$$
$$x^{0\prime} - x^0 = -\delta\vec{v}\cdot\vec{r}$$

General 6-parameter infinitesimal transformation.

To find : Appropriate transformation matrix T that will induce this.

$$T\sigma_\nu T^\dagger = \underline{\sigma_\lambda a^\lambda{}_\nu} \Rightarrow x^{\lambda\prime} = a^\lambda{}_\nu x^\nu$$
$$\text{or } x^{\lambda\prime} = \underline{a_\lambda{}^\nu \, x_\nu}$$

$\rightarrow$ Inverse transformations.

To secure the transformation $\sigma_\lambda{}' = a_\lambda{}^\nu \sigma_\nu$ we require not T but T^{-1}

$$T^{-1}\sigma_\nu T^{-1\dagger} = \sigma_\lambda \tilde{a}^\lambda{}_\nu = a_\nu{}^\lambda \sigma_\lambda$$

We know $a_\nu{}^\lambda$
solve for T^{-1}

Infinitesimally: $T = 1 + \Sigma\sigma^\nu c_\nu$ $\qquad$ $\det T = \det(1 + \Sigma\sigma^\nu c_\nu)$ $\qquad$ $c_\nu = O(\delta\vec{\omega}, \delta\vec{v})$

$$\det(1 + \lambda A) \cong 1 + \lambda \operatorname{Tr} A + \cdots$$
λ small

precisely: $\frac{d}{d\lambda}\det(1+\lambda A)\big|_{\lambda=0} = \operatorname{Tr} A$

So $\det T = 1 + \operatorname{Tr}\Sigma\sigma^\nu c_\nu = 1 + 2c_0 \Rightarrow c_0 = 0$

since σ^j, $j \neq 0$ are traceless.

So $T = 1 + \vec{\sigma}\cdot\vec{c}$

$\vec{c} = \vec{A} + i\vec{B}$, arbitrary complex infinitesimal vector.

$$T = 1 + \vec{\sigma}\cdot(\vec{A} + i\vec{B})$$

$=$

$T^{-1} = 1 - \vec{\sigma}\cdot(\vec{A} + i\vec{B})$ since $T^{-1}\cdot T = 1$ to 1st order

$T^{-1}\sigma^{\lambda}T^{-1\dagger} = a^{\lambda}{}_{\nu}\sigma^{\nu}$ Know right side: Find T^{-1} or $\vec{A},\vec{B}$

$(1-\sigma\cdot(A+iB))\,\sigma^{\lambda}(1-\sigma(A-iB)) = a^{\lambda}{}_{\nu}\sigma^{\nu}$

$\lambda=1,2,3$ $\quad T^{-1}\sigma^{j}T^{-1\dagger} = \sigma^{j} - \{\sigma\cdot A, \sigma^{j}\}_{+} - i[\sigma\cdot B, \sigma^{j}]_{-} = \sigma^{j} - 2A^{j} + 2(\sigma\times B)^{j}$

since $(\sigma\cdot a)(\sigma\cdot b) = a\cdot b + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

for $\lambda=0$: $T^{-1}T = 1 - 2\vec{\sigma}\cdot\vec{A}$

since $x^{0\prime} - x^{0} = -\delta\vec{v}\cdot\vec{r}$

$\vec{r}\to\vec{\sigma}$ so $\vec{A} = \frac{\delta\vec{v}}{2}$

$T^{-1}\sigma^{j}T^{-1\dagger} = \sigma^{j} - 2A^{j} + 2(\sigma\times B)^{j}$

since $x^{j\prime} = x^{j} - \delta v^{j}x^{0}$, $\quad B = \frac{\delta\vec{\omega}}{2}$

So $\underline{T = 1 + \frac{1}{2}\sigma\cdot(\delta\vec{v} + i\delta\vec{\omega})}$: general six-parameter infinitesimal Lorentz trans.

for the rotations, T is unitary, for the boosts it is not.

Summary: $Q = t - \vec{\sigma}\cdot\vec{r} = \sigma^{\nu}x_{\nu}$ $\quad \sigma^{0}=1$

Instead, if we had chosen:

$$\mathcal{Q} = t + \sigma\cdot r = -\sigma^{\nu}x_{\nu}, \quad \sigma^{0} = -1$$

Let $\mathcal{Q}' = \mathcal{T}\mathcal{Q}\mathcal{T}^{\dagger}$

these transformations have the same properties as before. Go through same calculations.

Q and $\mathcal{Q}$ are not unitarily equivalent.

$\sigma_1^{*} = \sigma_1$, $\sigma_3^{*} = \sigma_3$, $\sigma_2^{*} = -\sigma_2$

$\sigma^{2}\vec{\sigma}^{*}\sigma^{2} = -\vec{\sigma}$ $\quad$ so $\mathcal{Q} = \sigma^{2}Q^{*}\sigma^{2}$

This is the 1-1 correspondence between Q and $\mathcal{Q}$

$$\mathcal{Q}' = \sigma^2 Q'^* \sigma^2 = \sigma^2 (TQT^\dagger)^* \sigma^2$$

$$= \sigma^2 T^* Q^* T^{\dagger *} \sigma^2 = \sigma^2 T^* \sigma^2 \sigma^2 Q^* \sigma^2 \sigma^2 T^{\dagger *} \sigma^2$$

$$= \mathcal{T} \mathcal{Q} \mathcal{T}^\dagger$$ so $\mathcal{T} = \sigma^2 T^* \sigma^2$

$$T = 1 + \tfrac{1}{2}\sigma(\delta\vec{v} + i\delta\vec{\omega})$$

so $\mathcal{T} = 1 - \tfrac{1}{2}\vec{\sigma}(\delta\vec{v} - i\delta\vec{\omega})$

<u>To construct wave-equations</u>: Weyl equation.

Stationary state wave eq. $(E - H(\vec{p},\sigma))\,|\psi\rangle = 0$

Viewed from another Lorentz frame:

$$(E' - H(\vec{p}\,',\sigma))\,|\psi\rangle' = 0$$

want $|\psi\rangle' = A|\psi\rangle$, A a linear trans.

require that H only involve $\vec{p}$ linearly, (only plausible)

Choose $H = \vec{\sigma}\cdot\vec{p}$: $E - \vec{\sigma}\cdot\vec{p}\,|\psi\rangle = 0$

Let $Q = E - \sigma\cdot\vec{p}$ $\quad Q|\psi\rangle = 0$ and $Q'|\psi\rangle' = 0$

$Q' = TQT^\dagger$, $\quad TQT^\dagger|\psi\rangle' = 0 \Rightarrow QT^\dagger|\psi\rangle' = 0$

so $T^\dagger|\psi\rangle' = \text{const.}\,|\psi\rangle$

$$\underline{|\psi\rangle' = \text{const}\,T^{\dagger -1}|\psi\rangle}$$

If we choose $H = -\vec{\sigma}\cdot\vec{p}$

Then $\mathcal{Q}|\psi\rangle = 0$, then $\underline{|\psi\rangle' = \text{const.}\,\mathcal{T}^{\dagger -1}|\psi\rangle}$

So we have two wave eqs.

$$(E + \sigma\cdot p)(E - \sigma\cdot p)|\psi\rangle = 0$$

$$\left(E^2 - (\sigma\cdot p)^2\right)|\psi\rangle = 0$$

$$\left(E^2 - p^2\right)|\psi\rangle = 0$$

$$\left(E^2 - p^2\right) = 0 \quad \Rightarrow \quad m = 0$$

$E = \pm|\vec p|$ may contain spurrious roots.

$$\underbrace{\frac{\sigma\cdot p}{|\vec p|}}_{\text{Helicity}} |\psi\rangle = +|\psi\rangle$$

helicity here is alway +1

for the 2 eq. $\frac{\sigma\cdot p}{|\vec p|}|\psi\rangle = -|\psi\rangle$

Only possible for $v = c$

~

This is the orthochronous proper subgroup of the Lorentz group, we have been discussing 2-dim. representations

$Q(x) = t - \vec\sigma\cdot\vec r \qquad Q(p) = p^0 - \vec\sigma\cdot\vec p$

$Q' = TQT^{\dagger} \qquad \sigma_0 = \sigma^0 = 1 \qquad \det T = 1$

$T\sigma_\nu T^{\dagger} = \sum \sigma_\lambda a^\lambda_\nu \qquad x'^\lambda = \sum_\nu a^\lambda_\nu x^\nu \qquad T = 1 + \frac{1}{2}\sigma\cdot(i\delta\vec\omega + \delta\vec v$

Inequivalent form:

$\mathcal{Q} = t + \vec\sigma\cdot\vec r$

$\mathcal{Q}' = \mathcal{T}\,\mathcal{Q}\,\mathcal{T}^{\dagger}$

correspondence: $\mathcal{Q} = \sigma^2 Q^* \sigma^2$

$\mathcal{T} = \sigma^2 T^{\dagger} \sigma^2 = 1 + \frac{1}{2}\sigma\cdot(i\delta\vec\omega - \delta\vec v)$

<u>Weyl Eqn:</u>

$E - H(p, \sigma)|\psi\rangle = 0$

$E' - H(p', \sigma)|\psi\rangle' = 0$

Choose $H = \sigma \cdot p$

$$Q(p)|\psi\rangle = 0$$

$$Q(p')|\psi\rangle' = Q'(p)|\psi\rangle' = TQT^{\dagger}|\psi\rangle' = 0$$

$$|\psi\rangle = T^{\dagger}|\psi\rangle' \qquad |\psi\rangle' = T^{\dagger -1}|\psi\rangle$$

Write $\psi(x) = \langle x|\psi\rangle = \underset{\searrow \text{2-comp. spinor.}}{u}\, e^{i(\vec{p}\cdot\vec{r} - Et)}$ $\quad \vec{p}, E$ are eigenvalu

So: $(E - \sigma \cdot \vec{p})\, u = 0$

$$(\sigma \cdot \vec{p})\, u = E u$$ ~~the~~

The eigenvalues of $\vec{\sigma}\cdot\vec{p}$ are $\pm|\vec{p}|$

Two solutions

1) u^{+}: $\quad E = +|\vec{p}| \qquad$ Helicity $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = +1$

2) u^{-} $\quad E = -|\vec{p}| \qquad$ " $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = -1$

these are negative energy solutions.

Alternatively choose: $H = -\vec{\sigma}\cdot\vec{p}$

1) $E = +|\vec{p}| \qquad$ Helicity $= -1$

2) $E = -|\vec{p}| \qquad$ " $= +1$

$$(E - \sigma \cdot p)\psi(x) = \sigma^{\nu} p_{\nu} \psi(x) = i\sigma^{\nu}\frac{\partial}{\partial x^{\nu}}\psi(x) = 0$$

adj. eq: $\frac{\partial}{\partial x^{\nu}}\psi^{\dagger}\sigma^{\nu} = 0$

So there is a conserved current

$$j^{\nu} = \text{const.}\ \psi^{\dagger}\sigma^{\nu}\psi$$

$(E + \sigma \cdot p)\psi(x) = 0 \Rightarrow j^{\nu} = \text{const.}\ \psi^{\dagger}\sigma^{\nu}\psi$ where $\sigma^{0} = 1$

$$\mathcal{Q} = \begin{pmatrix} Q & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} p^0 - \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & p^0 + \vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$$\mathcal{Q}' = \Pi \mathcal{Q} \Pi^{-1}$$

In particular chose $\Pi = \begin{pmatrix} T & 0 \\ 0 & \tilde{J} \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & \sigma^2 T^{\dagger} \sigma^2 \end{pmatrix}$

Spatial inversion corresponds to $Q \rightleftarrows 2$

So choose $\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

analog to σ_1 analog to σ_2

So $\mathcal{Q}$ corresponds to a reducible rep. of the proper orthochronous subgroup.
$\mathcal{Q}$ corresponds to an irreducible rep. of the proper Lorentz subgroup.

$$\det\begin{pmatrix} T & 0 \\ 0 & \tilde{J} \end{pmatrix} = \det T \det \tilde{J} = 1$$

$$\det\begin{pmatrix} T & 0 \\ 0 & \tilde{J} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & T \\ \tilde{J} & 0 \end{pmatrix} = \det \tilde{J} \det T = 1$$

← since this is 4-dim matrix.

Require $\det \Pi = 1$

More general Weyl eq.

$$\mathcal{Q}\,\Psi(x) = 0$$

$$\begin{pmatrix} Q & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \psi(x) \\ \varphi(x) \end{pmatrix} = \begin{pmatrix} E - \sigma\cdot p & 0 \\ 0 & E + \sigma\cdot p \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0$$

$$(E - H)\,\Psi(x) = 0 \quad \text{if} \quad H = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}\cdot\vec{p}$$

$$\sigma^0 = 1,\ \sigma^1, \sigma^2, \sigma^3$$

$$\rho^0 = 1 = \rho_0 \qquad \rho_1, \rho_2, \rho_3 \ ; \ \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \rho_2$$

Set $\sigma^\nu \rho_\nu$: 16 4x4 matrices, all linearly independent.

Proof of linear independence:

$$\sum c_{\mu\nu}\,\sigma^{\mu}\rho_{\nu} = 0 \Rightarrow c_{\gamma\delta} + \sum \cdots \sigma\rho = 0$$

left multiply by $\sigma^{\gamma}\rho_{\delta}$ $\quad \frac{1}{4}\operatorname{Tr} c_{\gamma\delta} + 0 = 0$

$$\underline{c_{\gamma\delta} = 0}$$

<u>Define</u>: $\alpha^0 = \alpha_0 = \rho_0\sigma^0 = 1$

$$\alpha^j = \rho_3\sigma^j \qquad j = 1,2,3 \qquad \sigma^j = \begin{pmatrix}\sigma^j & 0\\ 0 & \sigma^j\end{pmatrix}$$

Then $\alpha^{j\dagger} = \alpha^j \quad \alpha^{j\,2} = 1 \quad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta_{ij}$ $\quad i,j = 1,2,3$

$$Q = E - \begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}\cdot p = p^0 - \rho_3\vec{\sigma}\cdot\vec{p} = p^0 - \vec{\alpha}\cdot\vec{p}$$

$$= \alpha^{\mu} p_{\nu} \qquad \alpha_{\nu} = (\alpha_0, -\vec{\alpha})$$

$$Q' = \Pi Q \Pi^{\dagger}$$

$\Pi\,\alpha_{\nu}\,\Pi^{\dagger} = \sum \alpha_{\lambda}\, a^{\lambda}{}_{\nu}$ $\quad$ induces $X'^{\lambda} = \sum a^{\lambda}{}_{\nu} X^{\nu}$

this is a Lorentz transformation since $\det \Pi = 1$

This does not include time inversions, since $\Pi\Pi^{\dagger} = -1$ so $\operatorname{Tr}\Pi\Pi^{\dagger} = -4$ but here, obviously. $\operatorname{Tr}\Pi\Pi^{\dagger} \geq 0$

Infinitesimal transformations

$$\Pi = 1 + \tfrac{i}{2}\vec{\sigma}\cdot\delta\vec{\omega} + \tfrac{1}{2}\rho_3\vec{\sigma}\cdot\delta\vec{v}$$

$$= 1 + \tfrac{i}{2}\vec{\sigma}\cdot\delta\vec{\omega} + \tfrac{1}{2}\vec{\alpha}\cdot\delta\vec{v}$$

Note that ρ_1 is invariant under these transformations

$$\Pi\rho_1\Pi^{\dagger} = (1 + \tfrac{i}{2}\sigma\cdot\delta\omega + \tfrac{1}{2}\rho_3\sigma\cdot\delta v)\,\rho_1\,(1 - \tfrac{i}{2}\sigma\cdot\delta\omega + \tfrac{1}{2}\rho_3\sigma\cdot\delta v)$$

$$= \rho_1(1 + \tfrac{i}{2}\sigma\cdot\delta\omega \cancel{- \tfrac{1}{2}\alpha\cdot\delta v})(1 - \tfrac{i}{2}\sigma\cdot\delta\omega \cancel{+ \tfrac{1}{2}\alpha\cdot\delta v})$$ cancel to 1st order

Likewise ρ_2 ..... etc.

