Modern Geometry II: Problem Set 1

Due Monday, February 21

Problem 1: If θ^i is the i'th component of the canonical 1-form θ on a frame bundle $F(M)$, and $s : U \to \pi^{-1}(U)$ is the local section over a coordinate patch U given by the coordinate frame, show that, for Y a vector field on U, $s^*\theta^i(Y)$ gives the i'th component of Y with respect to the coordinate frame basis.

Problem 2: Show that

$$\nabla\theta = \Theta$$

where ∇ is the covariant differential on $\mathbf{R}^n$ valued forms, θ is the canonical 1-form, and Θ is the torsion 2-form.

Problem 3: Let $B(\zeta)$, for $\zeta \in \mathbf{R}^n$ be the horizontal vector field defined in class. Show that

a) If $\zeta \neq 0$, $B(\zeta)$ is a nowhere zero vector field on $F(M)$.

b) $(R_g)_*(B(\zeta)) = B(g^{-1}\zeta)$, where $g \in GL(n, \mathbf{R})$ and R_g is the right action by g on $F(M)$.

Problem 4: If X_j is the j-th component of a coordinate frame on a coordinate patch $U \subset M$, and Γ^i_{jk} are the Christoffel symbols of a connection on $F(M)$ in the coordinates given by U, show that the horizontal lift of X_j to $\pi^{-1}(U)$ is given by

$$X_j^{horiz} = \frac{\partial}{\partial x^j} - \sum_{i,k,l} \Gamma^i_{jk} X^k_l \frac{\partial}{\partial X^i_l}$$

Here (x^i, X^j_k) are the local coordinates on $\pi^{-1}(U)$ defined in class.

Problem 5: If one has two coordinate patches: U, with coordinates x^i, and $\tilde{U}$ with coordinates $\tilde{x}^i$, show that on the overlap of the two patches the Christoffel symbols are related by:

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \sum_{i,j,k} \Gamma^i_{jk} \frac{\partial x^j}{\partial \tilde{x}^\beta} \frac{\partial x^k}{\partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\alpha}{\partial x^i} + \sum_i \frac{\partial^2 x^i}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\alpha}{\partial x^i}$$

Problem 6:

If the components R^i_{jkl} of the curvature tensor with respect to a coordinate frame X_i are defined by

$$R(X_j, X_l)X_j = \sum_i R^i_{jkl} X_i$$

show that

$$R^i_{jkl} = \frac{\partial}{\partial x^k}\Gamma^i_{lj} - \frac{\partial}{\partial x^l}\Gamma^i_{kj} + \sum_m (\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm})$$