Topics in Representation Theory: Fourier Analysis and the Peter Weyl Theorem

1 Fourier Analysis, a review

We'll begin with a short review of simple facts about Fourier analysis, before going on to interpret these in terms of representation theory of the group $G = U(1)$.

Consider the space of complex-valued functions on $\mathbf{R}$, periodic with period 2π, or, equivalently, the space of complex valued functions on the circle S^1. One can consider various possible classes of such functions, a convenient one to choose is $L^2(S^1)$, the (Lebesgue) square-integrable functions on the circle. Any such function $f(\theta)$ has a convergent Fourier expansion with coefficients $a_n \in \mathbf{C}$ (another common notation is $\hat{f}(n)$, for the n'th coefficient of the Fourier expansion of f)

$$f(\theta) = \sum_{n=-\infty}^{n=+\infty} a_n e^{in\theta}$$

with

$$\sum_{n=-\infty}^{n=+\infty} |a_n|^2 < \infty$$

We can interpret this in terms of group representation theory, for the group $G = U(1)$, which is just S^1, acting on itself by rotation. Group elements are given explicitly by $g = e^{i\theta}$ and group multiplication is just complex multiplication. Since the group is Abelian, all irreducible representations are one-dimensional. They are indexed by an integer n and given by

$$\pi_n(e^{i\theta}) = e^{in\theta} \in U(1) \subset GL(1, \mathbf{C})$$

Just like in the finite group case, on any $U(1)$ representation, we can construct a $U(1)$-invariant inner product by picking any particular one, and then averaging over the group. In this case we average over the group not by taking a sum, but by integrating, i.e.

$$\frac{1}{|G|}\sum_{g\in G} \rightarrow \frac{1}{2\pi}\int_0^{2\pi}$$

The same argument as in the finite group case shows that $U(1)$ representations are completely reducible. The representation ring is just $R(U(1)) = \mathbf{Z}$, and the characters of the irreducible representations are just

$$\chi_n(e^{i\theta}) = e^{in\theta}$$

The space of characters is the space of all functions on the group, since it is Abelian so all functions are conjugation invariant. The characters are orthonormal with respect to the invariant integral on the group:

$$< \chi_n, \chi_m >= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{im\theta} = \begin{cases} 0 & if\ n \neq m \\ 1 & if\ n = m \end{cases}$$

The analog of the (left) regular representation here is the action on $L^2(S^1)$ given by translation:

$$\pi(e^{i\theta})f(\theta_0) = f(\theta_0 - \theta)$$

(the analog of the right regular representation is essentially the same, except shifting by a positive angle, so there's not much use in considering, $U(1)\times U(1)$, i.e. both the right and left actions in this case.)

Decomposing $L^2(S^1)$ into irreducible representations corresponds to the fact that

$$L^2(S^1) = \hat{\oplus}_n V_n$$

Here

$$V_n = \{f(\theta_0) : f(\theta_0 - \theta) = e^{in\theta}f(\theta_0)\} = \mathbf{C}e^{-in\theta_0}$$

and the direct sum $\oplus_n V_n$ is the space of "trigonometric polynomials" (finite linear combinations of $e^{in\theta}$), and we are taking the "completed direct sum", the completion in the inner product on the Hilbert space $L^2(S^1)$. This includes infinite sums, with coefficients a_n satisfying $\sum_n |a_n|^2 < \infty$. Other function spaces like $C^\infty(S^1)$ correspond to other conditions on the coefficients.

The Stone-Weierstrass theorem says we can uniformly approximate continuous or L^2 functions on S^1 by trignometric polynomials. The space of trigonometric polynomials is dense in $C^0(S^1)$ and thus in $L^2(S^1)$.