For the inversion op., we can have $\Pi = \pm\rho_1, \pm i\rho_1$

choose $\Pi = \rho_1$ $\qquad \rho_1\rho_1\rho_1^\dagger = \rho_1$

Conditions on Π are

1) $\det \Pi = 1$
2) require $\Pi\rho_1\Pi^\dagger = \rho_1$

Weyl eqns: $Q\Psi(x) = (E - \vec{\alpha}\cdot\vec{p})\,\Psi(x) = 0$

$$H = \vec{\alpha}\cdot\vec{p}.$$

Transformed system $\quad Q'\Psi' = 0 \qquad Q' = \alpha_\nu p^{\nu'} = \Pi Q \Pi^\dagger$

New state: $\Psi' = \Pi^{\dagger^{-1}}\Psi$

We can now add to H a multiple of ρ_1, and retain invarianc

$$H = \vec{\alpha}\cdot\vec{p} + \text{const}\cdot\rho_1$$

So: New eq: $(E - \alpha\cdot p - \text{const}\,\rho_1)\,\Psi(x) = 0$

$$(E + \alpha\cdot p + \text{const}\,\rho_1)(E - \alpha\cdot p - \text{const}\,\rho_1)\,\Psi(x) = 0$$

$$(E^2 - (\alpha\cdot p)^2 + \text{const}\,\underbrace{\cancel{\{\rho_1, \alpha\cdot p\}}}_{\to 0} - \text{const}^2\,\rho_1^2)\,\Psi(x) = 0$$

so $\{E^2 - p^2 - \text{const}^2\}\,\Psi(x) = 0$

so $\text{const}^2 = m^2$ $\qquad \text{const} = \pm m$

choose const $= +m$

$$\boxed{(E - \vec{\alpha}\cdot\vec{p} - m\rho_1)\,\Psi = 0} \qquad \text{Dirac: } \beta = \rho_1$$

This is the Dirac eqn.

$$\boxed{\begin{array}{l}(\alpha\cdot p + \beta m)\Psi = E\Psi \\ \text{where } \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta_{ij} \quad \alpha^{j\dagger} = \alpha^j \\ \beta^2 = 1 \quad \beta^\dagger = 1 \quad \beta\alpha^j + \alpha^j\beta = 0\end{array}}$$

Ψ a 4-comp. spinor column

$\alpha^{j^2} = \beta^2 = 1$ so they have eigenvalues ± 1

$$\mathrm{Tr}\,\alpha^j = \mathrm{Tr}\,\beta^2\alpha^j \overset{\text{by prop. of Tr}}{=} \mathrm{Tr}\,\beta\alpha^j\beta$$

$$\overset{\text{by } \{\}}{=} -\mathrm{Tr}\,\alpha_j = 0$$

$$\mathrm{Tr}\,\beta = \mathrm{Tr}\,\alpha_j^2\beta = -\mathrm{Tr}\,\beta = 0$$

Tr = 0 and eigenvalues ±1, implies equal numbers of each implies even dimension

2, no good, must use at least 4.

Weyl Representation

$$\vec\alpha = \begin{pmatrix} \vec\sigma & 0 \\ 0 & -\vec\sigma \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Standard Representation

$$\vec\alpha = \begin{pmatrix} 0 & \vec\sigma \\ \vec\sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

External Field: Minimal coupling:

$$p^0\psi = E\psi = (\alpha\cdot(\vec p - e\vec A) + \beta m + e\varphi)\psi$$

$$(p^0 - H)\Psi = \alpha^\nu\left(i\frac{\partial}{\partial x^\nu} + eA_\nu\right)\psi = 0 \qquad A_\nu = (\varphi, -\vec A) \quad \alpha^0 = 1$$

Adjoint eq.
ψ a row spinor

$$\left(-i\frac{\partial}{\partial x^\nu} + eA_\nu\right)\psi^\dagger\alpha^\nu = 0$$

multiply first by $\psi^\dagger$, 2nd by ψ, subtract.

$$\frac{\partial}{\partial x^\nu} \psi^\dagger \alpha^\nu \psi = 0 \quad ,$$

so $j^\nu(x) = \underbrace{\text{const}}\ \psi^\dagger \alpha^\nu \psi$ is a conserved current.

$j^0(x) = \rho(x) = \psi^\dagger \psi$ is positive definite

$\vec{j}(x) = \psi^\dagger \vec{\alpha} \psi$

"$\vec{\alpha}$ an operator representation of $\vec{v}$"

but α's only eigen~~states~~ values are $\pm c$

So you must supperimpose these solutions to get realistic ones : Zitterbewegung.

Weyl Rep.

$$\vec{j}(x) = (\psi^\dagger\ \varphi^\dagger) \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

$$= \psi^\dagger \vec{\sigma} \psi - \varphi^\dagger \vec{\sigma} \varphi$$

Infinitesimal Lorentz Transformations

$$\hat{\Pi} = 1 + \tfrac{1}{2} \vec{\sigma} \cdot \delta\vec{\omega} + \tfrac{1}{2} \vec{\alpha} \cdot \delta\vec{v} \qquad \text{Take } \delta\vec{\omega} = 0$$

not unitary.

$$\overset{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}{\hat{q}}$$

$$\Pi(v) \rightarrow \Pi(v + \delta v)$$

$\Pi(v+\delta v) \overset{?}{=} (1 + \tfrac{1}{2} \vec{\alpha} \cdot \hat{q}\, \delta v)\, \Pi(v)$ this is not true

but velocities don't add like this relativistically, δv should be relative velocity, not lab. velocity.

rel. v. $= \dfrac{v_1 - v_2}{1 - v_1 v_2}$ $\quad v_1 = v + \delta v,\ v_2 = v,$

$$= \frac{\delta v}{1 - v^2}$$

$$\Pi(v + \delta v) = \left(1 + \tfrac{1}{2} \vec{\alpha} \cdot \hat{q} \frac{\delta v}{1 - v^2}\right) T(v)$$

$$\frac{\partial}{\partial u}\Pi(u) = \frac{1}{2}(\vec\alpha\cdot\hat q)\frac{1}{1-u^2}\Pi(u)$$

$$\Pi(u) = e^{\frac{1}{2}\vec\alpha\cdot\hat q\int_0^u \frac{du'}{1-u'^2}}$$

$$= e^{\frac{1}{2}\vec\alpha\cdot\hat q\tanh^{-1}u}$$

$$\Pi(\vec v) = e^{\frac{1}{2}\frac{\alpha\cdot\vec v}{|\vec v|}\tanh^{-1}|\vec v|}$$

Velocities in different directions.

$\frac{v}{c} \ll 1$, i.e. for $c=1$ $|\vec v| \ll 1$

$$\Pi \approx 1 + \frac{1}{2}\vec\alpha\cdot\vec v$$

$$\Pi(v+\delta v) \approx (1+\frac{1}{2}\vec\alpha\cdot\delta\vec v)\Pi(v) = (1+\frac{1}{2}\alpha\cdot\delta v)(1+\frac{1}{2}\alpha\cdot v$$

$\delta v, v$ not collinear

$$\Pi(v+\delta v) = 1 + \frac{1}{2}\alpha\cdot(v+\delta v) + \frac{1}{2}(\alpha\cdot\delta v)(\alpha\cdot v)$$

Using $\alpha^x\alpha^y = i\sigma^z$ use this to define the σ's

since $(\vec\alpha\cdot\vec A)(\vec\alpha\cdot\vec B) = \vec A\cdot\vec B + i\vec\sigma(\vec A\times\vec B)$

$$\Pi(v+\delta v) = 1 + \frac{1}{2}\alpha\cdot(v+\delta v) + \frac{1}{4}\delta v\cdot v + \underbrace{\frac{i}{4}\vec\sigma\cdot(\delta\vec v\times\vec v)}_{\text{a rotation } \delta\vec\omega = \frac{1}{2}\delta\vec v\times\vec v}$$

If $\vec v = \vec v(t)$

angular velocity $\vec\omega = \frac{1}{2}\dot{\vec v}\times\vec v$: Thomas Precession.

Free particle Dirac eq:

$$i\frac{\partial}{\partial t}\psi(x) = (-i\alpha\cdot\nabla + \beta m)\psi(x)$$

$$\psi(x) = u(p)\,e^{-ipx} = u(p)\,e^{i(p\cdot r - p^0 t)}$$

At rest $\vec p = 0$ $\quad p^0 u = \beta m u$

Diagonalize $\beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Two Alternatives

$\beta = 1 \quad p^0 = m$:

Two sols : $U_{(0)}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \sqrt{2m} \quad U_{(0)}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sqrt{2m}$

$\Psi(x) = u(r) e^{-imt}$ $\uparrow$ spin in + dir. $\uparrow$ spin in − dir.

$\beta = -1 \quad p^0 = -m$

negative energy.

$\Psi(x) = e^{imt}$

$$U_{(0)}^{(-1)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad U_{(0)}^{(-2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Normalization : $U^{(r)\dagger} U^{(s)} = 2m\, \delta_{rs}$

for $\vec{p} \neq 0$

Transform to frame moving with velocity $-\vec{v}$

$$\Psi'(x') = \Pi^{+^{-1}}(-\vec{v})\, \Psi(x) \overset{\text{For this cae}}{=} \Pi(v)\, \Psi(x)$$

$$\Psi'(x') = e^{\frac{1}{2}\frac{\vec{\alpha}\cdot\vec{v}}{|v|}\tanh^{-1} v}\, \Psi(x)$$

$e^x = \cosh x + \tanh x$

since $\left(\frac{\alpha \cdot v}{|v|}\right)^2 = 1$

$\left(\frac{\alpha \cdot v}{|v|}\right)^3 = \frac{\alpha \cdot v}{|v|}$

$$= \cosh\left(\tfrac{1}{2}\tanh^{-1} v\right) + \frac{\alpha \cdot v}{|v|}\sinh\left(\tfrac{1}{2}\tanh^{-1} v\right)\}\Psi(x)$$

using ½ angle formules

$$= \left\{\sqrt{\tfrac{1}{2}(\cosh\tanh^{-1} v + 1)} + \frac{\alpha \cdot v}{|v|}\sqrt{\tfrac{1}{2}(\cosh\tanh^{-1} v - 1)}\right\}\Psi(x)$$

but $\cosh x = \frac{1}{\sqrt{1-\tanh^2 x}} = \frac{1}{\sqrt{1-v^2}}$ if $x = \tanh^{-1} v$

$= \frac{E(p')}{m}$

$(m, 0) \xrightarrow{} (p^{0\prime}, p')$

$p^{\nu} p_{\nu} = m^2$

$$\text{so } \Psi'(x') = \left\{\sqrt{\frac{E(p')+m}{2m}} + \frac{\vec{\alpha}\cdot\vec{p}'}{p'}\sqrt{\frac{E(\vec{p}')+m}{2m}}\right\}\underbrace{\Psi(x)}_{U_{(0)}^{(r)} e^{-i\epsilon_r m t}}$$

$\epsilon_r = \frac{r}{|r|}$

$\vec{p}'$ in direction of z axis

$\frac{\alpha \cdot p'}{|p'|} = \alpha_z \qquad \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$ (standard rep)

$$u^{(1)}(p') = \begin{pmatrix} \sqrt{E(p')+m} \\ 0 \\ \sqrt{E(p')-m} \\ 0 \end{pmatrix} \qquad u^{(2)}(p') = \begin{pmatrix} 0 \\ \sqrt{E(p')+m} \\ 0 \\ -\sqrt{E(p')-m} \end{pmatrix}$$

$$u^{(-1)}(p') = \begin{pmatrix} \sqrt{E(p')-m} \\ 0 \\ \sqrt{E(p')+m} \\ 0 \end{pmatrix} \qquad u^{(-2)}(p') = \begin{pmatrix} 0 \\ -\sqrt{E(p')-m} \\ 0 \\ \sqrt{E(p')+m} \end{pmatrix}$$

$u^{(r)\dagger} u^{(s)} = 2E(p')\,\delta_{rs}$ for r,s of same sign.

$$\psi'(x') = u^{(r)}(p')\, e^{-i\epsilon_r mt}$$

$$mt = p'x' = p^{0\prime}t - \vec{p}\,'\cdot\vec{r}$$

$$\psi'(x') = u^{(r)}(p')\, e^{-i\epsilon_r p'x'}$$

$$\psi(x) = u^{(r)}(\vec{p}\,')\, e^{-i\epsilon_r p'x} \qquad \epsilon_r = \begin{matrix} 1 & r = 1,2 & p^{0\prime} = E_{p'} \\ -1 & r = -1,-2 & p^{0\prime} = -E_{p'} \end{matrix}$$

$u^{(r)\dagger}(p')\, u^{(s)}(p') = 2E_{p'}\,\delta_{rs}$ r, s of same sign.

$u^{(r)\dagger}(0)\, u^{(s)}(0) = 2m\,\delta_{rs}$ all 4 r,s.

So orthogonality is not preserved by Lorentz transformation (it is not unitary)

$\psi^\dagger\psi$ is the 4th comp. of a 4-vector: it is not an invariant

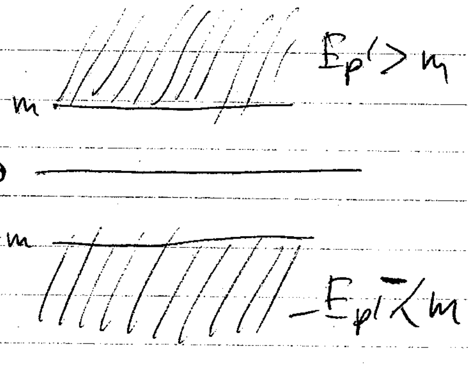
$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Define new ~~[illegible]~~ (not positive definite) scalar product $\psi^\dagger\beta\psi$

$u^{(r)\dagger}(p')\,\beta\, u^{(s)}(p') = 2m\,\epsilon_r\,\delta_{rs}$ all r,s.

This is an invariant.

$H = \alpha \cdot p + \beta m$



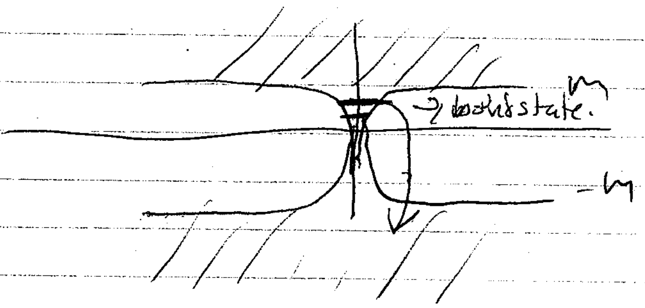
Eigen Values

Classically: no jumps, no worry about neg-energy states

Quantum: radiation field, jumps

Your ground state of H for instance, is not stable.

In coulomb field



for $r>0$ $(E_p - \vec{\alpha}\cdot\vec{p}' - \beta m)\, u^{(r)}(p') = 0$

$$\beta E_p - \beta\vec{\alpha}\cdot\vec{p}' - m)\, u^{(r)}(p') = 0$$

for $r<0$ $(E_p - \vec{\alpha}\cdot\vec{p} + \beta m)\, u^{(r)}(p') = 0$

$$(\beta E_{p'} - \beta\vec{\alpha}\cdot\vec{p}' + \beta m)\, u^{(r)}(p') = 0$$

let $v^{(i)} \cdot u^{(j)} = \delta_{ij}$

$u^{(j)} \Leftrightarrow v^{(j)}$

$u^{(j)}$ a complete set.

then $S = \sum_j u^{(j)} v^{(j)} = \mathbb{1}$

Proof: $S u^{(k)} = \sum_j u^j v^{(j)} \cdot u^{(k)} = u^{(k)}$

Take $v^{(r)} = \left(\frac{1}{2m} \mathrm{tr}\, u^{(r)\dagger} \beta\right)$

$v^{(r)} u^{(s)} = \delta_{rs}$ orthogonality

$\frac{1}{2m} \sum_r \mathrm{tr}\left(u^{(r)}(p')\, u^{(r)}(p')^{\dagger} \beta\right) = 1$ completeness.

$\sum_r u^{(r)} v^{(r)} = 1$

$\Lambda_\pm$: Projection operators on states of energy $\pm E_{p'}$

$$\Lambda_+ + \Lambda_- = 1 = \frac{1}{2m}\sum_r \epsilon_r \left(u^{(r)}(p')\, u^{(r)\dagger}(p')\, \beta\right)$$

$$\Lambda_+ = \frac{1}{2m}\sum_{r=1,2} u^{(r)}(p')\, u^{(r)\dagger}(p')\, \beta$$

$$\Lambda_- = -\frac{1}{2m}\sum_{r=-1,-2} u^{(r)}(p')\, u^{(r)\dagger}(p')\, \beta$$

from the Dirac equation:

$$\Lambda_+ = \frac{\beta E_{p'} - \beta\alpha\cdot p' + m}{2m} \qquad \Lambda_+ u^{(r)} = \begin{cases} u^{(r)} & r > 0 \\ 0 & r < 0 \end{cases}$$

$$\Lambda_- = -\frac{\beta E_{p'} - \beta\alpha\cdot p' - m}{2m}$$

<u>Angular Momentum</u>

Free particle $H = \alpha\cdot p + \beta m$

$L = \frac{r \times p}{\hbar} = \vec{r} \times \vec{p}$ Does $[H, L] = 0$

$$[L, H] = [\vec{r}\times\vec{p},\ \alpha\cdot p + \beta m] = [\vec{r},\ \alpha\cdot p] \times \vec{p}$$

$= i\,\vec{\alpha}\times\vec{p}$ so L is not a constant of the motion.

$$\alpha^1\alpha^2 = \rho_3\sigma^1\rho_3\sigma^2 = \sigma^1\sigma^2 = i\sigma^3 = i\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}$$

Define $\vec{\sigma} = \frac{1}{2i}\vec{\alpha}\times\vec{\alpha}$

$$[\sigma^1, H] = \frac{1}{i}[\alpha^2\alpha^3,\ \alpha\cdot p + \beta m] = \frac{1}{i}[\alpha^2\alpha^3,\ \alpha^2 p^2 + \alpha^3 p^3]$$

$$= \frac{1}{i}(-2\alpha^3 p^2 + 2\alpha^3 p^3) = -2i(\vec{\alpha}\times\vec{p})$$

Define $J = \vec{r}\times\vec{p} + \frac{1}{4i}\vec{\alpha}\times\vec{\alpha} = L + \underset{\uparrow}{\tfrac{1}{2}}\sigma$

Then $[J, H] = 0$

spin-½ particles

Pauli Notation:

$\alpha^\mu = (1, \vec{\alpha})$

$\beta\alpha^\mu = (\beta, \beta\vec{\alpha})$

$\gamma^\mu = (\gamma^0, \vec{\gamma}) = (\beta, \beta\vec{\alpha})$

Call $\gamma^\mu = \beta\alpha^\mu$ $\quad \gamma_\mu = (\gamma^0, -\vec{\gamma}) = (\beta, -\beta\vec{\alpha})$

$$\Lambda_+ = \frac{\gamma^\mu p_\mu + m}{2m} = \frac{\gamma p + m}{2m} = \frac{\not{p} + m}{2m}$$ (explicitly invariant)

$\gamma^\mu p_\mu = \not{p}$: Feynman notation

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ $\qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu$

$\gamma^{0\,2} = 1 \quad \gamma^{1\,2} = \gamma^{2\,2} = \gamma^{3\,2} = -1$

so γ^j $j = 1,2,3$ are skew hermitian: $\gamma^{j\dagger} = -\gamma^j$
$\gamma^{0\dagger} = \gamma^0$ is hermitian

$\Pi \alpha_\mu \Pi^\dagger = \alpha_\lambda a^\lambda{}_\mu$
$\Pi \alpha^\mu \Pi^\dagger = \alpha^\lambda a_\lambda{}^\mu$
} associated with $x^{\mu\prime} = a^\mu{}_\lambda x^\lambda$

$$\Pi^{-1} \alpha^\mu \Pi^{-1\dagger} = \alpha_\lambda a_\lambda{}^\mu(\Pi^{-1}) = a^\mu{}_\lambda(\Pi)\, \alpha^\lambda$$

$\Pi^{-1} \beta \Pi^{-1\dagger} = \beta$

Recall: $\psi'(x') = \Pi^{\dagger -1}\psi(x)$

Let $D = \Pi^{\dagger -1} = \Pi^{-1\dagger}$ $\qquad \Pi^{-1} = D^\dagger$

$D^\dagger \alpha^\mu D = a^\mu{}_\lambda a^\lambda$

$D^\dagger \alpha D = \beta$ $\qquad D^\dagger = \beta D^{-1} \beta$

$$j^\nu(x) = \psi^\dagger(x)\,\alpha^\nu\,\psi(x)$$

$$j^{\nu\prime}(x') = \psi'^\dagger(x')\,\alpha^\nu\,\psi'(x')$$

$$= (D\psi(x))^\dagger\,\alpha^\nu\,D\psi(x)$$

$$= \psi^\dagger(x)\,D^\dagger\alpha^\nu D\,\psi(x) \quad = a^\nu{}_\lambda\, j^\lambda(x)$$

So $j^\nu(x)$ transforms as a 4-vector.

