Problem 1: For the Lie group $G = SO(3)$, find an explicit basis for the Lie algebra $\text{Lie}(G)$ and identify $\text{Lie}(G)$ with $\mathbb{R}^3$. Explicitly construct the adjoint representations

$$Ad : SO(3) \to GL(3, \mathbb{R})$$

of the group, and

$$ad : \text{Lie} SO(3) \to M(3, \mathbb{R})$$

of the Lie algebra. Express $ad$ in terms of the vector cross-product on $\mathbb{R}^3$.

Problem 2: Consider the group $\text{Aff}(\mathbb{R})$ of affine transformations of $\mathbb{R}$. It can be identified with the subgroup of $GL(2, \mathbb{R})$ of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \neq 0$.

a) find the left invariant and right invariant 1-forms on this group.

b) find the left-invariant Maurer-Cartan form on the group and show that it satisfies the Maurer-Cartan equations.

c) find the left and right invariant 2-forms on the group.

Problem 3: Suppose that $P' \to M$ is a principal $H$ bundle and $H \subset G$ is a Lie subgroup. Show that $P' \times_H G \to M$ is naturally a principal $G$ bundle. A reduction of a $G$ bundle $P \to M$ to an $H$ bundle is a pair consisting of an $H$ bundle $P' \to M$ and an isomorphism of $G$ bundles $P' \times_H G \to P$. Show that a principal $G$ bundle reduces to the subgroup $H = \{1\}$ iff the $G$ bundle is trivial.

Problem 4: Prove that the first two definitions of a connection given in class (as a choice of horizontal subspace, as a 1-form) are equivalent.

Problem 5: Given a connection $\omega$ on a principal bundle $P$ and two local sections $s_1$ and $s_2$ defined on a coordinate patch $U$, derive the formula relating $s_1^* \omega$ and $s_2^* \omega$.

Problem 6: Consider the complex line bundles $L_n$ associated to the Hopf bundle (principal $U(1)$ bundle) $S^3 \to \mathbb{C}P^1$, using the representation of $U(1)$ on $\mathbb{C}$ by $e^{i\theta}$. Find the value of $n$ that corresponds to the tautological line bundle over $\mathbb{C}P^1$. Find the value of $n$ that corresponds to the tangent bundle.