Modern Geometry, Fall 2017: Problem Set 5

Due Thursday, October 12

Problem 1: Show that for smooth vector fields X and Y, one has for each $p \in M$

$$(L_X Y)_p \equiv \lim_{t\to 0} \frac{1}{t}(\Phi^X_{-t,*} Y_{\Phi^X_t(p)} - Y_p) = [X,Y]_p$$

Problem 2:

For (M,ω) a symplectic manifold, a vector field X is called *symplectic* if $\mathcal{L}_X\omega = 0$, and *Hamiltonian* if there exists a function f on M such that

$$i_X\omega = df$$

Show that all symplectic vector fields on M are Hamiltonian iff $H^1_{deR}(M) = 0$.

Problem 3: Show that for a complex C^* of finite dimensional vector spaces, with C^n highest degree non-zero element, if the complex is exact then

$$\sum_{k=0}^{n} (-1)^k \dim C^k = 0$$

In general, defining the Euler characteristic of a complex by

$$\chi(C^*) = \sum_{k=0}^{n} (-1)^k \dim H^k$$

(where H^k is the cohomology of the complex in degree k) show that

$$\chi(C^*) = \sum_{k=0}^{n} (-1)^k \dim C^k$$

Defining $\chi(M)$ to be the Euler characteristic of the de Rham complex, use the Mayer-Vietoris sequence to show that if M has an open cover by U, V, then

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Problem 4: Use Mayer-Vietoris to calculate the cohomology $H^*_{deR}(T^2)$ of the torus $T^2 = S^1 \times S^1$.
Find de Rham representatives of all the non-zero cohomology classes, and use these to find the ring structure on $H^*_{deR}(T^2)$.