Substituting in $D^\dagger\alpha^\nu D =$ $\beta D^{-1}\beta\alpha^\nu D = a^\nu{}_\lambda\,\alpha^\lambda$

multiplying by β: $\underline{D^{-1}\gamma^\nu D = a^\nu{}_\lambda\,\gamma^\lambda}$ $\quad \nu = 0, \ldots 4$

so $D^\dagger = \gamma^0 D^{-1}\gamma^0$

$$j^\nu(x) = \psi^\dagger(x)\,\beta^2\alpha^\nu\,\psi(x)$$

Define $\bar\Psi(x) = \psi^\dagger(x)\beta$: Pauli adjoint spinor.

$$j^\nu(x) = \bar\psi(x)\,\gamma^\nu\,\psi(x)$$

$\psi'(x') = D\,\psi(x)$ $\quad \bar\psi'(x') = \psi'^\dagger(x')\beta = (D\psi(x))^\dagger\beta$

$$= \psi^\dagger(x)\,D^\dagger\beta = \psi^\dagger(x)\,\beta D^{-1}$$

$$\underline{\bar\Psi'(x') = \bar\Psi(x)\,D^{-1}}$$

so $j^{\nu\prime}(x') = \Psi(x)\,D^{-1}\gamma^\nu D\,\psi(x)$

$$= a^\nu{}_\lambda\, j^\lambda(x)$$

Construct $\bar\Psi(x)\psi(x) = \psi^\dagger\beta\psi$ is indefinite

$\bar\Psi'(x')\,\psi'(x') =$

$$\bar\psi(x)\,D^{-1}D\,\psi(x)$$

$= \bar\Psi(x)\Psi(x)$ a scalar invariant

Consider: $\bar{\psi}'(x')\gamma^\mu\gamma^\nu\psi'(x') = \bar{\psi}(x)\, D^{-1}\gamma^\mu\gamma^\nu D\,\psi(x)$

$$= a^\mu{}_\lambda a^\nu{}_\rho\,(\bar{\psi}\gamma^\lambda\gamma^\rho\psi)$$

so $\gamma^\mu\gamma^\nu$ is a 2nd-rank tensor.

Divide into symmetric, antisymetric parts.

Symmetric: $\frac{1}{2}\bar{\psi}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\psi = g^{\mu\nu}\bar{\psi}\psi$

Antisymmetric: $\frac{1}{2}(\bar{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi)$

$$\frac{1}{2}(\gamma^1\gamma^2 - \gamma^2\gamma^1) = \frac{1}{2}(\beta\alpha^1\beta\alpha^2 - \beta\alpha^2\beta\alpha^1) = \frac{1}{2}\cdot 2(-\alpha^1\alpha^2)$$

$$= -\rho_3\sigma^1\rho_3\sigma^2 = -i\sigma^3$$

So: $\sigma_3 = \frac{-1}{2i}[\gamma^1, \gamma^2]$

Define $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

So antisymmetric part $(\times i)$:

$\bar{\psi}\sigma^{\mu\nu}\psi$ is a 2nd-rank antisymmetric tensor.

$\sigma^{01} = \frac{i}{2}[\gamma^0, \gamma^1] = i\gamma^0\gamma^1 = i\beta\beta\alpha^1 = i\alpha^1$

$\sigma^{0j} = i\alpha^j$

Dirac equation: Success with fine structure in Hydrogen spectrum.

$(E - \alpha\cdot p - \beta m)\psi = 0$

multiply by β: $\gamma^\mu p_\mu - \psi = 0$ $\quad p_\mu = i\partial/\partial x^\mu$

$\boxed{(\not{p} - m)\psi = 0}$ - covariant form

$\boxed{(i\not{\partial} - m)\psi = 0}$

Adjoint equation: $\psi^\dagger(E-\alpha\cdot p-\beta m)=0$

$$\psi^\dagger\beta^2(E-\alpha\cdot p-\beta m)=0$$

$$\bar\Psi(\gamma^\nu p_\nu - m)=0 \qquad \bar\Psi(\not p - m)=0$$

or $-i\partial_\nu\psi^\dagger\gamma^0\underbrace{\gamma^0\gamma^{\nu\dagger}}_{\gamma^\nu\gamma^0} - m\psi^\dagger=0$

multiply from left by γ^0

Then: $-i\partial_\nu\bar\Psi\gamma^\nu - m\bar\Psi=0$

so $i\partial_\nu\bar\Psi\gamma^\nu + m\bar\Psi=0 \quad\Leftrightarrow\quad \bar\Psi(i\not\partial - m)=0$

Bilinear covariants.

$\bar\Psi\psi$ scalar $\bar\Psi\sigma^{\nu\upsilon}\psi$: antisymmetric tensor

$\bar\Psi\gamma^\nu\psi$ vector

Construct $\bar\Psi\gamma^\nu\gamma^\upsilon\gamma^\lambda\gamma^\rho\psi$ 256 comp. 4th order tensor, but 255 of these are outer products of $g^{\nu\upsilon}$ and lower order objects already studied.

Define $\gamma^5\equiv\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$${\gamma^5}^2=1, \quad \gamma^5 \text{ is hermitian; } \gamma^{5\dagger}=\gamma^5$$

$\gamma^5\gamma^\nu = -\gamma^\nu\gamma^5$ for $\nu=0,\cdots 3$

$\bar\Psi\gamma^5\psi$ transforms as:

$$\bar\Psi'\gamma_5\psi' = i\bar\Psi'\gamma^0\gamma^1\gamma^2\gamma^3\psi'$$

$$= a^0{}_\nu a^1{}_\upsilon a^2{}_\lambda a^3{}_\rho\, i(\bar\Psi\gamma^\nu\gamma^\upsilon\gamma^\lambda\gamma^\rho\psi)$$

$$\gamma^\upsilon\gamma^\nu = -\gamma^\nu\gamma^\upsilon + 2g^{\nu\upsilon}$$

Interchanging two γ matrices here:

gives additional term $2a^0{}_\nu a^1{}_\upsilon g^{\nu\upsilon}[a^2{}_\lambda a^3{}_\rho i(\bar\Psi\gamma^\lambda\gamma^\rho\psi)]$

Show: $2 a^0_\mu a^1_\nu g^{\mu\nu} = 0$

$$x^{\mu\prime} = a^\mu_\lambda x^\lambda = a^\mu_\lambda g^{\lambda\rho} x_\rho = \underbrace{a^\mu_\lambda g^{\lambda\rho} a^\nu_\rho} x'_\nu$$

$$x'_\rho = a_\rho^{\ \lambda} x_\lambda$$
$$x_\lambda = x'_\rho a^\rho_\lambda \rightarrow \text{Transpose} = \text{inverse.}$$

So we have shown that a is orthogonal: $a^\mu_\lambda a^\nu_\rho g^{\lambda\rho} = g^{\mu\nu}$

$$\text{So } 2a^0_\mu a^1_\nu g^{\mu\nu} = 2g^{01} = 0.$$

So $\bar{\Psi}'\gamma_5\Psi'$ is antisymmetric under any interchange of γ matrices

$$\bar{\Psi}'\gamma_5\Psi' = a^0_\mu a^1_\nu a^2_\lambda a^3_\rho \, i(-1)^P \, \bar{\Psi} \underset{\hookrightarrow \text{permutation}}{P}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\Psi$$

Sum over all $4! = 24$ permutations

$$\bar{\Psi}'\gamma^5\Psi' = a^0_\mu a^1_\nu a^2_\lambda a^3_\rho \frac{1}{4!}\sum_P (-1)^P i \, \bar{\Psi} P(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\Psi$$

$\frac{1}{4!}\sum_P (-1)^P P(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 0$ unless μ, ν, λ, ρ are all different, i.e. some permutation of 0, 1, 2, 3.

$$\gamma^5 = \frac{i}{4!}\sum_P (-1)^P P(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = i\,\epsilon^{\mu\nu\lambda\rho}\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\epsilon^{\mu\nu\lambda\rho}\begin{cases} = 0 \text{ unless all different} \\ (-1)^P \quad \mu\nu\lambda\rho = P(0,1,2,3) \end{cases}$$

$$\bar{\Psi}'\gamma^5\Psi' = a^0_\mu a^1_\nu a^2_\lambda a^3_\rho \, \epsilon^{\mu\nu\lambda\rho}(\bar{\Psi}\gamma^5\Psi)$$
$$= (\det a)\, \bar{\Psi}\gamma^5\Psi \qquad \det a = \pm 1$$

So $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar, (a tensor density)

Examine $\bar{\Psi}\gamma^\mu\gamma^\nu\gamma^\lambda\Psi$ 3rd rank tensor, 64-components.

This has 4-interesting components, others just lower order things.

Discuss $\gamma^\nu\gamma^5$

$$\bar\psi'\gamma^\nu\gamma^5\psi' = \bar\psi D^{-1}\gamma^\nu\gamma^5 D\psi =$$

$$= a^\nu_\nu (\det a)\,\bar\psi\gamma^\nu\gamma^5\psi$$

so $\bar\psi'\gamma^\nu\gamma^5\psi'$ is a pseudovector (axial vector)

Scalar : $\bar\psi\psi$	1	$\Gamma^S = 1$
vectors ; $\bar\psi\gamma^\nu\psi$	4	$\Gamma^V = \gamma_\nu$
tensor : $\bar\psi\sigma^{\mu\nu}\psi$	6	$\Gamma^T = \sigma_{\mu\nu}$
Axial vector ; $\bar\psi\gamma^\nu\gamma^5\psi$	4	$\Gamma^A = \gamma_\nu\gamma^5$
Pseudoscalar : $\bar\psi\gamma^5\psi$	1	$\Gamma^P = \gamma^5$

$\gamma^\nu, 1$ are the generators of a 16-element algebra,

$$\Gamma^{x^2} = \pm 1$$

$$\{\Gamma^P, \Gamma^V\} = 0 \qquad \{\Gamma^T_{\mu\nu}, \Gamma^V_\nu\} = 0 \qquad \{\Gamma^A, \Gamma^P\} = 0$$

each Γ anticommutes with another Γ, except $\Gamma^S = 1$

$$\mathrm{Tr}\,\underset{x\neq s}{\Gamma^x} = \pm\mathrm{Tr}(\Gamma^x\Gamma^{m2}) \qquad \text{where } \{\Gamma^m, \Gamma^x\} = 0,\ \Gamma^{m2} = \pm 1$$

$$= \pm\mathrm{Tr}(\Gamma^m\Gamma^x\Gamma^m) = \mp\mathrm{Tr}(\Gamma^{m2}\Gamma^x)$$

$$= -\mathrm{Tr}\,\Gamma^x = 0$$

The 16 Γ^x are linearly independent.

Assume $\sum_1^{16} c_{xy}\Gamma^x_y = 0$ $\quad\sum^{16}\mathrm{Tr}\, c_{xy}\Gamma^x_y = 0$

$\Rightarrow c_{xy} = 0$ for all x, y

They also form a complete basis for 4x4 matrices.

Theorem: All representations of the Dirac Matrices are unitarily equivalent.

$j^\mu(x) = e\bar{\Psi}\gamma^\mu\Psi$

$\Psi = \frac{1}{m} i\gamma^\nu\partial_\nu\Psi$ (using Dirac eq.

$\bar{\Psi} = -\frac{i}{m}\partial_\nu\bar{\Psi}\gamma^\nu$

$$j^\mu(x) = -\frac{ie}{2m}\{(\partial_\nu\bar{\Psi})\gamma^\nu\gamma^\mu\Psi - \bar{\Psi}\gamma^\mu\gamma^\nu\partial_\nu\Psi\}$$

$$= -\frac{ie}{2m}\underbrace{\{(\partial^\mu\bar{\Psi})\Psi - \bar{\Psi}\partial^\mu\Psi\}}_{\text{using } \nu=\mu\ \gamma^{\mu 2}=g^{\mu\mu}} + \sum_{\nu\neq\mu}\left(+\frac{ie}{2m}\right)\partial_\nu(\bar{\Psi}\gamma^\mu\gamma^\nu\Psi)$$

Define $M^{\mu\nu} = \frac{ie}{2m}\bar{\Psi}\gamma^\mu\gamma^\nu\Psi \quad \mu\neq\nu$

$= 0 \quad \mu = \nu$

$= \frac{e}{2m}\bar{\Psi}\sigma^{\mu\nu}\Psi$

for $\mu,\nu = 1,2,3$: $\vec{M} = \frac{e}{m}(\bar{\Psi}\,\frac{1}{2}\vec{\sigma}\,\Psi)$

↖ gyro magnetic ratio.

Gyromagnetic ratio is usually $e/2mc$, here e/mc,

Let $j^\mu_M = \partial_\nu M^{\mu\nu}$

Note that j^μ_M is separately conserved:

$\partial_\mu j^\mu_M = \partial_\mu\partial_\nu M^{\mu\nu} = 0$ since M is antisymmetric.

Let $j^\mu_c = \frac{-ie}{2m}\{(\partial^\mu\bar{\Psi})\Psi - \bar{\Psi}\partial^\mu\Psi\}$ a convection current

$j^\mu = j^\mu_c + j^\mu_M$

For a plane wave: $j^\mu_c = \frac{-ie}{2m}(2ip^\mu)(\bar{\Psi}\Psi) = \frac{ep^\mu}{m}(\bar{\Psi}\Psi)$

$\sim e\vec{v}\rho$

In a plane wave $j^\mu_M = 0$

This is the Gordon decomposition!

What is j^{ν}_{M}?

