Modern Geometry II: Problem Set 2

Due Monday, March 21

Problem 1: A Riemannian manifold is said to be flat if every point has a neighborhood that is isometric to a neighborhood of a point in $\mathbf{R}^n$ with the standard metric (i.e. there is a smooth 1-1 map between neighborhoods preserving the metric). Show that a Riemannian manifold is flat if and only if its Riemann curvature tensor vanishes. Hint: Spivak vol. II contains an extremely large number of different such proofs.

Problem 2: Let G be a compact Lie group, *Lie* G its Lie algebra.

a) Assume that one is given an Ad-invariant scalar product $< \cdot, \cdot >$ on *Lie* G (this always exists, the negative of the Killing form), i.e. for $g \in G$, $X, Y \in$ *Lie* G,

$$< Ad(g)X, Ad(g)Y > = < X, Y >$$

Show that

$$< [X, Y], Z > = < X, [Y, Z] >$$

Hint: Let $\gamma(t) = exp(tX)$ (exp is the Lie group exponential map, not the Riemannian one). Compute the t-derivative at $t = 0$ of $< Ad(\gamma(t))X, Ad(\gamma(t))Y >$ using facts that $Ad(\gamma(t)) = (R_{\gamma(-t)})_*(L_{\gamma(t)})_*$ and $R_{\gamma(-t)}$ is the flow of $-X$.

b) Show that such an invariant scalar product gives a metric defined by

$$g(u, v)_a = < (L_{a^{-1}})_* u, (L_{a^{-1}})_* v >$$

(where $u, v \in T_a G$) that is bi-invariant, i.e. left and right invariant.

c) Show that for left-invariant vector fields X and Y on G, the Riemannian connection for a bi-invariant metric is given by

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

c) Show that the geodesics of G starting at the identity are exactly the one-parameter subgroups, so the Lie group exponential map coincides with the Riemannian exponential map at the identity.

d) Show that for a bi-invariant metric and $X, Y, Z \in$ *Lie* G, the curvature tensor is given by

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$$

and the Ricci curvature is given by

$$Ric(X, Y) = \frac{1}{4}\sum_i < [X, E_i], [Y, E_i] >$$

where E_i is an orthonormal basis for *Lie* G.

Problem 3: Consider the Schwarzschild metric

$$g(\cdot,\cdot) = -(1-\frac{2M}{r})dt\otimes dt + \frac{1}{1-\frac{2M}{r}}dr\otimes dr + r^2 d\theta\otimes d\theta + r^2 sin^2\theta d\phi\otimes d\phi$$

Calculate the Christoffel symbols for this metric and a) Show that they satisfy the vacuum Einstein eqs.
b) Write down the equations for a geodesic in these coordinates.

Problem 4: Show that, for $M = \mathbf{R}^n$, the Hodge Laplacian $\triangle$ acts on a k-form

$$\omega = f\ dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

as

$$\triangle\omega = -\sum_{j=1}^{n}\frac{\partial^2 f}{\partial {x^j}^2}dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$