Modern Geometry I: Problem Set 3

Due Monday, November 8

Problem 1:

Given two smooth manifolds M and N, two smooth maps

$$f_1 : M \to N; \quad f_2 : M \to N$$

are said to be smoothly homotopic if there is a smooth map

$$F : M \times [0,1] \to N$$

such that

$$F(x,0) = f_1(x), \quad F(x,1) = f_2(x)$$

Show that smoothly homotopic maps induce the same map on cohomology.

Problem 2:

Consider the unit sphere S^n in $\mathbf{R}^{n+1}$. Find a differential form ω of degree n on S^n that represents a non-trivial class in $H^n(S^n)$. Compute the integral

$$\int_{S^n} \omega$$

Problem 3:

Given a basis X_i of a Lie algebra L of dimension n, the structure constants of L are the numbers c_{ij}^k defined by

$$[X_i, X_j] = \sum_{i=1}^{n} c_{ij}^k X_k$$

Compute the values of c_{ij}^k for $L = Lie(GL(n,\mathbf{R}))$ and $L = Lie(O(n))$.

Problem 4:

Show that the Lie bracket of two left-invariant vector fields is left-invariant.

Problem 5:

Consider the Lie group $G = GL(n,\mathbf{R})$. Show that the space of left-invariant vector fields on G can be identified with $M(n,\mathbf{R})$, the n by n real matrices, and that under this identification the Lie bracket becomes the commutator of matrices.

Problem 6:

Show that under the identification of vector fields and matrices found in Problem 5, the exponential map for a given left-invariant vector field X corresponds to the exponential of the corresponding matrix.