Modern Geometry I: Final Exam

Due Wednesday, December 15, Noon (leave in my mailbox)

Problem 1: Let M be a smooth manifold of dimension $2n$, with Ω a closed 2-form such that the product

$$\Omega \wedge \Omega \wedge \cdots \wedge \Omega$$

of n copies of Ω is a non-zero $2n$-form everywhere on M.
a) Given a smooth vector field X on M, show that the 1-form $\omega_X = i_X \Omega$ is closed if and only if the Lie derivative of Ω satisfies $L_X \Omega = 0$.
b) Show that the map $h : X \to \omega_X$ gives an isomorphism of $\Gamma(TM)$ and $\Omega^1(M)$
c) If $\alpha, \beta \in \Omega^1(M)$ are 1-forms, and $X_\alpha = h^{-1}(\alpha), X_\beta = h^{-1}(\beta)$, let $(\alpha, \beta) = h([X_\alpha, X_\beta])$. If α is closed, prove that

$$L_{X_\alpha} \beta = (\alpha, \beta)$$

and that (α, β) is exact if α and β are closed.
d) If f and g are smooth functions on M, let

$$(f, g) = \Omega(X_{dg}, X_{df})$$

Show that $(df, dg) = d(f, g)$, and that f is constant along the integral curves of X_{dg} if and only if g is constant along the integral curves of X_{df}.

Problem 2: On $\mathbf{R}^4$, with coordinates x_1, x_2, x_3, x_4, consider the vector fields

$$X = x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}$$

and

$$Y = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}$$

a) On which submanifold M of $\mathbf{R}^4$ do the vector fields X and Y define a field of two-planes (two-dimensional distribution)? Is this field of two-planes integrable on M?
b) Consider a third vector field

$$Z = -x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4}$$

Find an integral curve for this vector field.
c) Show that X, Y, Z define a field of 3-planes for each point in M. Is this field integrable?
d) Find a differential 1-form ω such that for all $p \in M$, $W \in T_p(M)$ is in the field of 3-planes of part c) if $\omega(W) = 0$.

Problem 3: Let $\pi : P \to M$ be a principal G-bundle, V a representation of G and $E = P \times_G V$ the associated vector bundle.
a) Show that the pull-back bundle $\pi^* P \to P$ over P is isomorphic to the trivial bundle $P \times G$ over P.
b) Show that the pull-back bundle $\pi^* E \to P$ over P is isomorphic to the trivial bundle $P \times V$ over P.

Problem 4: Let G be a Lie group, with H a closed subgroup. Consider G as a principal H-bundle, with base space G/H. Suppose that there is a direct sum decomposition *Lie* $G =$ *Lie* $H \oplus M$, where M is a subspace of *Lie* G invariant under the adjoint action of $H \subset G$ on *Lie* G.
a) Prove that the *Lie* H component of the Maurer-Cartan 1-form on G determines a connection on the bundle.
b) Show that this connection is invariant under the action of G on the total space of the bundle by left multiplication.
c) Show that the curvature of this connection

$$\Omega(X, Y) = -[X, Y]_{Lie\ H}$$

where X and Y are in M, and the subscript indicates the *Lie* H component.

Problem 5: Consider the Hopf bundle $S^3 \subset \mathbf{C}^2 \to \mathbf{CP}(1)$.

a) In terms of coordinates (z_1, z_2) on $\mathbf{C}^2$, show that the 1-form

$$\omega = \overline{z_1} dz_1 + \overline{z_2} dz_2$$

is a connection 1-form on this bundle.
b) Compute the curvature Ω of this connection and show that it is the pull-back under the projection of a 2-form $\tilde{\Omega}$ on the base space.
c) Show that the first Chern number of the bundle is given by

$$\int_{\mathbf{CP}^1} -\frac{\tilde{\Omega}}{2\pi i} = -1$$

Problem 6: For an $SU(2)$ principal bundle over a base space M we showed in class that $c_2(P)$ is represented by

$$\frac{1}{8\pi^2} tr(\tilde{\Omega} \wedge \tilde{\Omega})$$

where $\tilde{\Omega}$ is the pull-back of the curvature 2-form Ω of some connection ω to the base using some local section.
a) Show that on the total space of the bundle

$$tr(\Omega \wedge \Omega) = dCS(A)$$

where

$$CS(A) = tr(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)$$

is called the Chern-Simons form of the connection.
b) If one uses a local section to pull-back $CS(A)$ to a 3-form $\widetilde{CS(A)}$ on the base, under change of section by some local function on a coordinate patch U, $\Phi : U \rightarrow SU(2)$, show that $\widetilde{CS(A)}$ changes by the addition of two terms, one proportional to the exact form

$$d(tr((d\Phi)\Phi^{-1} \wedge \omega))$$

the other proportional to

$$tr(\Phi^{-1}d\Phi \wedge \Phi^{-1}d\Phi \wedge \Phi^{-1}d\Phi)$$

(this form is closed and represents the non-trivial class in $H^3(SU(2))$).