Problem 1: Show that for smooth vector fields $X$ and $Y$, one has for each $p \in M$

$$(L_X Y)_p \equiv \lim_{t \to 0} \frac{1}{t} (\Phi^X_{-t, *}(Y_{\Phi^X_t(p)}) - Y_p) = [X, Y]_p$$

Problem 2:
For $(M, \omega)$ a symplectic manifold, a vector field $X$ is called symplectic if $L_X \omega = 0$, and Hamiltonian if there exists a function $f$ on $M$ such that

$$i_X \omega = df$$

Show that all symplectic vector fields on $M$ are Hamiltonian iff $H^1_{deR}(M) = 0$.

Problem 3: Show that for a complex $C^*$ of finite dimensional vector spaces, with $C^n$ highest degree non-zero element, if the complex is exact then

$$\sum_{k=0}^{n} (-1)^k \dim C^k = 0$$

In general, defining the Euler characteristic of a complex by

$$\chi(C^*) = \sum_{k=0}^{n} (-1)^k \dim H^k$$

(where $H^k$ is the cohomology of the complex in degree $k$) show that

$$\chi(C^*) = \sum_{k=0}^{n} (-1)^k \dim C^k$$

Defining $\chi(M)$ to be the Euler characteristic of the de Rham complex, use the Mayer-Vietoris sequence to show that if $M$ has an open cover by $U, V$, then

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Problem 4: Use Mayer-Vietoris to calculate the cohomology $H_{deR}^*(T^2)$ of the torus $T^2 = S^1 \times S^1$.

Find de Rham representatives of all the non-zero cohomology classes, and use these to find the ring structure on $H_{deR}^*(T^2)$. 