A standard notation for the set of isomorphism classes of irreducible unitary representations of a group G is $\hat{G}$. So we have seen that $\widehat{U(1)} = \mathbf{Z}$. The ring structure on this corresponding to tensor product of representations is just the standard ring structure on $\mathbf{Z}$. One can associate to the irreducible representation with character $\chi_n(\theta) = e^{in\theta}$ the monomial z^n. Then the representation ring $R(U(1))$ is the ring one gets by taking sums of these with integral coefficients, i.e.

$$R(U(1)) = \mathbf{Z}[z, z^{-1}]$$

For Abelian groups, $\hat{G}$ is itself an Abelian group, and one can show (Pontryagin duality) that $\hat{\hat{G}} = G$. In this case this corresponds to the fact that unitary representations of $\mathbf{Z}$ are determined by choosing the phase $e^{i\theta}$ by which $1 \in \mathbf{Z}$ acts. For general Abelian groups, if G is compact, $\hat{G}$ will be discrete, and vice-versa. Some groups that are "self-dual", such that $G = \hat{G}$, include $G = \mathbf{Z}_n$ and $G = \mathbf{R}$.

2 Convolution

There is another interesting product that one can define on $L^2(S^1)$, besides the usual point-wise multiplication:

Definition 1 (Convolution)**.** *The convolution of two functions f_1 and f_2 in $L^2(S^1)$ is the function*

$$f_1 * f_2 = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta - \theta')f_2(\theta')d\theta'$$

This product is commutative and associative (the generalization we will see later will be commutative only for commutative groups).

One use of the convolution product is to construct an orthogonal projection

$$L^2(S^1) \rightarrow V_n$$

using convolution with an irreducible character. It is easy to show that if

$$f = \sum_n a_n e^{in\theta}$$

then

$$f * \chi_n = a_n e^{in\theta}$$

and characters provide idempotents in the algebra $(L^2(S^1), *)$, satisfying

$$\chi_n * \chi_m = \begin{cases} 0 & if\ n \neq m \\ \chi_n & if\ n = m \end{cases}$$

We have

$$f = \sum_n f * \chi_n$$

The construction of the convolution, with the same properties as above, generalizes to the case of non-abelian groups with an invariant integral

Definition 2. *The convolution product on $L^2(G)$ is*

$$(f_1, f_2) \rightarrow f_1 * f_2 = \int_G f_1(gh^{-1})f_2(h)dh$$

With this product, the functions on G become an algebra, called the *Group Algebra*, which is non-commutative when the group is non-commutative. Given a representation (π, V) of G, one can make V into a module over the group algebra, defining an algebra homomorphism

$$\tilde{\pi} : L^2(G) \rightarrow End(V)$$

by

$$\tilde{\pi}v = \int_G f(g)\pi(g)vdg$$

One can check that this satisfies

$$\tilde{\pi}(f_1 * f_2) = \tilde{\pi}(f_1)\tilde{\pi}(f_2)$$

An alternate approach to representation theory of groups is to think of it as the theory of these algebras and their modules. In the finite group case this is especially effective since the algebras are finite-dimensional. As an example, the general structure theory of finite-dimensional algebras over $\mathbf{C}$ shows

$$\mathbf{C}G = \oplus_i M(n_i, \mathbf{C})$$

i.e. that such algebras are sums of matrix algebras. In our case the n_i are the dimensions of the irreducible representations of G.

Finally, it is useful to think of general group representation theory as reflecting two generalizations of the Fourier transform, which differ only in the non-commutative case. These are:

- Conjugation-invariant functions on G can be in expanded in terms of characters:

$$f = \sum_i (dimV_i) f * \chi_{V_i}$$

- Arbitrary functions on G can be expanded in terms of matrix elements of operators on irreducible representations of G. Recall that

$$\mathbf{C}G = \oplus_i V_i^* \otimes V_i = \oplus_i End(V_i)$$

  One can define an operator-valued generalization of the Fourier transform, such that the Fourier transform $\mathcal{F}$ of a function on G will be an operator-valued function on $\hat{G}$:

$$\mathcal{F}f([V_i] = \tilde{\pi}_{V_i}(f) = \int_G f(g)\pi_{V_i}(g)dg$$

  As in the case of the usual Fourier transform, it may be convenient to normalize this differently, for instance by multiplying by a factor of $\sqrt{dimV_i}$.

