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Chern-Simons Numbers and Universal Bundles on the Lattice

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ABSTRACT. The problems involved in defining the Chern-Simons number of a lattice gauge field are examined. A proposal is advanced for doing so by constructing a classifying map into a universal bundle over the lattice. Such a classifying map also provides a natural way of interpolating gauge fields on the lattice and defining the topological charge.

Chern-Simons secondary characteristic classes[1] have recently found many applications in quantum field theories, ranging from providing a topological mass term in Yang-Mills theory in 2+1 dimensions[2,3] to being involved in the calculation of the variation of the determinant of the Dirac operator in an anomalous gauge theory[4]. Since lattice gauge theory is the main formalism in which non-perturbative gauge theory calculations can be performed, it would be useful to better understand how to define Chern-Simons classes within this formalism. In this paper we would like to propose such a definition, one that may be computationally practical in certain circumstances. The geometric framework involved, that of universal bundles, has an independent interest in terms of explaining the similarity between Chern-Simons and Wess-Zumino terms, providing a natural way of interpolating gauge fields and giving a new and potentially useful way of calculating the topological charge of a lattice configuration.

We will be considering the Chern-Simons 3-form ω which is defined as

$$\omega[A] = \frac{-1}{8\pi^2} tr(A \wedge F - \frac{1}{3} A \wedge A \wedge A)$$

A is the SU(N) connection 1-form, F is the curvature 2-form. The Chern-Simons form has two important properties:

1. the 4-form $q = d\omega$ is the topological charge density, integrated over a compact closed 4-manifold X it gives the second Chern number of the SU(N) bundle. If X is a manifold with boundary M=∂X, the integral of the topological charge over X will be given by the integral of the Chern-Simons form over M (assuming that a single gauge can be chosen over X). The integral of the Chern-Simons form over a compact 3-manifold M will be called the Chern-Simons number n_{CS} of the SU(k) bundle over M.

2. Under a gauge transformation the Chern-Simons form changes as

$$\omega \to \omega + \frac{1}{8\pi^2} dtr(A \wedge gdg^{-1}) + \frac{1}{24\pi^2} gdg^{-1} \wedge gdg^{-1} \wedge gdg^{-1}$$

The Chern-Simons number transforms under a gauge transformation as

$$n_{CS} \to n_{CS} + N$$

where N is the winding number of the gauge transformation. It is this number that we wish to find a suitable lattice version of, in particular we are only interested in its fractional part. The integral part is just an artifact of the gauge chosen.

Naively one might try and construct a simple combination of link and plaquette variables that agrees with the continuum expression for the Chern-Simons number as the lattice spacing is taken to zero. Such a construction will not have the gauge transformation property described above, and is unlikely to correctly reproduce the physics of the Chern-Simons number for the same reason that similar constructions in the case of the topological charge density q(x) give misleading results. For the coupling constant values for which Monte-Carlo calculations can be performed, the plaquette variables are not particularly close to the identity, and the link variables will for no coupling be typically close to the identity.

If one had a continuum configuration that agreed with the lattice configuration data in the sense that

$$U_\mu(n) = e^{ig\int_n^{n+\mu} A_\mu(x)dx^\mu}$$

then one could apply the continuum integral formula for the Chern-Simons number to this interpolating configuration. In the case of the topological charge, the final result is a topological invariant, and thus is insensitive to the details of the interpolation, as long as the plaquette variables are sufficiently close to the identity [5,6,7]. The situation here is quite different because the Chern-Simons number is not a topological invariant. Different interpolations in principle may give wildly different values for n_{CS}. This ambiguity reflects an inherent ambiguity in defining the local topological charge density of a lattice configuration. Seiberg [8] has given a definition of the Chern-Simons number on the lattice in terms of an integral formula that uses the interpolation of reference 5.

Physically perhaps the most sensible interpolation is given by choosing the interpolation that minimizes the Yang-Mills action, since the one clearly bad property that an interpolation can have is that of carrying a large action which is not reflected in the values of the traces of the plaquette variables. However, it may be possible to use any convenient interpolation as long as the value of its action is not too different from that given by the Wilson action when the plaquette variables are not too large. One will have to analyze this question for whatever physical quantities involving the Chern-Simons number one wishes to calculate. The essence of our proposal will be an interpolation method using variables in terms of which the Chern-Simons number has a simple geometrical interpretation as a Wess-Zumino type[9] term. This interpolation method involves constructing a classifying map into a universal bundle, a concept to which we now turn.

