BRST and Dirac Cohomology

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Outline

Physics Problem

Quantum field theories used to describe elementary particles have a symmetry under an infinite dimensional group, the "gauge group". How does one handle this?
"BRST": Technology using anticommuting variables, developed early 1970s by physicists Becchi, Rouet, Stora and Tyutin.

New Ideas in Representation Theory

Algebraic approach to representations of semi-simple Lie algebras: use Casimir operators and their action on reps. (Harish-Chandra homomorphism) to characterize irreducible reps. Lie algebra cohomology.
New approach: introduce a Clifford algebra, spinors, and take a "square root" of the Casimir. This is a sort of "Dirac operator", use to construct a "Dirac cohomology" that will characterize representations.
Highly recommended reference: *Dirac Operators in Representation Theory*, Huang and Pandzic, Birkhauser 2006.

Idea on sale here: think of BRST as Dirac cohomology

Quantum Mechanics

Quantum mechanics has two basic structures:

I: States

The state of a physical system is given by a vector $|\psi>$ in a Hilbert space $\mathcal{H}$.

II: Observables

To each observable quantity Q of a physical system corresponds a self-adjoint operator O_Q on $\mathcal{H}$. If

$$O_Q|\psi>= q|\psi>$$

(i.e. $|\psi>$ is an eigenvector of O_Q with eigenvalue q) then the observed value of Q in the state $|\psi>$ will be q.

Lie Groups and Lie Algebras

Let G be a Lie group, with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$.

Crucial Example

$G = SU(2)$, the group of 2 by 2 unitary matrices of determinant one.
$\mathfrak{su}(2)$ is trace-less 2 x 2 skew-Hermitian matrices ($X^\dagger = -X$, $tr\ X = 0$).
Physicists like the basis of "Pauli matrices" $\frac{i}{2}\vec{\sigma}$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Geometric point of view: G is a manifold (S^3 for $SU(2)$), and elements $X \in \mathfrak{g}$ are left-invariant vectors fields on G, i.e. left-invariant first-order differential operators.

Universal Enveloping Algebra

The universal enveloping algebra $U(\mathfrak{g})$ is the associative algebra over **C** of left-invariant differential operators (of ALL orders).

Representations and Quantum Mechanics

A representation of a Lie algebra $\mathfrak{g}$ is just a module for the algebra $U(\mathfrak{g})$. Such a module is given by a complex vector space V, and a homomorphism $\pi : \mathfrak{g} \rightarrow End(V)$ satisfying

$$[\pi(X_1), \pi(X_2)] = \pi([X_1, X_2])$$

Explicitly, for each $X \in \mathfrak{g}$ we get a linear operator $\pi(X)$ and π takes the Lie bracket to the commutator. If V is a Hilbert space, and π is a unitary representation, then the $\pi(X)$ are skew-Hermitian operators (these exponentiate to give unitary operators on V).

Implications

- To any unitary representation (π, V) of a Lie algebra $\mathfrak{g}$ corresponds a quantum system with state space V and algebra of observables $U(\mathfrak{g})$.
- Any quantum system with a symmetry group G gives a unitary representation of $\mathfrak{g}$

Examples

- Translations in space: $G = \mathbf{R}^3$. $\vec{X} \in \mathfrak{g} = \mathbf{R}^3$ acts as the momentum operators $\pi(\vec{X}) = i\vec{\mathbf{P}}$. Eigenvalues of $\vec{\mathbf{P}}$ are $\vec{p}$, components of the momentum vector.
- Translation in time: $G = \mathbf{R}$. $X \in \mathfrak{g} = \mathbf{R}$ acts as the energy (Hamiltonian) operator $\pi(X) = iH$. Eigenvalue is the energy E. This is the Schrödinger equation $i\frac{d}{dt}|\psi >= H|\psi >$.
- Rotations in space: $G = Spin(3) = SU(2)$, $\vec{X} \in \mathfrak{so}(3) \simeq \mathfrak{su}(2)$ acts as the angular momentum operators $\pi(\vec{X}) = i\vec{\mathbf{J}}$. Note: the components of $\vec{\mathbf{J}}$ are not simultaneously diagonalizable, can't characterize a state by an angular momentum vector.
- Phase transformations of $|\psi >$: $G = U(1)$ and there is an integral lattice of points in $\mathfrak{g} = \mathbf{R}$ that exponentiate to the identity in G. If L is a generator of this lattice, on a 1-d representation, $\pi(L)|\psi >= i2\pi Q|\psi >$ for some integer Q, the "charge" of the state.

Semi-Simple Lie Algebras and Casimir Operators

For $G = U(1)$ irreducible representations are one dimensional, characterized by an integer Q ("weight" to mathematicians, "charge" to physicists). What about larger, non-Abelian Lie groups such as $SU(2)$?