$\nu=1$ $\quad j^1_M = \partial_0 M^{10} + \partial_2 M^{12} + \partial_3 M^{13}$

$$= \partial_0 \frac{ie}{2m}(\bar\psi\gamma^1\gamma^0\psi) + \partial_2 \frac{e}{2m}(\bar\psi\sigma^{12}\psi) + \partial_3 \frac{e}{2m}(\bar\psi\sigma^{13}\psi)$$

$$= \partial_0 \frac{-ie}{2m}(\bar\psi\alpha\psi) + \left(\vec\nabla\times\frac{e}{2m}(\bar\psi\vec\sigma\psi)\right)^1$$

Define $\vec P = -\frac{ie}{2m}(\bar\psi\vec\alpha\psi)$

$\vec M = \frac{e}{2m}(\bar\psi\vec\sigma\psi)$

$$\vec j_M = \frac{\partial}{\partial t}\vec P + \vec\nabla\times\vec M$$

In Media: $\vec P$ = polarization vector
$\vec M$ = magnetization vector

so $\vec P$ = effective polarization density
$\vec M$ = eff. magnetisation density

$$j^0_M = \partial_1 M^{01} + \partial_2 M^{02} + \partial_3 M^{03}$$

$$= \partial_1 \frac{ie}{2m}\bar\psi\gamma^0\gamma^1\psi + \cdots$$

$$= -\vec\nabla\cdot\vec P$$

In media: $\vec\nabla\cdot\vec P$ = Bound charge density.

j^0_M = effective charge density

Gordon decomposition

$$j^\nu = j_c^\nu + j_M^\nu$$

$$j_c^\nu = \frac{-ie}{2m}\left\{(\partial^\nu \bar{\psi})\psi - \bar{\psi}\,\partial^\nu \psi\right\} \Rightarrow \vec{j} = \frac{+ie}{2m}\left\{(\nabla\vec{\bar{\psi}})\psi - \bar{\psi}\nabla\psi\right\}$$

since $\partial^j = \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x^j}$

$$j_M^\nu = \partial_\nu M^{\nu\nu} \text{ where } M^{\nu\nu} = \frac{e}{2m}\bar{\psi}\,\sigma^{\nu\nu}\psi \Rightarrow \vec{M} = \frac{e}{2m}\bar{\psi}\vec{\sigma}\psi$$

$$\vec{j}_M = \partial_0\vec{P} + \nabla\times\vec{M} \qquad \vec{P} = -ie/_{2m}\,\bar{\psi}\vec{\alpha}\psi$$

$$j_M^0 = -\nabla.\vec{P}$$

Magnetic moment: $\vec{m} = \frac{1}{2c}\int \vec{r}\times\vec{j}\,d\vec{r} \quad : c=1$

$$= \frac{1}{2}\int \vec{r}\times\vec{j}_c\,d\vec{r} + \frac{1}{2}\int \vec{r}\times\vec{j}_M\,d\vec{r}$$

Assume a stationary state

$\partial_0 P = 0$

$$\int \vec{r}\times\vec{j}_M\,d\vec{r} = \int \vec{r}\times(\vec{\nabla}\times\vec{M})\,d\vec{r}$$

$$\int(\vec{r}\times\vec{j}_m)_1\,d\vec{r} = \int\left\{y\left(\frac{\partial}{\partial x}M_y - \frac{\partial}{\partial y}M_x\right) - z\left(\frac{\partial}{\partial z}M_x - \frac{\partial}{\partial x}M_z\right)\right\}d\vec{r}$$

$$= \int\left\{\frac{\partial}{\partial x}(yM_y + zM_z) - \frac{\partial}{\partial y}yM_x - \frac{\partial}{\partial z}zM_x + 2M_x\right\}d\vec{r}$$

$$= \int \cancel{\frac{\partial}{\partial x}(r.M)} - \cancel{\nabla.(\vec{r}M_x)} + 2M_x\,d\vec{r}$$

$\to 0$ when integrated.

$$\int(\vec{r}\times\vec{j}_m)_1\,d\vec{r} = 2\int M_x\,d\vec{r}$$

so: $\frac{1}{2}\int \vec{r}\times\vec{j}_m\,d\vec{r} = \int \vec{M}\,d\vec{r}$

$$\frac{1}{2}\int \vec{r}\times\vec{j}_c\,d\vec{r} = \frac{ie}{4m}\int \vec{r}\times(\nabla\bar{\psi}\psi - \bar{\psi}\nabla\psi)\,d\vec{r}$$

$$\int(r\times\nabla\bar{\psi})\psi\,d\vec{r} = -\int(\nabla\bar{\psi}\times r)\psi\,d\vec{r} = -\int\bar{\psi}\,r\times\nabla\psi\,d\vec{r}$$

so $\frac{1}{2}\int r\times j_c\,d\vec{r} = \frac{-ie}{2m}\int\bar{\psi}\,r\times\nabla\psi\,d\vec{r}$

$$\vec{m} = \frac{e}{2m}\int \bar\Psi\left\{r \times \frac{1}{i}\nabla + \vec\sigma\right\}\Psi\, d\vec r$$

$$= e/2m \int \bar\Psi\left\{\vec r \times \vec p + \underset{\nwarrow}{\vec\sigma}\right\}\varphi\, d\vec r$$

<u>Recall</u>: $J = \vec r \times \vec p + 1/2\,\vec\sigma$

So the electron has <u>twice</u> the gyromagnetic ratio of the orbital component, the spin is doubly weighted.
Contribution of spin is one Bohr magneton: $\frac{e\hbar}{2mc}$

<u>Non-relativistic limit</u>

$$E\Psi = (\vec\alpha\cdot\vec\pi + \beta m + V)\Psi$$

$\vec\pi = \vec p - e\vec A$
$V = eA^0$
$\vec E = -\nabla V = -e\nabla A^0$

Choose representation: Standard representation

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_3 \qquad \vec\alpha = \begin{pmatrix} 0 & \vec\sigma \\ \vec\sigma & 0 \end{pmatrix} = \rho_1\vec\sigma$$

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \begin{matrix} \text{upper} \\ \text{lower} \end{matrix}$$

for $E>0$, φ large, χ small
$E<0$, φ small, χ large.

$$\beta\begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \qquad \beta \to 1$$

$$\beta\begin{pmatrix} 0 \\ \chi \end{pmatrix} = -\begin{pmatrix} 0 \\ \chi \end{pmatrix} \qquad \beta \to -1$$

$$(E-V)\varphi = \sigma\cdot\pi\,\chi + m\varphi$$
$$(E-V)\chi = \sigma\cdot\pi\,\varphi - m\chi$$

$$\chi = \underbrace{\frac{1}{E-V+m}\vec\sigma\cdot\vec\pi}_{\frac{\text{momentum}}{2m}}\varphi \sim O(v/c)\varphi$$

$$(E-m-V)\varphi = \sigma\cdot\pi \frac{1}{E-V+m}\sigma\cdot\pi\,\varphi$$

Alternate method: Foldy-Wouthuysen transformation, a canonical transformation, See Bjorken + Drell Vol. I.

Let $E-m=\epsilon$

$$(\epsilon-V)\varphi = \sigma\cdot\pi\frac{1}{2m+\underbrace{\epsilon-V}}\sigma\cdot\pi\,\varphi$$

$\rightarrow$ non-relativistic: this is negligible compared to 2m

Non-relativistic approximation:

$$(\epsilon-V)\varphi = \frac{1}{2m}(\sigma\cdot\pi)^2\varphi = \frac{1}{2m}\{\pi^2 + i\sigma\cdot(\pi\times\pi)\}\varphi$$

$$\pi\times\pi = (\tfrac{1}{i}\nabla - e\vec{A})\,(\tfrac{1}{i}\nabla - e\vec{A})$$

$$= -\frac{e}{i}\nabla\times A = ie\vec{B}$$

$$\boxed{(\epsilon-V)\varphi = \{\tfrac{1}{2m}(\vec{p}-e\vec{A})^2 - \tfrac{e}{2m}\vec{\sigma}\cdot\vec{B}\}\varphi}$$, the Pauli equation

<u>No approximation</u>: $(\epsilon-V)\varphi = (\sigma\cdot\pi)^2\frac{1}{2m+\epsilon-V}\varphi + (\sigma\cdot\pi)\,\sigma\cdot[\pi, \frac{1}{2m+\epsilon-V}]\varphi$

$$(\epsilon-V)\varphi = (\sigma\cdot\pi)^2\frac{1}{2m+\epsilon-V}\varphi + \sigma\cdot\pi\frac{\sigma\cdot\frac{1}{i}\nabla V}{(2m+\epsilon-V)^2}\varphi$$

$$(\epsilon-V)\varphi = \frac{1}{2m+\epsilon-V}(\sigma\cdot\pi)^2\varphi + \sigma\cdot[\pi, \frac{1}{2m+\epsilon-V}]\,\sigma\cdot\pi\,\varphi$$

$$= \frac{1}{2m+\epsilon-V}(\sigma\cdot\pi)^2\varphi + \frac{\sigma\cdot\frac{1}{i}\nabla V}{(2m+\epsilon-V)^2}\sigma\cdot\pi\,\varphi$$

$$= \frac{1}{2m}(\sigma\cdot\pi)^2\varphi - \frac{1}{(2m)^2}(\epsilon-V)(\sigma\cdot\pi)^2\varphi - \frac{i\,\sigma\cdot(+\vec{E})(\sigma\cdot\pi)}{(2m)^2}\varphi$$

$$\longrightarrow (\epsilon-V)\varphi = \frac{1}{2m}(\sigma\cdot\pi)^2\varphi - \frac{1}{(2m)^2}(\sigma\cdot\pi)^2(\epsilon-V)\varphi - \frac{i(\sigma\cdot\pi)(\sigma\cdot\vec{E})}{(2m)^2}\varphi$$

Take ½ the sum of these two eqs.

$$(\epsilon-V)\varphi = \frac{1}{2m}(\sigma\cdot\pi)^2\varphi \left\{ \begin{array}{l} -\frac{1}{2(2m)^2}\{(\sigma\cdot\pi)^2, \epsilon-V\} \\ +\frac{i}{8m^2}[\sigma\cdot\vec{E}, \sigma\cdot\vec{\pi}]\varphi \end{array} \right.$$

Relativistic corrections

$$(\sigma\cdot\vec{E})(\sigma\cdot\vec{\pi}) = E\cdot\pi + i\sigma\cdot(E\times\pi)$$

$$(\sigma\cdot\vec{\pi})(\sigma\cdot E) = \pi\cdot E + i\sigma\cdot(\pi\times E)$$

$\pi = \frac{1}{i}\nabla - \cancel{e}\vec{A}$ neglect A in typical applications

$$[\sigma\cdot E, \sigma\cdot\pi] \approx -\frac{1}{i}\nabla\cdot E + i\sigma\cdot(E\times\pi - \pi\times E)$$

$$\approx (E\times p - p\times E)$$

$$\approx 2\vec{E}\times p$$

$\nabla\times E = 0$ in electrostatic field.

Central field: $\vec{E} = -\nabla V(|\vec{r}|)$

$$= -\frac{\partial V}{\partial r}\frac{\vec{r}}{|\vec{r}|}$$

$$\vec{E}\times\vec{p} = -\frac{\vec{r}}{|r|}\frac{\partial V}{\partial r}\times p = -\frac{1}{r}\frac{\partial V}{\partial r}\vec{L}$$

So:

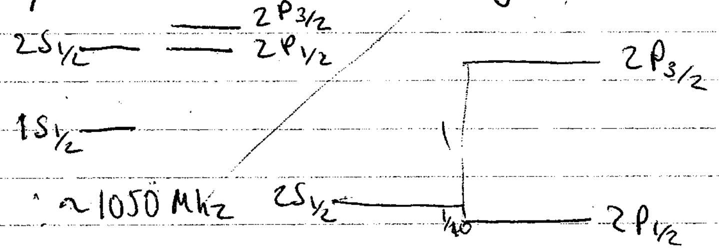
$$\boxed{\begin{aligned}(\epsilon - V)\varphi &= \frac{1}{2m}\left\{(p-eA)^2 - \frac{e}{2m}\sigma\cdot B\right\}\varphi - \frac{1}{8m^2}\left\{(\sigma\cdot\pi)^2, \epsilon - V\right\} \\ &\underbrace{-\frac{1}{8m^2}\nabla\cdot\vec{E}\varphi}_{\text{Darwin term}} + \underbrace{\frac{1}{4m^2}\frac{1}{r}\frac{\partial V}{\partial r}(\vec{\sigma}\cdot\vec{L})\varphi}_{\text{spin-orbit coupling.}}\end{aligned}}$$

factor of 2 less than what expected Thomas precession

Darwin term: $\nabla\cdot\vec{E} = -\rho_{\text{nucleus}}$

affects s-states only, keeps s-state degenerate with a p-state.

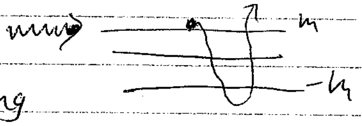
1947: Lamb Shift : ~1050 MHz



Scattering of light by electrons

Classical: Thomson Scattering
Quantum: $H = \frac{1}{2m}(p-eA)^2$ gives Thomson scattering in lowenergy limit. (1^{st} order effect)

Dirac: $H = \alpha\cdot(p-eA)+\beta m$
No first order effect
Second-Order: Compton scattering → Thomson scattering in low freq. limit.
Klein Nishina formula:
Its derivation requires considering negative energy states.



Dirac: 1930 Hole theory.
All negative states in vacuum are filled.

Compton effect in hole theory:
Electron in neg. energy state absorbs photon, goes up to pos. energy state. Electron in pos. energy state drops to fill hole, giving off a photon.

Reference: Heitler: The Quantum Theory of Radiation.

Anticommuting fields: Developed 1932-33, not widely used until 40's

1932: Observation of positron. Carl Anderson

In this theory you can get light-light [redacted] scattering, scattering of light by Coulomb field (Delbrück)

Solid state: hole theory
excitons ⇄ positronium

$(i\not{\partial} - m)\psi = 0$: Eqn. of motion.

$\mathcal{L} = \pm \bar{\Psi}(i\not{\partial} - m)\psi$

$$\delta \int \mathcal{L} = \delta \int_{t_0}^{t_1} \pm \bar{\Psi}(i\not{\partial} - m)\psi \, d^4x = 0$$

$\psi, \bar{\Psi}$: complex fields, independently variable.

Varying $\bar{\Psi}$ $\pm \int (\delta\bar{\Psi})(i\not{\partial} - m)\psi \, \partial^4 x = 0$

so $\pm(i\not{\partial} - m)\psi = 0$ $\qquad \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = 0$

Varying ψ; $\int \pm \bar{\Psi}(i\not{\partial} - m)(\delta\psi)\,\partial^4 x = 0$

so $\mp (i\partial_\nu \bar{\Psi}\gamma^\nu + m\psi)\,\delta\psi\,\partial^4 x = 0$

∴ Adjoint eq: $i\partial_\nu \bar{\Psi}\gamma^\nu + m\bar{\Psi} = 0$ $\qquad \partial^\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\nu \psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi}$

Momenta: $\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \pm i \bar{\Psi}\gamma^0$

$\Pi_{\bar{\Psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\Psi})} = 0$

Hamiltonian

$$\mathcal{H} = \Pi_\psi \partial_0 \psi - \mathcal{L} = \pm i \bar{\Psi}\gamma^0 \partial_0 \psi \mp \bar{\Psi}(i\gamma^\nu \partial_\nu - m)\psi$$

$$= \mp \Psi^\dagger \beta (i\beta\alpha^\nu \partial_\nu - m\psi)$$

$$= \pm \psi^\dagger (\alpha \cdot p + \beta m)\psi$$

So to get pos. energies for pos. states : choose +.

Define: $F_{\vec{p}}^{(r)}(x) = u_{\vec{p}}^{(r)} e^{-i\epsilon_r p x}$ $\qquad r = \pm 1, \pm 2$ $\qquad \epsilon_r = \begin{cases} 1 & r > 0 \\ -1 & r < 0 \end{cases}$

Energy eigenstates: $(\alpha \cdot p + \beta m) F_p^{(r)}(x) = \epsilon_r E_p F_p^{(r)}(x)$

$E_p = \sqrt{m^2 + p^2}$

Recall: $u^{(r)\dagger}(p)\, u^{(s)}(p) = 2E_p\, \delta_{rs}$ for r, s of same sign

$\psi'(x') = \left\{ \sqrt{\frac{E_{p'}+m}{2m}} + \frac{\alpha \cdot p'}{|p'|} \sqrt{\frac{E_{p'}-m}{2m}} \right\} \psi(x)$ $\qquad \frac{\alpha \cdot \vec{p}'}{|p|} = \alpha_z$ for p' in $+z$ dir.

$\Rightarrow u^{(r)\dagger}(\vec{p})\, u^{(s)}(-\vec{p}) = 0$ $\qquad \frac{\alpha \cdot \vec{p}'}{|p|} = -\alpha_z$ for p' in $-z$ dir.

So: $\underline{u^{(r)\dagger}(\vec{p})\, u^{(s)}(\epsilon_r \epsilon_s \vec{p}) = 2E_p\, \delta_{rs}}$, for all r, s.

Define a scalar product:

$$(F^{(r)}_{p}, F^{(s)}_{p'}) \equiv \int u^{(r)\dagger}(p)\, u^{(s)}(p')\, e^{i(\epsilon_r p - \epsilon_s p')x}\, d\vec{r}$$

$$= u^{(r)\dagger}(p)\, u^{(s)}(p')\, (2\pi)^3 \delta(\vec{p}' - \epsilon_r \epsilon_s \vec{p})\, e^{i(\epsilon_r - \epsilon_s)p_0 x^0}$$

$$= 2E_p\, \delta_{rs}\, (2\pi)^3\, \delta(\vec{p}' - \vec{p})$$

Expand ψ: $\psi(x) = \int \sum_{\text{all } r} \frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}\, b^{(r)}_{p}\, F^{(r)}_{\vec{p}}(x)\, d\vec{p}$

$$\psi(x) = \int \frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}\, b^{(r)}_{p}\, u^{(r)}(\vec{p})\, e^{-i\epsilon_r p x}\, d\vec{p}$$

$$(F^{(r)}_{p}, \psi) = (2\pi)^{3/2}\sqrt{2E_p}\, b^{(r)}_{p}$$

so: $b^{(r)}_{p} = \frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}\, (F^{(r)}_{p}, \psi)$

$b^{(r)\dagger}_{p} = \frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}\, (\psi, F^{(r)}_{p})$

$$H = \int \mathcal{H}\, d\vec{r}$$

$$= \int \psi^\dagger(x)\, (\alpha \cdot p + \beta m)\, \psi(x)\, d\vec{r}$$

$$= \iint d\vec{p}\, d\vec{p}' \sum_{rs} b^{(r)\dagger}_{p}\, b^{(r)}_{p'} \frac{1}{(2\pi)^3 \sqrt{2E_p 2E_{p'}}} \left(F^{(r)\dagger}_{p}\, \epsilon_s E_{p'}\, F^{(s)}_{p} \right) e^{i(\epsilon_r p_r - \epsilon_s p_s)}\, d\vec{r}$$

$$H = \int d\vec{p} \sum_r b^{(r)\dagger}_{p}\, b^{(r)}_{p}\, \epsilon_r E_p$$

$$H = \int \sum_{r>0} E_p \left(b^{(r)\dagger}_{p}\, b^{(r)}_{p} - b^{(-r)\dagger}_{p}\, b^{(-r)}_{p} \right)$$

$$Q = \int j^0(x)\, dx = \int \psi^\dagger(x)\, \psi(x)\, d\vec{r}$$

$$= \int \sum_{r>0} \left(b^{(r)\dagger}_{p}\, b^{(r)}_{p} + b^{(-r)\dagger}_{p}\, b^{(-r)}_{p} \right)$$

Quantization of the field.

