Modern Geometry, Fall 2017: Problem Set 9

Due Thursday, November 16

Problem 1: Differentiating (with respect to G) the right action of G on a principal bundle P gives a map

$$\sigma : \mathfrak{g} \to Vect(P)$$

Show that for elements $Z_1, Z_2 \in \mathfrak{g}$ one has

$$\sigma([Z_1, Z_2]) = [\sigma(Z_1), \sigma(Z_2)]$$

If $R_g : P \to P$ is the right action map (now g is fixed), show that

$$(R_g)_* \sigma(Z) = \sigma(Ad(g^{-1})Z)$$

Problem 2: Show that the definitions of a connection on a principal bundle P as

- A decomposition
  $$T_pP = V_pP \oplus H_pP$$
  of the tangent bundle of P such that V_pP is the subspace of vertical vectors and H_pP satisfies
  $$R_{g*}H_p(P) = H_{pg}(P)$$
- A Lie algebra valued one-form ω satisfying
  $$\omega(\sigma(Z)) = Z$$
  and
  $$R_g^*\omega = Ad(g^{-1})\omega$$

are equivalent.

Problem 3: For the case of the trivial bundle $P = M \times G$, show that

$$\pi_G^* \omega_G$$

(for π_G projection onto G, and ω_G the Maurer-Cartan form) is a connection on P, with curvature $\Omega = 0$.

Problem 4: Show that if X is a horizontal vector field on a principal G-bundle P, then for all Z in the Lie algebra of G,

$$[\sigma(Z), X]$$

is horizontal.

Problem 5: Prove the structural equation relating the torsion and the canonical one-form:

$$d\theta(X, Y) + \omega(X)\theta(Y) - \omega(Y)\theta(X) = \Theta(X, Y)$$

Problem 6: Show that for any two standard horizontal vector fields $B(v), B(v')$ ($v, v' \in \mathbf{R}^n$)

- If the curvature two-form $\Omega = 0$, then $[B(v), B(v')]$ is horizontal.
- If the torsion two-form $\Theta = 0$, then $[B(v), B(v')]$ is vertical.