Casimir Operators

For semi-simple Lie groups G, one has a bi-invariant quadratic element $\mathcal{C}$ in $U(\mathfrak{g})$ called the "Casimir element". It is in the center of $U(\mathfrak{g})$, so it acts by a scalar on irreducible representations (Schur's lemma).

Example: $SU(2)$

For $G = SU(2)$, the Casimir element is $J_1^2 + J_2^2 + J_3^2$ and its eigenvalues are given by $j(j+1)$, where j is called the "spin" and takes values $j = 0, 1/2, 1, 3/2, \ldots$.

Note: as a second-order differential operator on the manifold G, the Casimir is just the Laplacian operator.

Gauge Symmetry

Geometrical framework of modern quantum field theory:

- A principal bundle P over 3+1d spacetime M, with fiber a Lie group G (in Standard Model, $G = SU(3)xSU(2)xU(1)$).
- Connections A on P (physics interpretation: a field with photons, gluons, etc. as quanta). The space $\mathcal{A}$ is an inf. dim. linear space.
- Vector bundles on M, constructed from P and a representation of G. Sections of these vector bundles are field with quanta electrons, quarks, neutrinos, etc.

The gauge group $\mathcal{G}$

The infinite dimensional group of automorphisms of P that preserve the fibers is called the gauge group $\mathcal{G}$. Locally on M, $\mathcal{G} = Map(M, G)$. It acts on the above structures.

Conventional Wisdom

By general philosophy of quantum systems, the state space $\mathcal{H}$ of a gauge theory should carry a unitary representation of $\mathcal{G}$.

Mathematical Problem

For $dim(M) > 1$, little is known about the irreducible unitary reps of $\mathcal{G}$.

Conventional Wisdom

This doesn't matter. $\mathcal{G}$ is just a symmetry of change of coordinates on a geometric structure, all physics should be invariant under this. The physical states should just transform as the trivial representation of $\mathcal{G}$. The observable corresponding to the Lie algebra of $\mathcal{G}$ should be an operator **G** that gives "Gauss's law": $\mathbf{G}|\psi>= 0$ for every physical state.

The BRST Formalism

In quantum field theory, one would like to have a "covariant" formalism, with space and time treated on the same footing, the Lorentz symmetry group $SO(3,1)$ acting manifestly. But in such a formalism, one can show that there are no invariant states: $\mathbf{G}|\psi >= 0 \Rightarrow |\psi >= 0$

Gupta-Bleuler, 1950

Write $\mathbf{G} = \mathbf{G}^+ + \mathbf{G}^-$, only enforce $\mathbf{G}^+|\psi >= 0$. This requires that $\mathcal{H}$ have an indefinite inner product. A suitable decomposition exists for $U(1)$ gauge theory, but not for the non-abelian case.

BRST (Becchi-Rouet-Stora-Tyutin), 1975

Extend $\mathcal{H}$ to $\mathcal{H} \otimes \Lambda^*(Lie\ \mathcal{G})$. Construct operators Q, N on this space such that N has integer eigenvalues and gives a $\mathbf{Z}$ grading, and Q is a differential: $Q^2 = 0$. Then define the physical Hilbert space as the degree-zero part of

$$\mathcal{H}^{phys} = KerQ/ImQ$$

A Toy Model: Gauge theory in 0+1 dimensions

Simplest possible situation: 0 space and 1 time dimensions ($M = S^1$). G a connected simple compact Lie group, V an irreducible representation.
Physical interpretation: quantum mechanics of the internal degrees of freedom of an infinitely massive particle, coupled to a gauge field.

Connections on S^1

For G connected, all principal G-bundles P on S^1 are trivial. Choose a trivialization $P \simeq G \times S^1$.
Gauge-transformations: changes of trivialization, $\phi \in Maps(S^1, G)$.
Connections: Use trivialization to pull-back connections from P to S^1, then $A(t) \in Map(S^1, \mathfrak{g})$. Under a gauge transformation

$$A \to \phi^{-1} A \phi + \phi^{-1} \frac{d}{dt} \phi$$

Choosing a Gauge

Standard tactic in quantum gauge theory: use gauge freedom to reduce to a smaller set of variables before "quantizing", imposing some "gauge condition".

In our toy model, can always find a gauge transformation that makes $A(t) = const.$, so impose $\frac{d}{dt}A(t) = 0$.

Residual Gauge Symmetry

In this gauge, space of connections is just $\mathfrak{g}$, gauge transformations are $\phi \in G$ acting on connections by the adjoint action

$$A \to \phi^{-1} A \phi$$

Want to fix gauge by demanding $A \in \mathfrak{t}$ (Lie algebra of maximal torus), no component in $\mathfrak{g}/\mathfrak{t}$.