Bose–Einstein Quantization

$$[\psi_\alpha(r,t), \underbrace{\Pi_{\psi_\beta}(r',t)}_{i\psi^\dagger(r',t)}] = i\delta_{\alpha\beta}\,\delta(\vec{r}-\vec{r}\,')$$: Canonical Commutation relations

$$[\psi_\alpha(r,t), \psi^\dagger(r',t)] = \delta(\vec{r}-\vec{r}\,')\,\mathbb{1}$$

$$\Rightarrow [b^{(r)}_p, b^{(s)\dagger}_{p'}] = \delta_{rs}\,\delta(\vec{p}-\vec{p}\,')$$

All this would give ① an indefinite Hamiltonian, no ground state

② Q becomes positive definite

③ No exclusion principle.

Instead: $\{\psi(r,t), \psi^\dagger(r',t)\} = \delta(\vec{r}-\vec{r}\,')\,\mathbb{1}$

$$\Rightarrow \{b^{(r)}_p, b^{(r)}_{p'}\} = \delta_{rs}\,\delta(\vec{p}-\vec{p}\,')$$

let $c^{(r)}_p = b^{(-r)\dagger}_p \qquad c^{(r)\dagger}_p = b^{(-r)}_p$: Hole theory.

$$\{\psi_\alpha(\vec{r},t), \psi^\dagger_\beta(\vec{r}\,',t)\} = \delta(\vec{r}-\vec{r}\,')\,\mathbb{1}_{\alpha\beta} = \delta(\vec{r}-\vec{r}\,')\,\delta_{\alpha\beta}$$

$$\psi(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}}}} \sum_r b^{(r)}_p F^{(r)}_p(x)$$ where $F^{(r)}_p(x) = v^{(r)}_{\vec{p}}\, e^{-i\epsilon_r p x}$

$$\{b^{(r)}_p, b^{(s)\dagger}_{p'}\} = \delta_{rs}\,\delta(\vec{p}-\vec{p}\,')$$ all r, s.

$$\{\psi(r,t), \psi(r',t)\} = 0 = \{\psi^\dagger(r,t), \psi^\dagger(r',t)\}$$

Vacuum: $|0\rangle$

$b^{(r)}_p|0\rangle = 0$ for $r>0$

$b^{(r)\dagger}_p|0\rangle = 0$ for $r<0$

Note, this is wrong see change near end of notes.

Define $c^{(r)}_{p} = b^{(-r)\dagger}_{p}$: annihilation and creation operators

$c^{(r)\dagger}_{p} = b^{(-r)}_{p}$ for positrons.

$\{ c^{(r)}_{p}, c^{(s)\dagger}_{p'} \} = \delta_{rs}\,\delta(\vec{p}-\vec{p}\,')$ all r,s.

$\{ c^{(r)}_{p}, c^{(s)}_{p'} \} = 0 = \{ c^{(r)\dagger}_{p}, c^{(s)\dagger}_{p'} \}$

$\{ b^{(r)}_{p}, c^{(s)}_{p} \} = \{ b^{(r)\dagger}_{p}, c^{(s)\dagger}_{p'} \} = 0$ $r,s>0$

Define $v^{(r)}(\vec{p}) \equiv u^{(-r)}(\vec{p})$: spinor for positrons

Now $\Psi(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \{ b^{(r)}_{p}\, u^{(r)}(\vec{p})\, e^{-ipx} + c^{(r)}_{p}\, v^{(r)}(\vec{p})\, e^{ipx} \}$

$H = \int \mathcal{H}\, d\vec{r}$

$= \int \sum_{r>0} E_p \left(b^{(r)\dagger}_{\vec{p}}\, b^{(r)}_{\vec{p}} - b^{(-r)\dagger}_{\vec{p}}\, b^{(-r)}_{p} \right) d\vec{p}$

but $b^{(-r)\dagger}_{p}\, b^{(-r)}_{p} = c^{(r)}_{p}\, c^{(r)\dagger}_{p} = -c^{(r)\dagger}_{p}\, c^{(r)}_{p} + \delta(0)$

$H = \int \sum_{r=1,2} E_p \left(b^{(r)\dagger}_{\vec{p}}\, b^{(r)}_{\vec{p}} + c^{(r)\dagger}_{p}\, c^{(r)}_{p} \right) d\vec{r} + \underbrace{2\int \delta(0) \int E_p\, d\vec{p}}_{\text{Energy of the vacuum; ignorable.}}$

Normal ordering: Use only $b^{(r)}, b^{(r)\dagger}, c^{(r)}, c^{(r)\dagger}$ for $r>0$

$Q = \int j^0(x)\, d\vec{r} = \int \Psi^\dagger(x)\Psi(x)\, d\vec{r}$

$= \int \sum_{r>0} \left(b^{(r)\dagger}_{p}\, b^{(r)}_{p} + b^{(-r)\dagger}_{p}\, b^{(-r)}_{p} \right) d\vec{p}$

$= \int \sum_{r>0} \left(b^{(r)\dagger}_{p}\, b^{(r)}_{p} - c^{(r)\dagger}_{p}\, c^{(r)}_{p} \right) d\vec{p} + 2\delta(0)\int d\vec{p}$

To eliminate infinity: ① Symmetrize w.r.t. electrons, positrons

② Normal order

③ Redefine the current: $j^\mu(x) = \frac{1}{2}\left[\bar{\Psi}(x)\gamma^\mu, \Psi(x) \right]$

$$j^{\nu}(x) = \tfrac{1}{2} \sum \gamma^{\nu}_{\alpha\beta} (\bar{\psi}_\alpha \psi_\beta - \psi_\beta \bar{\psi}_\alpha)$$

$$\langle 0 | [\bar{\psi}_\alpha, \psi_\beta] | 0 \rangle = 0 \quad \text{so} \quad \langle 0 | j^{\nu}(x) | 0 \rangle = 0$$

Free-field Anticommutator:

$$\{\psi(x), \bar{\psi}(y)\} = \iint \frac{dp\, dp'}{(2\pi)^3 \sqrt{2E_p 2E_{p'}}} \sum_{rs} F_p^{(r)}(x)\, \overline{F_{p'}^{(s)}}(y) \underbrace{\{b_p^{(r)}, b_{p'}^{(s)\dagger}\}}_{\delta_{rs}\,\delta(p-p')}$$

$$= \int \frac{dp}{(2\pi)^3 2E_p} \sum_{r>0} \{ u^{(r)}(p)\, \overline{u^{(r)}}(p)\, e^{-ip(x-y)} + u_p^{(-r)}\, \overline{u_p^{(-r)}}\, e^{ip(x-y)} \}$$

$$\sum_{r>0} u_p^{(r)} \overline{u_p^{(r)}} = 2m\Lambda_+ = 2m \frac{\not{p}+m}{2m} = \not{p}+m$$

$$\sum_{r>0} u_p^{(-r)} \overline{u_p^{(-r)}} = -2m\Lambda_- = (-2m)\left(\frac{\not{p}-m}{2m}\right) = \not{p}-m$$

$$\{\psi(x), \bar{\psi}(y)\} = \int \frac{d\vec{p}}{(2\pi)^3 2E_p} \sum_{r>0} \{ (\not{p}+m) e^{-ip(x-y)} + (\not{p}-m) e^{ip(x-y)} \}$$

Since $\not{p} = \gamma^{\nu} p_{\nu}$ $\quad (\not{p}+m) = i\not{\partial} + m$ (applied to $e^{-ip(x-y)}$)

$(\not{p}-m) = -(i\not{\partial}+m)$ (applied to $e^{ip(x-y)}$)

$$\{\psi(x), \bar{\psi}(y)\} = (i\not{\partial}+m) \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2E_p} \{ e^{-ip(x-y)} - e^{ip(x-y)} \}$$

$$= (i\not{\partial}+m)\, i\Delta(x-y, m)$$

$$= -i\, S(x-y)$$

Vacuum expectation : Sums over final particle states

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{dp\, dp'}{(2\pi)^3 \sqrt{2E_p 2E_{p'}}} \sum_{rs} F_p^{(r)}(x)\, \overline{F_{p'}^{(s)}}(y)\, \langle 0 | b_p^{(r)} b_{p'}^{(s)\dagger} | 0 \rangle$$

$\langle 0 | b_p^{(r)} b_{p'}^{(s)\dagger} | 0 \rangle = \quad 0 \quad$ for $s<0$, or $r<0$

$$= \langle 0 | \underbrace{\cancel{-b_{p'}^{(s)\dagger} b_p^{(r)}}}_{\substack{\text{zero for}\\ r\geq 0\\ \text{on } s\geq 0}} + \underbrace{\{b_p^{(r)}, b_{p'}^{(s)\dagger}\}}_{\delta_{rs}\, \delta(\vec{p}-\vec{p}')} | 0 \rangle$$

$$\langle 0|\psi(x)\bar\psi(y)|0\rangle = \frac{1}{(2\pi)^3}\int\frac{dp}{2E_p}\underbrace{\sum_{r\geq 0}U^{(r)}(p)\,\bar U^{(r)}(p)}_{\not p+m}\,e^{ip(x-y)}$$

$$= (i\not\partial+m)\frac{1}{(2\pi)^3}\int\frac{d\vec p}{2E_p}\,e^{-ip(x-y)}$$

$$= (i\not\partial+m)\,i\Delta^{(+)}(x-y) \equiv -iS^{(+)}(x-y)$$

Explicitly: $\langle 0|\psi_\alpha(x)\bar\psi_\beta(y)|0\rangle = (i\not\partial+m)_{\alpha\beta}\,i\Delta^{+}(x-y)$

$$= -iS^{(+)}_{\alpha\beta}(x-y)$$

<u>Note:</u> Since Q is conserved:

$Q =$ # of particle $-$ # of antiparticles is conserved.

~~$\langle 0|\bar\psi(x)\psi(y)|0\rangle = \iint\frac{dp\,dp'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}$~~

$$\langle 0|\bar\psi_\beta(y)\psi_\alpha(x)|0\rangle = \iint\frac{dp\,dp'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}\sum_{r,s}\bar F^{(r)}_{p_\beta}(x)\,F^{(s)}_{p'_\alpha}(x)\,\langle 0|\underbrace{b^{(r)\dagger}_p\,b^{(s)}_{p'}}_{c_p^{(r)}\,c_{p'}^{\dagger(s)}}|0\rangle$$

Vanishes for $s>0$ or $r>0$

$$= \iint\frac{dp\,dp'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}\sum_{r,s}\bar F^{(r)}_{p_\beta}(x)\,F^{(s)}_{p'_\alpha}(x)\,\delta_{rs}\,\delta(\vec p-\vec p')\ \text{for } r<0$$

$$= \frac{1}{(2\pi)^3}\int\frac{dp}{2E_p}\sum_{r<0}U^{(r)}_\alpha(p)\,\bar U^{(r)}_\beta(p)\,e^{-i\epsilon_r p(x-y)} \qquad \epsilon_r = -1$$

$$= \frac{1}{(2\pi)^3}\int\frac{dp}{2E_p}(\not p-m)_{\alpha\beta}\,e^{ip(x-y)}$$

$$= -(i\not\partial+m)_{\alpha\beta}\frac{1}{(2\pi)^3}\int\frac{d\vec p}{2E_p}\,e^{ip(x-y)}$$

$$= -(i\not\partial+m)_{\alpha\beta}(-i)\,\Delta^{(-)}(x-y) = i(\not\partial+m)_{\alpha\beta}\Delta^{(-)}(x-y)$$

$$= -iS^{(-)}_{\alpha\beta}(x-y)$$

So, omitting indices

$$\langle 0|\bar\psi(y)\psi(x)|0\rangle = -i\overset{\text{Transpose.}}{\tilde S}{}^{(-)}(x-y)$$

Customarily: $\langle 0|\bar\psi(y)\psi(x)|0\rangle = -iS^{(-)}(x-y)$
then use convention of reversing order of subscripts.

$\langle 0|\Psi(x)\Psi(y)|0\rangle$ contains $\langle 0|b^{(r)}_p b^{(s)}_{p'}|0\rangle$
which vanishes for $s>0$, $r<0$

but $\langle 0|b^{(r)}_p b^{(s)}_{p'}|0\rangle = -\langle 0|b^{(s)}_{p'} b^{(r)}_p|0\rangle$ vanishes for $r>0$, $s<0$,

So $\langle 0|\Psi(x)\Psi(y)|0\rangle = \langle 0|\bar\Psi(x)\bar\Psi(y)|0\rangle = 0$

Measureability: Fermion fields are not hermitian, they are complex. They don't commute on spacelike surfaces, they anti-commute. So there is a correlation between diff. points: gives the Pauli principle.

So it is not possible to measure $\Psi(r,t)$ at two diff. r's since the measurement of one will affect the other.

Bilinear quantities: $\bar\Psi(r,t)\Psi(r,t)$ will commute on spacelike surfaces (two anticommutations): so these are observable quantities.

Fermi field has no classical limit. Occupation numbers never get large, they are at most 1.

$$\Psi = \underset{\text{contains } b\text{'s}}{\Psi^{(+)}} + \underset{\text{contains } c^\dagger\text{'s}}{\Psi^{(-)}}$$

Normal Ordering:

$$: b_p^{(r)} b_{p'}^{(s)\dagger} : = -b_{p'}^{(s)\dagger} b_p^{(r)}$$

$$: b_p^{(r)} \cdots : = (-1)^P \{\text{Normally rearranged product}\}$$

P = parity of the permutation.

$$\left.\begin{aligned} : b b^\dagger c c^\dagger : &= -b^\dagger c^\dagger b c \\ &= b^\dagger b\, c^\dagger c \end{aligned}\right\} \text{either of these could be used.}$$
ambiguity:

Convention: push all adjoints to the left.

Chronological Ordering

$$T(\Psi(x_1) \cdots \bar{\Psi}(x_n)) = (-1)^P (\text{Time ordered product})$$

We will be time ordering bilinear products for the S-matrix, i.e. $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^\mu\Psi$, so the parity will be even the − sign will not be needed.

$$T(\Psi(x), \bar{\Psi}(y)) = \begin{cases} \Psi(x)\bar{\Psi}(y) & x^0 > y^0 \\ -\bar{\Psi}(y)\Psi(x) & y^0 > x^0 \end{cases}$$

<u>Wick's Thm:</u> $T\{\Psi(x)\bar{\Psi}(y)\} = :\Psi(x)\bar{\Psi}(y): +$ c-number function.

since $\langle 0 | : \; : | 0 \rangle = 0$ $\quad = :\Psi(x)\bar{\Psi}(y): + \langle 0 | T(\Psi(x)\bar{\Psi}(y)) | 0 \rangle$

Consider $\Psi^{(+)}(x), \Psi^{(+)}(y)$

$$T\{\Psi^{(+)}(x)\,\Psi^{(+)}(y)\} = \begin{cases} \Psi^{(+)}(x)\Psi^{(+)}(y) & x^0 > y^0 \\ -\Psi^{(+)}(y)\Psi^{(+)}(x) = \Psi^{(+)}(x)\Psi^{(+)}(y) & y^0 > x^0 \end{cases}$$

$$:\Psi^{(+)}(x)\,\Psi^{(+)}(y): \; = \Psi^{(+)}(x)\,\Psi^{(+)}(y). \qquad \langle 0 | \Psi^{(+)}(x)\,\Psi^{(+)}(y) | 0 \rangle = 0$$

This shows that we need both the $(-1)^P$ in the <u>Time-ordered</u> <u>and</u> the normal-ordered definitions.

$$\langle 0 | T\{\Psi(x)\bar{\Psi}(y)\} | 0 \rangle = \langle 0 | \Psi(x)\bar{\Psi}(y) | 0 \rangle \theta(x^0 - y^0) - \langle 0 | \bar{\Psi}(y)\Psi(x) | 0 \rangle \theta(y^0 - x^0)$$

$$= (i\not{\partial} + m)\, i\Delta^{(+)}(x-y)\,\theta(x^0 - y^0) + (i\not{\partial} + m)\, i\Delta^{(-)}(x-y)\,\theta(y^0 - x^0)$$

Since $\Delta_F(x-y) = \begin{cases} \Delta^{(+)}(x-y) & x^0 > y^0 \\ \Delta^{(+)}(y-x) = -\Delta^{(-)}(x-y) & y^0 > x^0 \end{cases}$

$$(i\not{\partial} + m)\, i\Delta^{(-)}(x-y) = -(i\not{\partial}_x + m)\, i\Delta^{(+)}(y-x)$$

