Problem 1: Show that the Lie bracket of vector fields satisfies the Jacobi identity, i.e. for three vector fields $X_1, X_2, X_3$

$$[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] = 0$$

Problem 2:
Let $O(n) = \{ A \in M_n(\mathbb{R}) | AA^t = I \}$ be the space of orthogonal $n$ by $n$ matrices. Define a function

$$f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

by

$$f(A) = AA^t$$

so that $O(n) = f^{-1}(I)$.

a) Show that $f$ is a smooth function and that its derivative is given by

$$f_*(A)V = VA^t + AV^t$$

b) Use this to show that $O(n)$ is a smooth submanifold of $\mathbb{R}^{n^2}$.

Problem 3:
Given a smooth map $f : M \to N$ between smooth manifolds $M$ and $N$, show that the pull-back map $f^* : \Omega^*(N) \to \Omega^*(M)$ satisfies

a) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$ for $\omega_1, \omega_2 \in \Omega^*(N)$

b) $f^*d = df^*$

Problem 4: Show that, for $M$ a smooth manifold, $X_1, \ldots, X_{k+1}$ smooth vector fields on $M$, and $\omega \in \Omega^k(M)$

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{k+1} X_i(\omega(X_1, \ldots, \hat{X_i}, \ldots, X_{k+1}))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots, X_{k+1})$$

Problem 5: Prove the two Cartan formulas relating the Lie derivative $L_X$, the interior product $i_X$ and the exterior derivative $d$

a) $L_X = di_X + i_X d$
b) $L_X i_Y - i_Y L_X = i_{[X,Y]}$

**Problem 6:**

Using the definition of the Poisson bracket of two functions in terms of a symplectic 2-form $\omega$:

$$\{f, g\} = -\omega(X_f, X_g)$$

(where $X_f$ is determined by $i_{X_f} \omega = -df$)

show that it satisfies the Jacobi identity, i.e. for three functions $f_1, f_2, f_3$ one has

$$\{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\}$$