The notion of a universal bundle is well known to mathematicians as the standard tool for studying the topology of bundles[10]. It has been discussed in the physics literature by Dubois-Violette and Georgelin[11] who used it to study instanton solutions. Consider the Grassmanian Gr(k,N) of of complex k-planes in $\mathbf{C}^N$. There is a complex vector bundle E_0 over Gr(k,N), known as the tautological bundle, whose fiber at a point x is just the complex k-plane in $\mathbf{C}^N$ determined by x. A basic theorem used in the topological classification of vector bundles says that any vector bundle E with fiber $\mathbf{C}^k$ over a compact manifold M is the pullback f^*E_0 of E_0 by some classifying map

$$f : M \to Gr(k, N)$$

for N large enough.

The bundle E_0 has a standard connection γ_0 that is invariant under the action of U(N) on the bundle. The theorem of Narasimhan and Ramanan [13] states that, again for N large enough, a classifying map f can be found such that the connection A on E is the pullback by f of the connection γ_0 on E_0. The connection γ_0 is a 1-form on the bundle of frames in E_0, also known as the Stiefel manifold of St(k,N) of unitary k-frames in $\mathbf{C}^N$. This one form takes values in the Lie-algebra of U(k) and is constructed as follows. A point in St(k,N) will be given by an N×k matrix V whose entries are the coordinates of the k-frames with respect to the standard basis in $\mathbf{C}^N$. The connection γ_0 is given by

$$\gamma_0 = V^\dagger dV$$

If a gauge is chosen, in other words we have a section

$$s : Gr(k, N) \rightarrow St(k, N)$$

then this connection form exists not just on the bundle St(k,N) but also on the base space Gr(k,N) as the pullback $s^*\gamma_0$.

Once a classifying map f has been found such that $A = f*\gamma_0$, the Chern-Simons number can be written

$$n_{CS} = \int_M \omega[A] = \int_M f^*\omega[s^*\gamma_0] = \int_{f(M)} \omega[s^*\gamma_0] = \int_Z q[\gamma_0]$$

where $\partial Z = f(M)$ and $q[\gamma_0]$ is both the topological charge density of the bundle E_0 with connection γ_0 and a representative of a deRham cohomology class in H^4(Gr(k,N)). Thus expressed in terms of the classifying map f, the Chern-Simons number is a Wess-Zumino term which occurs due to the fact that H^4(Gr(k,N))$\neq 0$ and H^3(Gr(k,N))=0.

The connection γ_0 on the bundle E_0 over Gr(k,N) has another important property. Parallel transport of a frame from x to y (along the shortest geodesic connecting x and y) using this connection is exactly the same thing as the process of orthogonal projection (using the standard metric on $\mathbf{C}^\mathbf{N}$) of the frame in the $\mathbf{C}^\mathbf{k}$ determined by x to the $\mathbf{C}^\mathbf{k}$ determined by y, followed by rescaling of the frame so it is again unitary. Thus, once we have chosen a section s of the bundle St(k,N)$\rightarrow$Gr(k,N), to any two points x and y on Gr(k,N) we can associate an SU(k) group element $U_{x,y}$ which gives the rule for parallel transport from x to y along the geodesic between them. For some choices of x and y orthogonal projection will give zero, this corresponds to the fact that there will be an infinity of different same length geodesics between these points.

We will now define a classifying map for an SU(k) lattice gauge theory configuration on a lattice L to be a map

$$f(n) : \text{vertices of L} \rightarrow Gr(k, N)$$

for some N and such that

$$U_\mu(n) = U_{f(n),f(n+\mu)}$$

for all the links in the lattice L (a link is labelled by a vertex n and a direction μ). Unfortunately, it is very difficult to decide how large a value of N will be required, and there is no known constructive procedure for finding such maps. The theorem of Narasimhan and Ramanan gives a rather weak upper bound on N for the problem of finding such maps in the continuum, nothing is known about this problem in the lattice case. One could try and approach this problem numerically by starting with some random map f(n) and changing it so as to minimize the difference between $U_\mu(n)$ and $U_{f(n),f(n+\mu)}$.

This lack of a constructive procedure for finding classifying maps is the major practical weakness of the Chern-Simons number definition we are proposing. It is simple to recognize a classifying map when one has one, but more work is required to develop efficient methods for constructing such maps. Alternatively, one could perform Monte-Carlo calculations directly in terms of the classifying map variables. The Wilson action is easily expressed this way, but the standard measure on the space of link variables becomes very complicated in terms of the f(n)'s.