So: $\langle 0 | T\{\Psi(x)\bar{\Psi}(y)\} | 0 \rangle = i(i\not{\partial} + m)\,\Delta_F(x-y) = iS_F(x-y)$

$$= i(i\not{\partial} + m)\frac{1}{(2\pi)^4}\int \frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon}\, d^4p$$

$$\langle 0|T\{\psi(x)\,\bar\psi(y)\}|0\rangle = \frac{i}{(2\pi)^4}\int \frac{(\not p+m)}{p^2-m^2+i\epsilon}\, e^{ip(x-y)}\, d^4p$$

$$(\not p)^2 = p^2 = p_0^2 - \vec{p}^{\,2}$$

$m^2 \to m^2 - i\epsilon$ is equivalent to $m \to m - \frac{i\epsilon}{2m} = m - i\epsilon'$

To add an infinitesmal to mass in numerator would thus make no difference.

$$p^2-m^2+i\epsilon = (\not p+m-i\epsilon')(\not p-m+i\epsilon')$$
$$= p^2-(m-i\epsilon')^2 = p^2-m^2+2im\epsilon'$$

$$\frac{1}{p^2-m^2+i\epsilon} = \frac{1}{(\not p+m-i\epsilon')(\not p-m+i\epsilon')}$$

$$\frac{\not p+m-i\epsilon'}{p^2-m^2+i\epsilon} = \frac{1}{\not p-m+i\epsilon}$$

So: $$iS_F(x-y) = \frac{i}{(2\pi)^4}\int \frac{e^{ip(x-y)}}{\not p-m+i\epsilon}\, d^4p$$

$$\langle 0|T\{\psi(x)\,\bar\psi(y)\}|0\rangle = iS_F(x-y)$$
$$= -\langle 0|T\{\bar\psi(y)\,\psi(x)\}|0\rangle$$

This leads to the Feynman rule that a closed loop introduces a minus sign.

$$T\{:\bar\psi(x_1)\psi(x_1)::\bar\psi(x_2)\psi(x_2):\} = :\bar\psi_1\psi_1\bar\psi_2\psi_2: + :\bar\psi_1\overbrace{\psi_1\bar\psi_2}\psi_2:$$
$$+ :\overbrace{\bar\psi_1\psi_1\bar\psi_2\psi_2}: + \overbrace{\bar\psi_1\overbrace{\psi_1\bar\psi_2}\psi_2}$$

So all contractions at same time give zero expectation values.

$$= :\bar\psi_1\psi_1\bar\psi_2\psi_2: + iS_F(x_1-x_2) :\bar\psi_1\psi_2: - iS_F(x_2-x_1) :\psi_1\bar\psi_2:$$

note

$$+ S_F(x_1-x_2)\, S_F(x_2-x_1)$$

note $\ominus i^2$

Using indices, if we are dealing with $\bar{\Psi}_\alpha \Psi_\alpha$, scalar products

$$T\{:\bar{\Psi}_\alpha(x_1)\Psi_\alpha(x_1)::\bar{\Psi}_\gamma(x_2)\Psi_\gamma(x_2):\} = :\bar{\Psi}_1\Psi_1\bar{\Psi}_2\Psi_2: + iS_F(1,2):\bar{\Psi}_1\Psi_2:$$

$$+ iS_F(2,1):\Psi_1\bar{\Psi}_2: + \underbrace{S_{F\alpha\gamma}(1,2)\,S_{F\gamma\alpha}(2,1)}_{\mathrm{Tr}\{S_F(1,2)S_F(2,1)\}}$$

Ψ is a <u>complex</u> field.
So the lines in a Feynman diagram are directed lines.

In momentum space

① Each internal fermion line:

Propagator $\frac{i}{(2\pi)^4}\frac{1}{\not{p}-m+i\epsilon}$ and integration over $\int d^4p$

$$\xrightarrow{p}$$

Calculate using: $\frac{i}{(2\pi)^4}\frac{1}{\not{p}-m+i\epsilon} = \frac{i}{(2\pi)^4}\frac{\not{p}+m}{p^2-m^2+i\epsilon}$

② External fermion lines:

Single initial particle $|1_{r\vec{p}}\rangle = b^{(r)+}_{\vec{p}}|0\rangle$ $r\geq 0$

$$\langle 0|\Psi(x)|1_{rp}\rangle = \langle 0|\Psi(x)\,b^{(r)+}_{\vec{p}}|0\rangle = \frac{u^{(r)}(\vec{p})\,e^{-ipx}}{(2\pi)^{3/2}\sqrt{2E_p}}$$

Particle in initial state: factor $u^{(r)}(p)\frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}$

" in final state: $\bar{u}^{(r)}(p)\frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}e^{+ipx}$

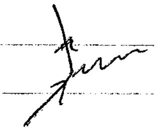
Antiparticle in initial state: $\bar{v}^{(r)}(p)\frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}e^{-ipx}$

" in final state: $v^{(r)}(p)\frac{1}{(2\pi)^{3/2}\sqrt{2E_p}}e^{ipx}$

Can change normalization to get rid of factors.

③ <u>Vertex</u>: Q.E.D: $\mathcal{H}_I = ej^\nu(x)A_\nu(x) = e\bar{\Psi}(x)\gamma^\nu\Psi(x)A_\nu(x)$

At each vertex: $-ie\gamma^{\nu}(2\pi)^4\delta(\Sigma p_i - \Sigma p_{out})$



Pseudoscalar Meson theory:

Pseudoscalar coupling: $\mathcal{H} = gi\bar{\Psi}\gamma^5\Psi\varphi$

Each vertex: $\underbrace{(-i)(ig)}_{g\gamma^5}\gamma^5(2\pi)^4\delta(\Sigma p_{in} - \Sigma p_{out})$

Normalization of external lines

$\langle 0|\Psi(x)|\vec{p}\,r\rangle = \frac{F_p^{(r)}(x)}{(2\pi)^{3/2}\sqrt{2E_p}}$ $\qquad |\vec{p}\,r\rangle = b_p^{(r)\dagger}|0\rangle$

Completeness relation: (1-particle states)

$$P_1 = \sum_r \int |pr\rangle\langle pr|\, d\vec{p} \qquad \text{since } \langle pr|p's\rangle = \delta_{rs}\delta(\vec{p}-\vec{p}')$$

$$\langle pr|\Psi^\dagger\Psi|pr\rangle = \langle pr|\Psi^\dagger|0\rangle\langle 0|\Psi|pr\rangle$$

$$= \frac{F_p^{(r)\dagger}F_p^{(r)}}{(2\pi)^3\, 2E_p} = \frac{1}{(2\pi)^3} \rightarrow \text{not } 4^{th} \text{ comp. of 4-vector}$$

Introduce $b^{(r)}(\vec{p}) = (2\pi)^{3/2}\sqrt{2E_p}\, b_{\vec{p}}^{(r)}$

Now $|pr\rangle = b^{(r)\dagger}(p)|0\rangle$

Anticommutation relation:

$$\{b^{(r)}(p), b^{(s)\dagger}(p')\} = (2\pi)^3\, 2E_p\,\delta_{rs}\,\delta(\vec{p}-\vec{p}')$$

$\langle 0|\Psi(x)|pr\rangle = F_p^{(r)}(x) = u^{(r)}(\vec{p})e^{-ipx}$

Completeness: $P_1 = \frac{1}{(2\pi)^3}\int |pr\rangle\langle pr|\frac{d\vec{p}}{2E_p}$

Normalization: $\langle pr|:\Psi^\dagger\Psi:|pr\rangle = 2E_p$: covariant.

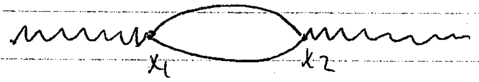
This normalization will give invariant S-matrix elements, invariant cross-sections, etc.

Problem Set:

① Finish discussion of Vacuum Polarization

② μ-meson decay.

Vacuum Polarization



See calculation of $T(:\bar\Psi\Psi\bar\Psi\Psi:)$ 2 pgs ago.

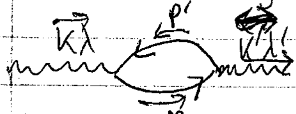
$$S^{(2)} = \frac{(-ie)^2}{2!}\int :A_\nu(x_1)A_\mu(x_2): \mathrm{Tr}\Big\{\underbrace{(-i)S_F(x_2-x_1)}_{\langle 0|T(\psi_2\bar\psi_1)|0\rangle}\gamma^\mu \underbrace{iS_F(x_1-x_2)}_{\langle 0|T(\psi_1\bar\psi_2)|0\rangle}\gamma^\nu\Big\}\, d^4x_1\, d^4x_2$$

$$\langle \vec{k}'\lambda'|S^{(2)}|\vec{k}\lambda\rangle = (-ie)^2\int e_\mu^{(\lambda')*}(k')\, e_\nu^{(\lambda)}(k)\, \mathrm{Tr}\Big\{-\frac{i}{(2\pi)^4}\frac{1}{\not p - m + i\epsilon}\gamma^\mu \frac{i}{(2\pi)^4}\frac{1}{\not p' - m + i\epsilon}\gamma^\nu\Big\}$$

$$(2\pi)^4\delta^4(k-p+p')(2\pi)^4\delta^4(k'-p+p')\, d^4p\, d^4p'$$

minus sign comes from closed loop.

$\langle|A\,A|\rangle$ will give two identical terms depending on which does absorbing



$$= -(-ie)^2(2\pi)^4\delta^4(k-k')\; e_\mu^{(\lambda')*}(k)\, e_\nu^{(\lambda)}(k)\, I_{\mu\nu}$$

$I_{\mu\nu}$ normalization is $(2\pi)^4$ diff than last semester

$$I_{\mu\nu} = -\frac{1}{(2\pi)^4}\int \mathrm{Tr}\Big\{\frac{1}{\not p - m + i\epsilon}\gamma^\mu \frac{1}{\not p - \not k - m + i\epsilon}\gamma^\nu\Big\}\, d^4p$$

$$= \cancel{-\frac{1}{(2\pi)^4}\int \mathrm{Tr}\{\not p + m}$$

$$= -\frac{1}{(2\pi)^4}\int \cancel{\mathrm{Tr}}\; \frac{\mathrm{Tr}\{(\not p + m)\gamma^\mu(\not p - \not k + m)\gamma^\nu\}}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\, d^4p$$

Traces of γ matrices

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$$

so $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \mathrm{Tr}\,g^{\mu\nu} = g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} = 4g^{\mu\nu}$

$$\tfrac{1}{4}\operatorname{Tr}\not{a}\not{b} = a_\mu b_\nu \cdot \tfrac{1}{4}\operatorname{Tr}\gamma^\mu\gamma^\nu = a_\nu b^\nu = a\cdot b$$

for n odd:

$$\operatorname{Tr}\not{a}_1\cdots\not{a}_n = \operatorname{Tr}\not{a}_1\cdots\not{a}_n\gamma_5\gamma_5$$

$$= \operatorname{Tr}\gamma^5\not{a}_1\cdots\not{a}_n\gamma^5$$

$$= (-1)^n \operatorname{Tr}\not{a}_1\cdots\not{a}_n = 0 \quad n \text{ odd.}$$

for n even:

$$\operatorname{Tr}\not{a}_1\not{a}_2\cdots\not{a}_n = 2(a_1a_2)\operatorname{Tr}\not{a}_3\cdots\not{a}_n - \operatorname{Tr}\not{a}_2\not{a}_1\not{a}_3\cdots\not{a}_n$$

$$= 2(a_1a_2)\operatorname{Tr}\not{a}_3\cdots\not{a}_n - 2(a_1a_3)\operatorname{Tr}\not{a}_2\not{a}_4\cdots\not{a}_n$$

$$+ \operatorname{tr}(\not{a}_2\not{a}_3\not{a}_1\cdots\not{a}_n)$$

$$\operatorname{Tr}\not{a}_1\not{a}_2\cdots\not{a}_n = 2(a_1a_2)\operatorname{Tr}(\not{a}_3\cdots\not{a}_n) - 2(a_1a_3)\operatorname{Tr}\not{a}_2\not{a}_4\cdots\not{a}_n + \cdots$$

$$+ 2(a_1a_n)\operatorname{Tr}\not{a}_2\cdots\not{a}_{n-1} - \underbrace{\operatorname{Tr}\not{a}_2\cdots\not{a}_n\not{a}_1}_{= -\operatorname{Tr}\not{a}_1\cdots\not{a}_n}$$

so $\underline{\operatorname{Tr}(\not{a}_1\not{a}_2\cdots\not{a}_n) = (a_1a_2)\operatorname{Tr}\not{a}_3\cdots\not{a}_n - (a_1a_3)\operatorname{Tr}\not{a}_2\not{a}_4\cdots\not{a}_n \cdots + (a_1a_n)\operatorname{Tr}\not{a}_2\cdots\not{a}_{n-}}$

$$\tfrac{1}{4}\operatorname{Tr}\not{a}\not{b}\not{c}\not{d} = (ab)(cd) - (ac)(bd) + (ad)(bc)$$

$\underline{I_{\mu\nu}}$

$$\tfrac{1}{4}\operatorname{Tr}(\not{p}+m)\gamma^\mu(\not{q}+m)\gamma^\nu = \tfrac{1}{4}\operatorname{Tr}\not{p}\gamma^\mu\not{q}\gamma^\nu + m^2 g^{\mu\nu}$$

$$= p^\mu q^\nu + q^\mu p^\nu - (pq)g^{\mu\nu} + m^2 g^{\mu\nu}$$

$$I_{\mu\nu} = -\frac{4}{(2\pi)^4}\int \frac{p^\mu(p-k)^\nu + (p-k)^\mu p^\nu + g^{\mu\nu}(m^2 - p(p-k))}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\, d^4p$$

To do this integral:

① Use Feynman's famous formula

② $\int \frac{q^\nu}{(q^2+a^2)^n} d^4q$ → vanishes since $\frac{q \text{ is odd}}{\text{denom. is even}}$

$\int \frac{q^\mu q^\nu}{(q^2+a^2)^n} d^4q$: express as derivatives

Vacuum polarization: ① renormalized charge
② Additional δ-function interaction to Coulomb interaction.

Coulomb Scattering

$$S^{(1)} = -ie\int j^\nu A_\nu\, d^4x$$

$$= -ie\int \bar{\psi}(x)\gamma^\nu\psi(x)A_\nu(x)\,d^4x \qquad A_\nu(x) = \delta_{\nu 0}\frac{-Ze}{4\pi r}$$

Initial state $|\vec{p},s\rangle$: pos. energy particle state.

$$\langle f|S^{(1)}|\vec{p},s\rangle = -ie\int \langle f|{:}\bar{\psi}\gamma^0\psi(x){:}|ps\rangle\left(-\frac{Ze}{4\pi r}\right)$$

$$W = \sum_f |\langle f|S^{(1)}|\vec{p},s\rangle|^2$$

$$= e^2\iint \frac{Ze}{4\pi r'}\langle ps|\psi^\dagger(x')\gamma^0\cancel{\gamma^0}\underbrace{\psi(x')\underbrace{\sum_f|f\rangle\langle f|}_{1}\bar{\psi}(x)}\gamma^0\psi(x)|ps\rangle\frac{Ze}{4\pi r}d^4x\,d^4x'$$

$$= e^2\iint\frac{Ze}{4\pi r'}\langle ps|\bar{\psi}(x')|0\rangle\gamma^0\ \langle 0|\psi(x')\bar{\psi}(x)|0\rangle\gamma^0\langle 0|\psi(x)|ps\rangle\frac{Ze}{4\pi r}d^4x\,d^4x'$$

$$= e^2\iint\frac{Ze}{4\pi r'}\bar{u}^{(r)}(p)e^{ipx'}\gamma^0\underbrace{(-i)S^{(+)}(x'-x)}_{\substack{\text{Automatic}\\ \text{summation over } E>0\\ \text{final states}}}\gamma^0 u^{(r)}(p)e^{-ipx}\frac{Ze}{4\pi r}d^4x\,d^4x'$$

$$= e^2\frac{1}{(2\pi)^3}\int\frac{d\vec{p}'}{2E_{p'}}\bar{u}^{(s)}(p)\gamma^0(\not{p}'+m)\gamma^0u^{(s)}(p)\left|\int e^{i(\not{p}-p)x}\frac{Ze}{4\pi r}d^4x\right|^2$$ where p' = final state mom.