Complex Semi-simple Lie algebras

Under the adjoint action $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus T. T acts trivially on $\mathfrak{t}$, non-trivially (by "roots") on $\mathfrak{g}/\mathfrak{t} \otimes \mathbf{C}$ and

$$\mathfrak{g}/\mathfrak{t} \otimes \mathbf{C} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

$\mathfrak{n}^+$ is the sum of the "positive root spaces", a Lie subalgebra of $\mathfrak{g} \otimes \mathbf{C}$. $\mathfrak{g}/\mathfrak{t}$ is a real even dim. vector space, which can be given various invariant complex structures, one for each choice of "positive" roots. The set of these choices is permuted by the Weyl group W.

Example: $SU(2)$

$$\mathfrak{t} = span \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathfrak{n}^+ = span \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathfrak{n}^- = span \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$W = \mathbf{Z}_2$, interchanges $\mathfrak{n}^+ \leftrightarrow \mathfrak{n}^-$

Highest Weight Theory of $\mathfrak{g}$ Representations

Given a choice of $\mathfrak{n}^+$, finite dimensional representations V are classified by

$$V^{\mathfrak{n}^+} = \{v \in V : \pi(X)v = 0 \ \forall X \in \mathfrak{n}^+\}$$

For V irreducible, $V^{\mathfrak{n}^+}$ is 1-dimensional, carries a representation of $\mathfrak{t}$, the "highest weight".

Irreducible reps. can be characterized by

- Eigenvalue of the Casimir operator $\mathcal{C}$. Indep. of choice of $\mathfrak{n}^+$
- Highest weight. Depends on choice of $\mathfrak{n}^+$.

Example: $SU(2)$

Irreducible spin j rep. labeled by

- $j = 0, 1/2, 1, 3/2, \ldots$, the eigenvalue of $\pi(\frac{i}{2}\sigma_3)$ on $V^{\mathfrak{n}^+}$
- $j(j+1)$, the eigenvalue of $\mathcal{C}$ on V.

Lie Algebra Cohomology and BRST

Motivation for homological algebra: study a module V over an algebra by replacing it by a complex of free modules using a "resolution"

$$0 \rightarrow (\cdots F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0) \xrightarrow{\partial} V \rightarrow 0$$

where $\partial^2 = 0$ and $Im\ \partial = Ker\ \partial$ (an "exact sequence").
Apply $(\cdot) \rightarrow (\cdot)^{\mathfrak{g}} = Hom_{U(\mathfrak{g})}((\cdot), \mathbf{C})$ to this resolution, get a new sequence, with differential d induced by ∂.
New sequence is not exact: $Ker\ d/Im\ d = H^*(\mathfrak{g}, V)$.
Using a standard resolution (Chevalley-Eilenberg) of V,

BRST cohomology $Ker\ Q/Im\ Q$ is just Lie algebra cohomology $H^*(\mathfrak{g}, V)$

Note: BRST wants just $V^{\mathfrak{g}} = H^0(\mathfrak{g}, V)$, but we get more.

Examples

Lie algebra $\mathfrak{g}$ of a compact Lie group G

$H^*(\mathfrak{g}, \mathbf{C}) = H^*_{deRham}(G)$ (for $G = SU(2)$, $H^3(\mathfrak{g}, \mathbf{C}) = H^0(\mathfrak{g}, \mathbf{C}) = \mathbf{C}$).
$H^*(\mathfrak{g}, V) = H^*(\mathfrak{g}, C) \otimes V^{\mathfrak{g}}$

$\mathfrak{n}^+$-cohomology

(Bott-Kostant) get highest weight space $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+}$, but also, for each $w \in W, \neq 1$, copies in $H^{l(w)}(\mathfrak{n}^+, V)$.
The Euler characteristic $\chi(H^*(\mathfrak{n}^+, V))$ (alternating sum of H^*) does not depend on the choice of $\mathfrak{n}^+$. As a T representation

$$\frac{\chi(H^*(\mathfrak{n}^+, V))}{\chi(H^*(\mathfrak{n}^+, \mathbf{C}))}$$

gives the Weyl character formula for the character of V.

Dirac Operators

Discovery of Dirac:

Given a Laplacian Δ, take its "square root", a Dirac operator $D : D^2 = \Delta$

To get a square root of

$$\Delta = -(\frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2})$$

need to introduce γ_i that satisfy $\gamma_i^2 = -1,\ \gamma_i\gamma_j = -\gamma_j\gamma_i$. These generate a Clifford algebra $Cliff(\mathbf{R}^n)$.