Assuming that a lattice classifying map has been found, it is now relatively simple to interpolate the configuration from the lattice vertices to the interior of the lattice cells. One could for instance apply the interpolation technique described in reference 6 where a method is given for interpolating a map into the group from the vertices throughout the lattice. The situation here is simpler since it is essentially the case described there but for a pure gauge configuration with all plaquettes the identity. The only difference is that one is interpolating in Gr(k,N), not SU(n), but the techniques for interpolating a map from a lattice into a non-linear space using geodesics are the same. Given the interpolated classifying map f(x) one now automatically has an interpolated connection $f^*\gamma_0$. This method for constructing an interpolated connection using the lattice data may turn out to be useful in many other contexts than the present one of understanding the Chern-Simons number.

The Chern-Simons number can now be defined as either

$$n_{CS} = \int_{f(M)} \omega[s^*\gamma_0]$$

or

$$n_{CS} = \int_Z q[\gamma_0]$$

where Z may be constructed by picking an arbitrary point p in Gr(k,N) and taking the space of all geodesics between p and f(M). Either definition involves performing a definite integral and thus will be numerically time consuming. Also note that given a classifying map for a four dimensional gauge field one could define the topological charge Q of the configuration as

$$Q = \int_{f(M)} q[\gamma_0]$$

This gives a new definition of the lattice topological charge density that may be useful in certain circumstances.

While the definition of the lattice Chern-Simons form we have given is inherently rather difficult to compute in general, in the special case of SU(2) it may be tractable. Since SU(2)=Sp(1), we can think of an SU(2) vector bundle with fiber $\mathbf{C}^2$ as being a quaternionic line bundle, with fiber $\mathbf{H}$. The formalism of universal bundles described above for complex vector bundles goes through in exactly the same way for quaternionic vector bundles. Quaternionic line bundles will be classified by maps into the quaternionic projective space $\mathbf{HP}^n$ for n large enough (Wess-Zumino terms in $\mathbf{HP}^n\,\sigma$ models were considered in reference 13). In particular, it may be possible to find lattice classifying maps for n as small as n=1. Maps f: L$\rightarrow$ $\mathbf{HP}^1$ which at least approximately satisfy the condition to be a lattice classifying map may not be too hard to construct. This case is very simple since $\mathbf{HP}^1 = \mathbf{S}^4$, the tautological line bundle over it is the instanton bundle, the connection γ_0 is the instanton solution to the Yang-Mills equations, and the topological charge density $q[\gamma_0]$ is just the volume 4-form on S^4.

The definite integral involved in defining the Chern-Simons number in this case may be more easily performed since one is just integrating a standard volume form on a sphere.

Furthermore, given a lattice classifying map into $\mathbf{HP}^1$ for a four dimensional lattice configuration, the topological charge would be extremely easy to compute since it would just be the winding number of the map. This calculation requires only the evaluation of a few 4 by 4 determinants per lattice cell using the technique for evaluating the winding number of a lattice mapping into a sphere first used in reference 14.

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References

1. S.S. Chern and J.Simons, Ann. Math **99** (1974), 48.
2. J. Schonfeld, Nucl. Phys. **B185** (1981), 157.
3. S. Deser, R. Jackiw and S. Templeton, Ann. of Physics **140** (1982), 372.
4. B. Zumino, in "Relativity, Groups and Topology II", Les Houches, 1984.
5. M. Luscher, Comm. Math. Phys. **85** (1982), 39.
6. P. Woit, Nucl. Phys. **B262** (1986), 284.
7. A. Phillips and D. Stone, Comm. Math. Phys. **103** (1986), 599.
8. N. Seiberg, Phys. Lett. **148B** (1984), 456.
9. E. Witten, Nucl. Phys. **B223** (1983), 422.
10. N. Steenrod, "Topology of Fiber Bundles", Princeton University Press, 1951.
11. M. Dubois-Violette and Y. Georgelin, Phys. Lett. **82B**, 251.
12. M. S. Narasimhan and S. Ramanan, Am. J. Math. **83** (1961), 563.
13. E. Rabinovici, A. Schwimmer and S. Yankielowicz, Nucl. Phys. **B248** (1984), 523.
14. P. Woit, Phys. Rev. Lett. **51** (1983), 638.