Problem 1: Given a connection 1-form $\omega$ on a principal bundle $P$ and the corresponding curvature 2-form

$$ \Omega = d\omega + \frac{1}{2}[\omega, \omega] $$

show that $\Omega(X, Y) = 0$ if either $X$ or $Y$ are in the vertical subspace $T^V P$.

Problem 2: Given a connection on a principal bundle $P$ over $M$, and a curve $\tau = x_t$ in $M$ parametrized by $t \in [0, 1]$. Parallel transport along the horizontal lift of $\tau$ gives a map

$$ \tilde{\tau}_0^1 : \pi^{-1}(x_0) \to \pi^{-1}(x_1) $$

If $\tau$ lies in some coordinate patch $U$, and one picks a trivialization of $P$ over $U$, $\tilde{\tau}_0^1$ can be identified with an element of $G$. How does this element of $G$ transform under change of trivialization over $U$?

Problem 3: For a vector bundle $E$ over $M$ constructed as an associated vector bundle to a principal bundle $P$, show that

$$ \nabla_X (g \phi) = g \nabla_X \phi + (Xg)\phi $$

where $X$ is a smooth vector field on $M$, $g \in C^\infty(M)$, and $\phi \in \Gamma(E)$.

Problem 4:

Given a vector bundle $E = P \times_G V$ constructed as an associated bundle to a principal $G$-bundle over $M$ using a representation $\rho$ of $G$ on $V$, show that

$$ \Omega^i(M, E) = \Omega^i_{basic}(P, V) $$

Problem 5:

Given a curvature form $\Omega$ on a principal bundle $P$ over $M$, and two vector fields $X$ and $Y$ on $M$, show that, on sections $\Gamma(E)$ of an associated vector bundle $E$, one has

$$ \rho_*(\Omega)(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} $$