Since beam is usually unpolarized, take ½ of sum over spin indizes.

$$e^2\frac{1}{(2\pi)^3}\int\frac{d\vec{p}'}{2E_{p'}}\sum_{\alpha\beta\delta\epsilon}\left(\bar{u}^{(s)}_\alpha(p)\gamma^0_{\alpha\beta}(\not{p}'+m)_\delta\gamma^0_{\delta\epsilon}u^{(s)}_\epsilon(p)\right)\left|\int e^{i(p'-p)x}\frac{Ze}{4\pi r}d^4x\right|^2$$

$$\sum_{s>0} u_t^{(s)}(p)\,\overline{u_\alpha^{(s)}}(p) = (\not{p}+m)_{t\alpha}$$

So $$\sum_{s>0} \left(\bar{u}_\alpha^{(s)}(p)\,\gamma^0\,(\not{p}'+m)\,\gamma^0\,u^{(s)}(p)\right)$$

$$= \mathrm{Tr}\{\gamma^0(\not{p}'+m)\gamma^0(\not{p}+m)\}$$

① $$W = \frac{e^2}{(2\pi)^3}\int \frac{d\vec{p}'}{2E_{p'}}\,\bar{u}^{(r)}(p)\,\gamma^0(\not{p}'+m)\,\gamma^0\,u^{(s)}(p)\,\left|\int e^{-i(p'-p)x}\frac{Ze}{4\pi|\vec{r}|}\,d^4x\right|^2$$

Second Procedure for same problem $$\sum_f |\langle f|S^{(1)}|\vec{p}s\rangle|^2 = \langle\vec{p}s|S^{(1)\dagger}P_1 S^{(1)}|ps\rangle$$

$|f\rangle$'s one particle states, so $\sum_f |f\rangle\langle f| = P_1$

This is equivalent to just using $1 = \sum P_n$ instead of P_1, since the many particle states don't contribute.

$$P_1 = \frac{1}{(2\pi)^3}\sum_t \int |\vec{p}'t\rangle\langle\vec{p}'t|\,\frac{d\vec{p}'}{2E_{p'}}$$

~~Use~~ $P_1 = P_1^{(+)} + P_1^{(-)}$

Use $$P_1^{(+)} = \frac{1}{(2\pi)^3}\sum_{t>0}\int |\vec{p}'t\rangle\langle\vec{p}'t|\,\frac{d\vec{p}'}{2E_{p'}}$$

$$W = \frac{1}{(2\pi)^3}\sum_{t>0}\int |\langle p't|S^{(1)}|ps\rangle|^2\,\frac{d\vec{p}'}{2E_{p'}}$$

$$[2\pi\,\delta(p^{0\prime}-p^0)]^2 \Rightarrow 2\pi T\,\delta(p^{0\prime}-p^0)$$

$$\langle p't|S^{(1)}|ps\rangle = \bar{u}^{(t)}(p')\,\gamma^0\,u^{(s)}(p)\int e^{i(p'-p)x}\,\frac{Ze^2}{4\pi|\vec{r}|}\,d^4x$$

$$= \bar{u}^{(t)}(p')\,\gamma^0 u^{(s)}(p)\,\frac{Ze^2}{|\vec{p}-\vec{p}'|^2}\,2\pi\,\delta(p_0'-p_0)$$

$$\frac{W}{T} = \frac{1}{(2\pi)^2}\sum_{t>0}\int |\bar{u}^{(t)}(p)\,\gamma^0\,u^{(s)}(p)|^2\left(\frac{Ze^2}{|\vec{p}-\vec{p}'|^2}\right)^2\delta(p_0^{0\prime}-p^0)\;\not{p}'^2\,\frac{d\Omega'\,dp'}{2E_{p'}}$$

Flux: density x velocity $= 2E_p v = 2|\vec{p}|$

$$\sigma = \frac{W}{T\,\text{flux}} = \frac{1}{(2\pi)^2} \sum_{t>0} \int |\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p)|^2 \left(\frac{Ze^2}{|\vec{p}-\vec{p}'|^2}\right)^2 \frac{p^2}{2E_p} \frac{1}{2E_p V} \frac{dp}{dp_0}\Big|_{p_0'=p_0} d\Omega$$

This is divergent

$$\cancel{\#} \; \frac{dp}{dp^0}\Big|_{p^{0\prime}=p^0} = 1/V$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{4\pi|\vec{p}-\vec{p}'|^2}\right)^2 \sum_{t>0} |\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p)|^2$$

For fixed s, t $\;|\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p)|^2 = (U^{(s)}(p)^{\dagger} \gamma^0 \gamma^{0\dagger} U^{(t)}(p'))(\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p))$

$$= (\bar{U}^{(s)}(p)\gamma^0 U^{(t)}(p'))(\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p))$$

Taking $\sum_{t>0}$: $\sum_{t>0} |\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p)|^2 = \bar{U}^{(s)}(p)\gamma^0 \sum_{t>0} U^{(t)}(p')\bar{U}^{(t)}(p')\gamma^0 U^{(s)}(p)$

$$= \bar{U}^{(s)}(p)\gamma^0(\not{p}'+m)\gamma^0 U^{(s)}(p)$$

So, this procedure gives the same result as ① above.

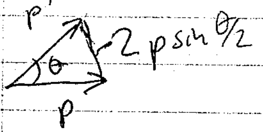
Unpolarized beam: 50% of each of any two orthogonal spin combinations.

Average over $s = 1, 2$

$$\Big\{\sum_{t>0} |\;\;|^2\Big\}_{\text{av. over } s} = \tfrac{1}{2} \operatorname{Tr}\Big\{\gamma^0(\not{p}'+m)\gamma^0 \sum_s U^{(s)}(p)\bar{U}^{(s)}(p)\Big\}$$

$$= \tfrac{1}{2} \operatorname{Tr}\{\gamma^0(\not{p}'+m)\gamma^0(\not{p}+m)\}$$

$$= 2\{\underbrace{p^{0\prime}p^0 + p^{0\prime}p^0 - (p', p)}_{\text{quadratic in } p \text{ terms}} \overset{\text{0 = pm terms}}{\uparrow} + m^2\} \qquad p^{0\prime} = p^0$$



$$|\vec{p}'| = |\vec{p}|$$

$$= 2\{2E_p^2 - E_p^2 + \vec{p}' \cdot \vec{p} + m^2\}$$

$P/E_p = V = \beta$

$$= 2\{2E_p^2 - p^2(1-\cos\theta)\}$$

$$= 2\{E_p^2 - 2p^2 \sin^2\theta/2\}$$

$$= 4E_p^2\{1 - \beta^2 \sin^2\theta/2$$

for unpolarized initial particles

$\frac{e^2}{4\pi\hbar c} = \alpha$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{4p^2 \sin^2\theta/2}\right)^2 4E_p^2 (1 - \beta^2 \sin^2\theta/2)$$

$$= \left(\frac{Z\alpha}{2p\beta}\right)^2 \frac{1}{\sin^4\theta/2} (1 - \beta^2 \sin^2\theta/2)$$; Mott Cross section.

Note: $\sum_{f\geq 0} | \quad |^2 = \sum_{\alpha\beta\delta\epsilon} \bar{u}^{(s)}_{\alpha}(p)\, \gamma^0_{\alpha\beta} (p\!\!\!/' + m)_{\beta\delta}\, \gamma^0_{\delta\epsilon}\, u^{(0)}_{\epsilon}(p)$

$$= \sum \gamma^0_{\alpha\beta} (p\!\!\!/' + m)_{\beta\delta}\, \gamma^0_{\delta\epsilon} \{ \underbrace{u^{(s)}_{\epsilon}(p)\, \bar{u}^{(s)}_{\alpha}(p)}_{1/2 (p\!\!\!/ + m)_{\epsilon\alpha}} \}_{av}$$

$$= \tfrac{1}{2} \mathrm{Tr} \{ \gamma^0 (p\!\!\!/' + m) \gamma^0 (p\!\!\!/ + m) \}$$

Spin:

$u^{(r)}(\vec{p})$: Lorentz transform of $u^{(r)}(0)$

$\sigma^3 = i\gamma^1\gamma^2$

Assume $\sigma^3 u^{(1)}_{(0)} = u^{(1)}_{(0)}$ $\qquad \sigma^3 u^{(2)}_{(0)} = -u^{(2)}(0)$

$\sigma^3 u^{(-1)}(0) = -u^{(-1)}(0) \qquad \sigma^3 u^{(-2)}(0) = u^{(-2)}(0)$

Projection operator: Let $P_{\pm} = \frac{1}{2}(1 \pm \sigma^3)$ $\qquad \boxed{P_{\pm}}^2 = P_{\pm}$

$$= \tfrac{1}{2}(1 \pm i\gamma^1_{\pm}\gamma^2)$$

$P_{\pm} = \frac{1}{2}(1 \pm i\gamma^0\gamma^1_{\pm}\gamma^2\gamma^0) = \frac{1}{2}(1 \mp i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^0)$ $\quad$ since $\gamma^{3^2} = -$

$= \frac{1}{2}(1 \mp \gamma^5\gamma^3\gamma^0)$ $\quad$ O.k. for particles at rest
No good for $\vec{p} \neq 0$

At rest $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \pm 1$

Define $\Sigma_{\pm} = \frac{1}{2}(1 \mp \gamma^5\gamma^3)$ $\qquad$ will be useful for $\vec{p} \neq 0$

$\Sigma_{+} u^{(1)}(0) = P_{+} u^{(1)}(0) = u^{(1)}(0)$

$\Sigma_{-} u^{(1)}(0) = P_{-} u^{(1)}(0) = 0$

$$\Sigma_+ U^{(-1)}(0) = P_- U^{(-1)}(0) = U^{(-1)}(0)$$

$$\Sigma_+ U^{(\pm 1)}(0) = U^{(\pm 1)}(0)$$

$$\Sigma_- U^{(\pm 2)}(0) = U^{(\pm 2)}(0) \quad \text{etc.}$$

Polarization along direction $\hat{n}$ in rest system.

Projection operator: $\Sigma(\hat{n}) = \frac{1}{2}(1 - \gamma^5 (\vec{\gamma} \cdot \hat{n}))$

After lorentz transformation: $n^\nu = a^\nu{}_\lambda (\hat{n})^\lambda$

In rest system $n^\nu = (0, \hat{n})$

length $n^\nu n_\nu = -(\hat{n})^2 = -1$: so it is a spacelike 4-vector.

$p^\nu = a^\nu{}_\lambda \, p_{rest}^\lambda \qquad p_{rest} = (m, 0)$

$n_\nu p^\nu = (n \cdot p) = \text{invariant} = 0$

Invariant Spin proj. operator: $\Sigma(n^\nu) = \frac{1}{2}(1 - \gamma^5 \vec{\gamma}_\nu \cdot \vec{n}^\nu)$

$$= \underline{\frac{1}{2}(1 + \gamma^5 \not{n})}$$

$$\Sigma^2 = \frac{1}{4}(1 - 2\gamma^5 \not{n} + \underbrace{\gamma^5 \not{n} \gamma^5 \not{n}}_{1}) = \Sigma$$

Beam polarized in direction n^ν
Final particles polarized in $n^{\nu\prime}$

: $\bar{U}^{(n')} \gamma^0 U^{(n)}$ will be the matrix element.

$$(U\bar{U})_{initial} = \Sigma(n^\nu) \sum_{S>0} U^{(s)}(p) \bar{U}^{(s)}(p) = \Sigma(n^\nu)(\not{p} + m)$$

$$(U\bar{U})_{final} = \Sigma(n^{\nu\prime})(\not{p}' + m)$$

$$(\bar{U}_{in} \gamma^0 U_{fin})(\bar{U}_{final} \gamma^0 U_{in}) = \mathrm{Tr}\{\gamma^0 \Sigma(n')(\not{p}' + m)\gamma^0 \Sigma(n)(\not{p} + m)\}$$

For $m=0$; Dirac eq: $\not{p}\psi = 0$

$$(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p})\psi = 0$$

multiply by $\gamma 5$ $(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p})\gamma^5\psi = 0$

$P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ | $P_\pm^2 = P_\pm$, $P_+ + P_- = 1$

Then $(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p})P_\pm\psi = 0$

$\gamma_5 P_\pm = \pm P_\pm$

so $(\gamma^0 p_0 \mp \gamma\cdot p\,\gamma^5)P_\pm\psi = 0$

$(p_0 \pm \gamma^0\gamma^5\gamma\cdot p)P_\pm\psi = 0$

$(p_0 \pm \gamma^0 i\gamma^0\gamma^1\gamma^2\gamma^3\,\gamma\cdot\vec{p})P_\pm\psi = 0$

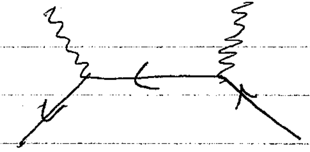
$(p_0 \mp i\gamma^1\gamma^2 p^3 \mp \cdots\cdots)P_\pm\psi$

$(p_0 \mp \sigma\cdot p)P_\pm\psi = 0$

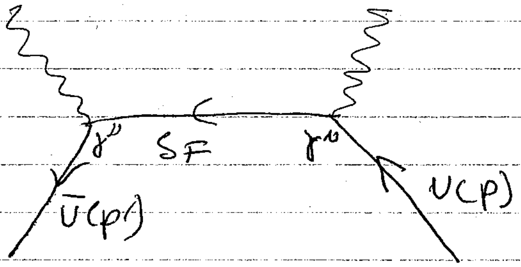
$P_\pm\psi$ satisfies the Weyl eqn: 2-comp. neutrino theory.

$p_0 = |\vec{p}|$ $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}P_\pm\psi = \pm P_\pm\psi$ so the helicity is ± 1

Feynman rules



Sense of progress here corresponds to the order of the Dirac matrices.

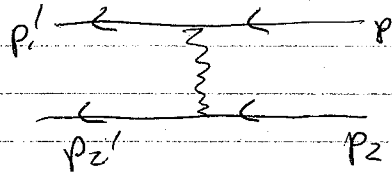


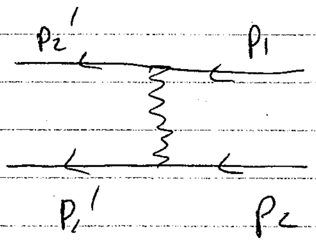
If this were a positron process

then $\bar{v}(p')\gamma^\mu S_F \gamma^\nu v(p)$

Closed loop involves a factor of -1

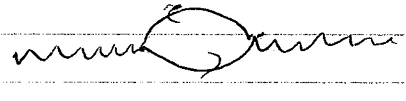
Exchange processes:





Exchanging two-fermion legs gives a minus sign, the overall minus sign is not measureable.

Vacuum Polarization



$\langle k'\lambda' | S^{(2)} | k\lambda \rangle$

$= -(-ie)^2 (2\pi)^4 \delta^4(k'-k)\, e_\mu^{(\lambda')*}(k')\, e_\nu^{(\lambda)}(k)\, I^{\mu\nu}$

$= -(2\pi)^4 i\, \delta^4(k'-k)\, e_\mu^{(\lambda')*}(k')\, e_\nu^{(\lambda)}(k)\, \Pi^{\mu\nu}$

$$I^{\nu\nu} = -\frac{1}{(2\pi)^4}\int \mathrm{Tr}\left\{\frac{1}{\not p - m + i\epsilon}\gamma^{\nu}\frac{1}{\not p - \not k + i\epsilon}\gamma^{\nu}\right\} d^4p$$

$$\Pi^{\nu\nu} = \frac{1}{1}(-ie)^2 I^{\nu\nu} \quad = ie^2 I^{\nu\nu}$$

$$= -\frac{ie^2}{(2\pi)^4}\int \frac{\mathrm{Tr}\{(\not p + m)\gamma^{\nu}(\not p - \not k + m)\gamma^{\nu}\}}{(p^2 - m^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)} d^4p$$

the denominator gives a logarithmic divergence
The numerator has two more powers, so the divergence is quadratic.

Use Feynman Tric

$$\Pi^{\nu\nu} = -\frac{ie^2}{(2\pi)^4}\int_0^1 dx \int \frac{\mathrm{Tr}\{(\not p + m)\gamma^{\nu}(\not p - \not k + m)\gamma^{\nu}\}}{[p^2 - 2(p\cdot k)x + k^2x - m^2 + i\epsilon]} d^4p$$

Let $q = p - kx$ (4 vectors)

$$\Pi^{\nu\nu} = -\frac{ie^2}{(2\pi)^4}\int_0^1 dx \int \frac{\mathrm{Tr}\{(\not q + \not k x + m)\gamma^{\nu}(\not q - \not k(1-x) + m)\gamma^{\nu}\}}{[q^2 + k^2x(1-x) - m^2 + i\epsilon]^2} d^4q$$

Since the denominator is even in q, drop terms odd in q!
Use the trace calculated before:

$$\Pi^{\nu\nu} = -\frac{ie^2}{(2\pi)^4}\int_0^1 dx \int \frac{4\{2q^{\nu}q^{\nu} - 2k^{\nu}k^{\nu}x(1-x) - g^{\nu\nu}[q^2 - k^2x(1-x) - m^2]\}}{[q^2 + k^2x(1-x) - m^2 + i\epsilon]^2} d^4q$$

First Term $= \int \frac{q^{\nu}q^{\nu}}{F(q^2)} d^4q$; quadratically divergent

Assume: $\int \frac{q^{\nu}q^{\nu}}{F(q^2)} d^4q = g^{\nu\nu}A$, A a constant.

