Modern Geometry, Fall 2017: Problem Set 7

Due Thursday, October 26

Problem 1: Show that $SL(n, \mathbf{R})$ and $SO(n, \mathbf{R})$ are submanifolds of $GL(n, \mathbf{R})$ by realizing them as $F^{-1}(c)$ for some $F : N \to M$, $c \in M$, and showing that F_* is surjective on $F^{-1}(c)$.

Problem 2: Using the coordinates on $GL(n, \mathbf{R})$ given by matrix entries, find explicitly in these coordinates the left-invariant vector field X^A corresponding (under the isomorphism between left-invariant vector fields and $T_e GL(n, \mathbf{R})$) to an element $A \in T_e GL(n, \mathbf{R}) = M(n, \mathbf{R})$.
Show that the Lie bracket of vector fields corresponds under this isomorphism to the commutator of matrices, i.e.

$$[X^A, X^B] = X^{AB-BA}$$

Problem 3: Prove that, for a left-invariant vector field X, the flow Φ_t^X is a homomorphism $\mathbf{R} \to G$, i.e. show

$$\Phi_{t_1}^X \circ \Phi_{t_2}^X = \Phi_{t_1+t_2}^X$$

Problem 4: For the group $SU(n) \subset GL(n, \mathbf{C}) \subset GL(2n, \mathbf{R})$, identify the Lie algebra as the Lie algebra of matrices satisfying certain conditions, with Lie bracket the matrix commutator.
Show that the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic (and note that the Lie groups are not).
Show that the exponential map

$$Lie(SL(2, \mathbf{R})) \to SL(2, \mathbf{R})$$

is not surjective.