Modern Geometry I: Problem Set 1

Due Monday, October 4

Problem 1:

Consider the stereographic projection maps ϕ, $\tilde{\phi}$ from $S^n \to \mathbf{R}^n$ defined in class.
a) Show that these two maps define coordinate charts that cover S^n.
b) Show that these two charts define a smooth structure on S^n.

Problem 2: Show that the $n+1$ coordinate charts on $\mathbf{RP}^n$ defined in class define a smooth structure on $\mathbf{RP}^n$.

Problem 3: Prove that the tautological bundle L over $\mathbf{RP}^n$ is a vector bundle.

Problem 4: Prove that the set of derivations of $C^\infty(\mathbf{R}^n)$ at a point $x = a$ is isomorphic to $\mathbf{R}^n$ (hint: do this by showing that directional derivatives are derivations, and all derivations are directional derivatives).

Problem 5: Given a coordinate chart

$$\phi : U \to \mathbf{R}^n$$

one can compose with projection on the i'th component to get a coordinate function $x^i : U \to \mathbf{R}$. Then a basis for the tangent vectors at $p \in M$ will be given by the $\frac{\partial}{\partial x^i}|_p$.
Given two coordinate charts both defined in a neighborhood of a point $p \in M$, what is the formula relating tangent vectors expressed in the two different bases corresponding to the two different coordinate charts?

Problem 6: Show that the space $T(M) = \bigcup_{p \in M} T_p(M)$ is a vector bundle.

Problem 7: A vector bundle E is trivial if one can find not only local trivialization maps $\Phi : E \to U \times \mathbf{R}^n$, but a global trivialization map $\Phi : E \to M \times \mathbf{R}^n$. Show that a rank n vector bundle E is trivial if and only if one can find n sections that are everywhere linearly independent.