Contract both sides with $g_{\nu\nu}$

$$\int \frac{q^2}{F(q^2)} d^4q = 4A$$

So we must "interpret"

$$\int \frac{q^{\nu}q^{\nu}}{F(q^2)} d^4q = \frac{g^{\nu\nu}}{4}\int \frac{q^2}{F} d^4q$$

$\Pi^{\mu\nu} =$ ~~$\frac{4ie^2}{(2\pi)^4}\int_0^1 dx \int 2k^\mu k^\nu x(1-x) - g^{\mu\nu}[\tfrac{1}{2}q^2$~~

$$= \frac{-4ie^2}{(2\pi)^4}\int_0^1 dx \int \frac{-(2k^\mu k^\nu - g^{\mu\nu}k^2)x(1-x) - g^{\mu\nu}[\tfrac{1}{2}q^2 - m^2]}{[q^2 + k^2x(1-x) - m^2 + i\epsilon]^2}\, d^4q$$

$\Pi^{\mu\nu} = k^\mu k^\nu \Pi^{(1)} + g^{\mu\nu}k^2\Pi^{(2)}$ $\quad \Pi^{(1)}, \Pi^{(2)}$ scalars, fns. of k^2

Gauge invariance: $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$

$e^{(\lambda)}_\mu(k) \to e_\mu^{(\lambda)}(k) + \eta k_\mu$ ∴ Result must be indep. of η, ξ for all η, ξ

$e^{(\lambda')*}_\nu(k') \to e_\nu^{(\lambda')*}(k') + \xi k'_\nu$

So $k_\mu \Pi^{\mu\nu} = 0$ $\quad \Pi^{\mu\nu}k_\nu = 0$ are required by gauge invariance.

$$k_\mu \Pi^{\mu\nu} = k^2 k^\nu \Pi^{(1)} + k^\nu k^2 \Pi^{(2)} = 0$$

$$\text{So } \Pi^{(2)} = -\Pi^{(1)}$$

Presume that this identity holds.

$$\Pi^{\mu\nu} = (k^\mu k^\nu - g^{\mu\nu}k^2)\,\Pi^{(1)}(k^2)$$

Just claim that $k_\mu \Pi^{\mu\nu} = 0$, or

$$\frac{-4ie^2}{(2\pi)^4}\int_0^1 dx \int \frac{-k^2x(1-x) - (\tfrac{1}{2}q^2 - m^2)}{[q^2 + k^2x(1-x) - m^2 + i\epsilon]}\, d^4q = 0$$

even though this a quadratically divergent integral.

Multiply by $g^{\mu\nu}$, subtract from expression for $\Pi^{\mu\nu}$

~~$\Pi^{\mu\nu} = \frac{8ie^2}{(2\pi)^4}\int_0^1 dx$~~

$$\Pi^{\mu\nu} = (k^\mu k^\nu - g^{\mu\nu}k^2)\,\frac{8ie^2}{(2\pi)^4}\int_0^1 dx\, x(1-x)\int \frac{d^4q}{[q^2 + k^2x(1-x) - m^2 + i\epsilon]^2}$$

$$\Pi^{(1)} = \frac{8ie^2}{(2\pi)^4} \int_0^1 \int \frac{x(1-x)}{[q^2+kx(1-x)-m^2+i\epsilon]^2} d^4q$$

$$\Pi^{\mu\nu} = (k^\mu k^\nu - g^{\mu\nu} k^2)\, \Pi_0^{(1)}(k^2)$$

For free transverse photons: $e_\mu^{(\lambda')*} \Pi^{\mu\nu} e_\nu^{(\lambda)} = 0$ since $e_\mu^{(\lambda)} k^\mu = 0$

If you didn't force things into this form, you would likely get $\Pi^{\mu\nu} = g^{\mu\nu} Q(k^2) + \cdots$ which will give the Hamiltonian a term of the form $c A_\mu A^\mu$, i.e., give the photon a rest mass.

$$\Pi^{(1)}_{(k^2,m^2)} = \frac{8ie^2}{(2\pi)^4} \int_0^1 dx\, x(1-x) \int \frac{d^4q}{[q^2+k^2x(1-x)-m^2+i\epsilon]^2}$$

$$\Pi^{(1)}(k^2,m^2) - \Pi^{(1)}(k^2,M^2) = -\int_{m^2}^{M^2} \frac{d}{d\lambda} \Pi^{(1)}(k^2,\lambda)\, d\lambda$$

$$= -\frac{8ie^2}{(2\pi)^4} \int_{m^2}^{M^2} d\lambda \int_0^1 dx\, x(1-x) \int \frac{2\, d^4q}{[q^2+k^2x(1-x)-\lambda+i\epsilon]^3}$$

This integral was done last semester by rotation of contour

$$* \quad \underline{\int \frac{d^4p}{[p^2+a]^n} = \frac{i\pi^2}{(n-1)(n-2)a^{n-2}}}$$

$$\Pi^{(1)}(k^2,m^2) - \Pi^{(1)}(k^2,M^2) = -\frac{8ie^2}{(2\pi)^4} \int_{m^2}^{M^2} d\lambda \int_0^1 dx\, x(1-x) \frac{2i\pi^2}{2[k^2x(1-x)-\lambda+i\epsilon]}$$

$$= -\frac{e^2}{2\pi^2} \int_0^1 dx\, x(1-x) \ln\left(\frac{M^2-k^2x(1-x)-i\epsilon}{m^2-k^2x(1-x)-i\epsilon}\right)$$

Choose $M^2 \gg k^2$

$$= -\frac{e^2}{2\pi^2} \int_0^1 dx\, x(1-x) \left\{ \ln\frac{M^2}{m^2} - \ln\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right) \right]$$

$$\int_0^1 x(1-x)\, dx = \tfrac{1}{6}$$

$$\Pi^{(1)}(k^2,m^2) - \Pi^{(1)}(k^2,M^2) = -\frac{e^2}{2\pi^2} \left\{ \frac{1}{6} \ln\frac{M^2}{m^2} - \int_0^1 dx\, x(1-x) \ln\left(1-\frac{k^2}{m^2}x(1-x)-i\epsilon\right) \right\}$$

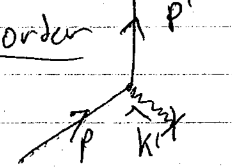
For all practical purposes, this regulated version of Π_1 is Π_1

(Possible justification: $\lim_{M\to\infty} \Pi^{(1)}(k^2,M^2) = 0$, which isn't true)

So: $\pi^{(1)} = -\frac{e^2}{2\pi^2}\left\{ \lim_{M\to\infty} \frac{1}{6} \ln\frac{M^2}{m^2} - \int_0^1 dx\, x(1-x) \ln\left(1-\frac{k^2}{m^2}x(1-x) - i\epsilon\right)\right\}$

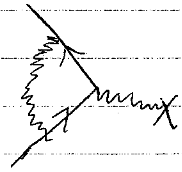
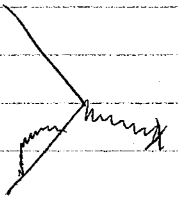
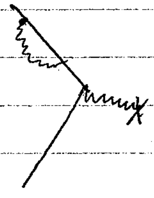
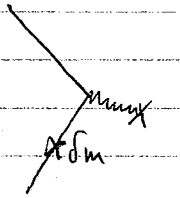
External field $A_\nu(x)$

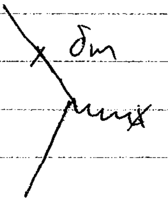
First order

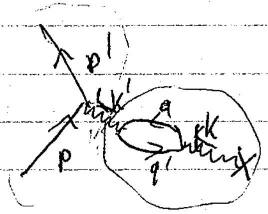


Matrix Element: $= \int \bar{u}^{(r)}(p')\gamma^\lambda u^{(s)}(p)\,(2\pi)^4\delta^4(p'-p-k')$
$\times \underbrace{\int e^{ik'x} A_\lambda(x)\, d^4x}_{\tilde{A}_\lambda(k)}\, d^4k'$

3rd order: Radiative corrections





: Matrix Element = First order matrix element with $\tilde{A}_\nu(k') \to -\frac{1}{(2\pi)^4}\frac{g_{\lambda\nu}}{k'^2+i\epsilon}\cdot(2\pi)^4$
$\times \int \delta(k'-k)\,\Pi^{\nu\nu}\tilde{A}_\nu(k)\, d^4k$

$\tilde{A}_\nu(k') \to \frac{g_{\lambda\nu}}{k'^2+i\epsilon}\Pi^{\nu\nu}(k'^2)\tilde{A}_\nu(k')$

$= \frac{g_{\lambda\nu}}{k'^2+i\epsilon}(k'^\nu k'^\nu - g^{\nu\nu}k'^2)\,\Pi^{(1)}(k'^2)\,\tilde{A}_\nu(k')$

External field: $k^\nu \tilde{A}_\nu(k) = 0$ by Lorentz condition

so $\tilde{A}_\nu(k') \to \Pi^{(1)}(k'^2)\tilde{A}_\lambda(k')$

So, in third order, the effective field is not $A_\nu(x)$, but a convolution integral of it with $\pi^{(1)}(k'^2)$

1rst + 3rd Orders: $(1+\Pi^{(1)}(k^2))A_\nu(k) = (\underbrace{1+\pi^{(1)}(0)}_{\text{charge renormalization factor}} + k^2\pi^{(1)'}(0) + \cdots)A_\nu$

$\pi^{(1)}(0) < 0$, and divergent, $1+\pi^{(1)}(0) < 1$

Like dielectric, putting in charges cause rearrangement of charges to cancel some of its effect, here this effect is formally infinite.

$$1 + \Pi^{(1)}(0) = 1 - \lim_{M \to \infty} \frac{e^2}{12\pi^2} \ln \frac{M^2}{m^2}$$

$$e_r^2 = (1 + \Pi^{(1)}(0))\, e^2$$

$$= Z_3\, e^2$$

The charge of the electron is the renormalized charge e_r, not the bare charge e

$$\Pi^{(1)\prime}(0) = \frac{-e^2}{2\pi^2 m^2} \underbrace{\int dx\, x^2 (1-x)^2}_{1/30} = \frac{-e^2}{60\pi^2 m^2} \qquad \alpha = \frac{e^2}{4\pi}$$

$$= -\frac{\alpha}{15\pi m^2}$$

$$\tilde{A}_\mu(k) \Rightarrow \left(1 + \Pi^{(1)}(0) - \frac{\alpha}{15\pi m^2} k^2 + O\left(\frac{k^3}{m^2}\right)^2 + \cdots\right\} \tilde{A}_\nu(k)$$

In coordinate space:

$$e A_\mu(x) \Rightarrow \left\{ 1 + \frac{\alpha_R}{15\pi} \frac{1}{m^2} \Box^2 + \quad \right\} \left[e A_\mu(x) \right]_{\text{Renormalized}}.$$

$$= Z_3\, e A_\mu(x)$$

Coulomb field: $[eA_\mu]_r = \frac{Z e_R^2}{4\pi r} \delta_{\mu 0}$ $\qquad \Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2$

$$e A_\mu(x) \to \delta_{\mu 0} \left\{ \frac{Z e_r^2}{4\pi r} + \frac{\alpha_R}{15\pi m^2} \nabla^2 \frac{Z e_r^2}{4\pi r} \right\}$$

$$= \delta_{\mu 0} \left\{ \frac{Z e_r^2}{4\pi r} + \frac{\alpha_R}{15\pi m^2} Z e_r^2\, \delta(\vec{r}) \right\}$$

So, a δ-function potential is added.

This contributes 27 Mhz to the Lamb Shift.

Charge Conjugation:

Bosons: φ a linear form of $b_k, c_k^{\dagger}$
$\varphi^{\dagger}$ " " " " $b_k^{\dagger}, c_k$

$$\varphi \to \varphi^{\dagger}$$

$$(\partial^{\nu} - ieA^{\nu}_{ext})(\partial_{\nu} + ieA_{\nu\, ext})\varphi + \mu^2\varphi = 0$$

$\varphi \to \varphi^{\dagger}$ reverse the sign of e.

C: Charge Conjugation operator

$$C\varphi(x)C^{-1} = \varphi^{\dagger}(x) \qquad C\, b_k C^{-1} = c_k \qquad C b_k^{\dagger} C^{1} = c_k^{\dagger}$$

$$j^{\nu} = ie(\partial^{\nu}\varphi\varphi^{\dagger} - \partial^{\nu}\varphi^{\dagger}\varphi)$$

$$C j^{\nu} C^{-1} = -j^{\nu}$$

for $A_{ext} = 0$
for $A_{ext} \neq 0$, have to flip sign of A too.

For φ hermitian: $C\varphi(x)C^{-1} = \varphi(x)$
these particles are self-conjugate, they are their own antiparticles: photons, π^0's.

Dirac Field: ψ a linear comb of $b_p^{(r)}, c_p^{(r)\dagger}$
$\overline{\psi}$ a linear comb of $b_p^{(r)\dagger}, c_p^{(r)}$

$$(\not{p} - e\not{A} - m)\psi(x) = 0 \qquad A = \text{ext. field.}$$

$$(p^{*} - eA_{\nu})\overline{\psi}\gamma^{\nu} - m\overline{\psi} = 0$$

since $p^{\dagger} = ip^{(1)}$

$$(p_{\nu} + eA_{\nu})\overline{\psi}\gamma^{\nu} + m\overline{\psi} = 0$$

Transpose: $[(p_{\nu} + eA_{\nu})\gamma^{\nu T} + m]\overline{\psi}^{T} = 0 \qquad \gamma^{T}_{\alpha\beta} = \gamma_{\beta\alpha} = 0$

Dirac eq. with $e \to -e$

$$(\not{p} + e\not{A} - m)\psi^{c} = 0$$

Let $\bar{C}$ be a 4X4 matrix such that $\bar{C}\gamma^{\mu T}\bar{C}^{-1} = -\gamma^{\mu}$

Choose Unitary C

$$[\bar{C}^2, \gamma^{\mu}] = 0$$

$$\psi^c(x) = \bar{C}\bar{\psi}^T(x)$$

In the standard representation: $\gamma^{\mu} = (\beta, \beta\vec{\alpha})$

γ^0 and γ^2 are symmetric

γ^1, γ^3 are antisymmetric

Want $\bar{C}$ that commutes with $\gamma^1\gamma^3$

anticommutes with γ^0, γ^2 $\qquad \bar{C} = i\gamma^2\gamma^0, \; \bar{C}^2 = -1$

In terms of 2^{nd} Quantization:

$$C\psi(x)C^{-1} = \psi^c(x) = \bar{C}\bar{\psi}^T(x)$$

$$C\bar{\psi}(x)C^{-1} = -\psi^T\bar{C}^{-1}$$

$$C\bar{\psi}\gamma^{\mu}\psi C^{-1} = -\psi^T\bar{C}^{-1}\gamma^{\mu}\bar{C}\bar{\psi}^T$$

$$= \psi^T\gamma^{\mu T}\bar{\psi}^T \neq \bar{\psi}\gamma^{\mu}\psi$$

But $\bar{\psi}\gamma^{\mu}\psi$ is not symmetric between particle, antiparticles

If we use $j^{\mu}(x) = [\bar{\psi}(x)\gamma^{\mu}, \psi(x)]$ which is symmetric

$$= \tfrac{1}{2}\left(\bar{\psi}(x)\gamma^{\mu}\psi(x) - \psi^T(x)\gamma^{\mu T}\bar{\psi}^T(x)\right)$$

So $C j^{\mu}(x) C^{-1} = -j^{\mu}(x)$

<u>Note</u> $\quad v^{(r)c}(p) = C\bar{v}^{(r)T}(p) = C\gamma^{0T}v^{(r)*}(p)$

$$= i\gamma^2\cancel{\gamma^0\gamma^0}\,v^{(r)*}(p)$$

We used $v^{(r)}$ real, $\gamma^2 = \beta\alpha^2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & \sigma^2 \\ \sigma^2 & \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$

so $i\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$

Change in definition

$-v^{(-1)} \to v^{(-2)}$

$v^{(-2)} \to v^{(1)}$ so now $v^{(-r)}(p) = v^{(r)c}(p)$

$v^{(r)}(p) = v^{(-r)}(p) = v^{(r)c}(p)$

$$\Psi(x) = \int \frac{d\vec{p}}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_{r>0} \left\{ b_p^{(r)} F_p^{(r)}(x) + c_p^{(r)\dagger} F_p^{(r)c}(x) \right\}$$

$$C \Psi(x) C^{-1} = \Psi^c(x)$$

$$C b_p^{(r)} C^{-1} = c_p^{(r)} \qquad C c_p^{(r)\dagger} C^{-1} = b_p^{(r)\dagger}$$

$j \to -j$ so $C A^\nu(x) C^{-1} = -A^\nu(x)$ so $j^\nu A_\nu$ is invariant.

$$C a_k C^{-1} = -a_k \qquad C a_k^\dagger C^{-1} = -a_k^\dagger$$

$C|0\rangle = |0\rangle$ vac is eigenstate $C=1$

$CQ = -QC$; so Simultaneous eigenstates only if $Q=0$

$$|\vec{k}_1 \cdots \vec{k}_n\rangle = a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$$

photons

$$C|\vec{k}_1 \cdots \vec{k}_n\rangle = (-1)^n |\vec{k}_1 \cdots \vec{k}_n |0\rangle$$

So n-photon states has charge parity $(-1)^n$

$$|e^- e^+\rangle = \int b_p^{(r)\dagger} c_{p'}^{(s)\dagger} F_{rs}(p,p')\, d^3p\, d^3p' |0\rangle$$

$$C|e^- e^+\rangle = \int c_p^{(r)\dagger} b_{p'}^{(s)\dagger} F_{rs}(p,p')\, d^3p\, d^3p' |0\rangle$$

$$= -\int b_{p}^{(s)\dagger} c_{p'}^{(r)\dagger} F_{rs}(p,p')\, d^3p\, d^3p' |0\rangle$$

$$= -\int b_p^{(r)\dagger} c_{p'}^{(s)\dagger} F_{sr}(p',p)\, d^3p\, d^3p' |0\rangle$$

If $F_{sr}(p',p) = F_{rs}(p,p')$: then $C=-1$

If $F_{sr}(p'p) \equiv -F_{rs}(p,p')$ then $C = 1$

Positronium states:

Para positronium :	^1S	$C = 1$	→ 2 photons
Ortho " :	^3S	$C = -1$	→ 3 photons
	^1P	$C = -1$	→ 3 photons

Conservation of momentum forbids 1 photon decays.