Then the Dirac operator is

$$D = \sum_{i=1,n} \gamma_i \frac{\partial}{\partial x_i}$$

Index theory is based on the Dirac operator, which generates all topological classes of operators (Bott periodicity).

Can find a Dirac operator on G such that $D^2 = \mathcal{C} + const.$. Gives analog of $H^*(\mathfrak{g}, V)$

The Dirac Operator: case of $\mathfrak{g}/\mathfrak{t}$

More interesting case, algebraic analog of Dirac operator on G/T. Will give analog of $H^*(\mathfrak{n}^+, V)$.

Clifford algebra $Cliff(\mathfrak{g}/\mathfrak{t})$

Algebra generated by $\{\gamma_i\}$ an orthonormal basis of $\mathfrak{g}/\mathfrak{t}$, relations $\gamma_i\gamma_j = -\gamma_j\gamma_i$ for $i \neq j$, $\gamma_i^2 = -1$.

Spinor space S

Unique irreducible module for $Cliff(\mathfrak{g}/\mathfrak{t})$.
Choosing a complex structure on $(\mathfrak{g}/\mathfrak{t})$, i.e. $(\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C} = \mathfrak{n}^+ + \mathfrak{n}^-$,
$Cliff(\mathfrak{g}/\mathfrak{t}) \otimes \mathbf{C} = End(\Lambda^*(\mathfrak{n}^+))$
$S \simeq \Lambda^*(\mathfrak{n}^+)$ up to a projective factor ("$\sqrt{\Lambda^{top}(\mathfrak{n}^-)}$").

(Kostant) One can construct an appropriate $D \in U(\mathfrak{g}) \otimes Cliff(\mathfrak{g}/\mathfrak{t})$, acts on $V \otimes S$.

Dirac Cohomology

Two versions of "Dirac Cohomology", on operators and on states.

Operators

D acts on $U(\mathfrak{g}) \otimes \mathit{Cliff}(\mathfrak{g}/\mathfrak{t})$ by $\mathbf{Z}_2$-graded commutator ($da = [D, a]$).
$D^2 = \mathcal{C} + \mathit{const}.$ is central so $d^2 = 0$. $\mathit{Ker}\ d/\mathit{Im}\ d = U(\mathfrak{t})$.

States

D acts on $V \otimes S$. One can define the Dirac cohomology $H_D(V)$ as $\mathit{Ker}\ D/(\mathit{Im}\ D \cap \mathit{Ker}\ D)$, even though $D^2 \neq 0$. Actually, in this case, D is skew self-adjoint, so $\mathit{Im}\ D \cap \mathit{Ker}\ D = 0$, and $H_D(V) = \mathit{Ker}\ D$.

New Features of Dirac Cohomology

- On operators, get a genuine differential ($d^2 = 0$), on states D^2 not zero, but an element of the center of $U(\mathfrak{g})$.
- Everything is $\mathbf{Z}_2$-graded, not $\mathbf{Z}$-graded
- Does not depend on a choice of complex structure on $\mathfrak{g}/\mathfrak{t}$
- $\mathfrak{g}/\mathfrak{t}$ is not a subalgebra. Dirac cohomology can be defined in contexts where there is no Lie algebra cohomology.

Generalization

Pick ANY orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, $\mathfrak{r}$ a subalgebra, but $\mathfrak{s}$ not. Then can define a Dirac operator $D \in U(\mathfrak{g}) \otimes \mathit{Cliff}(\mathfrak{s})$.

Relation to Hecke Algebras?

A general context for Dirac cohomology:
In many cases of interest in representation theory, given a group G and a subgroup K, one can characterize G representations V by considering V^K, the subspace of K-invariants, together with an action on V^K by an algebra $Hecke(G, K)$, a Hecke algebra.

Examples

- $Hecke(G(\mathbf{F}_q), B(\mathbf{F}_q))$, G an algebraic group, B a Borel subgroup
- $Hecke(G(\mathbf{Q}_p), G(\mathbf{Z}_p)) \simeq Rep_{\mathbf{C}} G^\vee$ Satake isomorphism
- $Hecke(G(\mathbf{F}_p((t)), G(\mathbf{F}_p[[t]]))$

Replacing F_p by $\mathbf{C}$ in the last one is part of the geometric Langlands story. Representations of loop groups parametrized by connections appear, as well as a version of BRST....

Conclusions

Work in progress....
Possible applications:

- Better non-perturbative understanding of how to handle gauge symmetry in quantum gauge theories, including in 3+1 dimensions.
- Insight into the correct framework for studying representations of gauge groups in $d > 1$.
- Geometric Langlands?