Modern Geometry, Fall 2017: Problem Set 8

Due Thursday, November 2

Problem 1: Using the general definition of the adjoint representation as the differential of the conjugation action

$$g_0 \to g g_0 g^{-1}$$

show that in the case of $GL(n, \mathbf{R})$, the adjoint representation of the group on the Lie algebra is given by matrix conjugation.

Problem 2: Given a basis E_i of the Lie algebra of left invariant vector fields of G satisfying

$$[E_i, E_j] = \sum_k c_{ij}^k E_k$$

show that the dual left-invariant one-forms θ_i satisfy

$$d\theta^i = -\sum_{j,k} c_{jk}^i \theta^i \wedge \theta^j$$

Also show that the condition $d^2 = 0$ is equivalent to the Jacobi identity and express this as a condition on the c_{jk}^i.

Problem 3:

- Find an expression for the right invariant vector fields on G in terms of their values at $e \in G$.
- Find a right-invariant version θ_R of the Maurer-Cartan 1-form (a Lie algebra valued 1-form equal to the identity at $e \in G$). Find a formula for $d\theta_R$ (right-invariant version of the Maurer-Cartan equation).
- Show that $R_A^* \theta = (AdA^{-1})\theta$ (for $A \in GL(n, \mathbf{R})$).

Problem 4 : Compute the Maurer-Cartan 1-form for the case of $G = GL(2, \mathbf{R})$. Show that on the subgroup $SO(2) \subset GL(2, \mathbf{R})$ this restricts to the one computed in class.