

# MODERN GEOMETRY

This is a one-year graduate-level introduction to modern geometry, what follows is a highly tentative syllabus. We'll see how much of this there is actually time for.

For the first part of the course, two textbooks that cover this material at a similar level are:

1. *Differential Geometry, Volume I*, by Michael Spivak
2. *Foundations of Differentiable Manifolds and Lie Groups*, by Frank Warner

For the later parts of the first semester, a good book to consult is *Geometry of Differential Forms*, by Shigeyuki Morita

- Differentiable Manifolds: Foundations
  - Examples of manifold,
  - Topological manifolds, smooth structures
  - Definition of a smooth manifold
  - The tangent space, various definitions
  - The tangent bundle, general vector bundles, vector fields
  - The differential of a smooth map
  - Implicit and inverse function theorems
- Tensors and Differential Forms
  - The exterior algebra
  - Differential forms and the exterior derivative
  - The Lie derivative, Cartan Formulas
  - Distributions, Frobenius integrability theorem
- de Rham Cohomology
  - Generalities on homological algebra
  - de Rham cohomology, other cohomology theories
  - Examples
- Integration on Manifolds
  - Orientation
  - Integrals over manifolds
  - Stokes' theorem

- Poincaré duality
- Lie Groups and Lie Algebras
  - Lie groups: definition and examples
  - Lie algebras: definition and examples
  - Left-invariant forms, Maurer-Cartan forms
  - Representations of Lie groups: the adjoint representation
  - Haar measure
  - de Rham cohomology of compact Lie groups, Lie algebra cohomology
- Principal Bundles and Associated Vector Bundles
  - Definition and Examples: Homogeneous spaces, Hopf fibration, frame bundles
  - Connections on principal bundles: various definitions
  - Space of connections, the gauge group
  - Curvature of a connection on a principal bundle
  - Parallel transport and holonomy
  - Covariant derivatives, connections on vector bundles
- Characteristic Classes and Chern-Weil Theory
  - $U(1)$  bundles and the first Chern class
  - Ad-invariant polynomials on Lie algebras
  - Chern classes: definition
  - Chern classes: properties
  - Pontryagin classes and the Euler class
- Frame Bundles and Cartan Connections
- Riemannian Geometry
  - Riemannian metrics
  - The Levi-Civita connection
  - Riemann curvature tensor and the Ricci tensor
  - Geodesics
  - Laplacian operator on differential forms, Hodge theory
  - Yang-Mills equation
  - General relativity
  - The Ricci flow
- Symplectic Geometry

- Symplectic manifolds
- Hamiltonian mechanics
- Hamiltonian group actions and moment maps
- Complex and Kähler Geometry
  - Complex structures
  - Complex and holomorphic vector bundles
  - Dolbeault cohomology
  - Kähler manifolds
- Spin Geometry
  - Clifford algebras
  - Spinors
  - The Dirac Operator
- The Atiyah-Singer Index Theorem
  - The index of an elliptic operator
  - The index theorem: heat equation proof