3 Compact Lie Groups and the Peter-Weyl Theorem

The arguments we gave concerning finite-dimensional unitary representations of finite groups all continue to hold in the case of general compact Lie groups, since they rely only on the existence of an invariant measure and the possibility of averaging over the group using it. In particular, just changing

$$\frac{1}{|G|}\sum_{g \in G} \rightarrow \int_G$$

we can derive the following facts about finite-dimensional unitary representations of compact Lie groups:

- These representations are completely reducible (direct sums of irreducible representations).
- Given two irreducible representations V_1, V_2, the intertwiners satisfy:

  $Hom_G(V_1, V_2) = \{0\}$ (the zero map) if V_1 is not isomorphic to V_2.

  $Hom_G(V_1, V_2) = \mathbf{C}$ if V_1 is isomorphic to V_2.
- Canonical decomposition theorem:

$$\mu = \oplus_i \mu_i : \oplus_i Hom_G(V_i, V) \otimes V_i \rightarrow V$$

  is an isomorphism.
- There is an inner product on $R(G)$, with respect to which irreducible representations are orthonormal

$$< V, W >= dim\ Hom_G(V, W)$$

- Representations can be studied using their characters, with the characters of irreducible representations orthonormal with respect to the inner product

$$< \chi_V, \chi_W >= \int_G \overline{\chi_V(g)} \chi_W(g) dg$$

- Matrix elements of irreducible representations give functions on G, orthogonal with respect to the inner product above.

The one place where the proofs used fail in the general compact Lie group case is that the regular representation is no longer finite dimensional. As a result one cannot just use complete reducibility and character computations to show that the regular representation decomposes as a $G \times G$ representation in the form

$$\mathbf{C}(G) = \oplus_i (V_i^* \otimes V_i)$$

We can replace $\mathbf{C}(G)$ by the function space $L^2(G)$, but the (left) regular representation is now no longer finite dimensional, and many of our arguments don't directly apply to infinite-dimensional representations. One can see this for example if one tries to take the character of the representation. In the case $G = U(1)$, we saw that the irreducible representations V_n are labeled by the integers, and have characters $\chi_n(\theta) = e^{in\theta}$. The regular representation $(\pi, L^2(G))$ should have a character

$$\chi_\pi(\theta) = \sum_n \chi_n(\theta) = \sum_n e^{in\theta}$$

but this sum does not converge to a function on G. It does converge to the distribution $\delta(0)$, the "delta function" at $\theta = 0$, and in general for infinite-dimensional representations, characters will be distributions: one needs to consider the distributional character

$$\chi_\pi(f) = Tr(\int_G f(g)\pi(g)dg)$$

As a distribution, the character of the regular representation will be a delta function at the identity

$$\chi_\pi(f) = f(e)$$

Just as in the finite group case, for each irreducible representation (π_i, V_i), with character $\chi_i(g)$, we can produce a $(dim\ V_i)^2$ subspace of $L^2(G)$. One way to think of this is as the set of all functions of the form

$$l(\pi_i(g)v),\ v \in V_i, l \in V_i^*$$

or, equivalently, all functions of the form

$$tr(\pi_i(g)X),\ X \in End(V_i)$$

The map

$$X \in End(V_i) \to tr(\pi_i(g)X) \in L^2(G)$$

embeds $End(V_i) = V_i^* \otimes V_i$ as a subspace of L^2. One can show that under this identification of $End(V_i)$ with a space of functions on G, matrix multiplication becomes convolution of functions. Orthogonal projection onto this subspace is given by convolution with the character

$$f \to f * \chi_i$$

and one can see that the identity map in $End(V_i)$ gets identified with the function $\chi_i(g)$.

The subspaces of functions corresponding to different V_i are orthogonal subspaces of $L^2(G)$. Taking their direct sum gives what is sometimes called the space of *representative functions*

$$C_{rf}(G) = \oplus_i End(V_i)$$

This is a subspace of $L^2(G)$, and it is a generalization of the space of trigonometric polynomials in the case $G = U(1)$. An obvious question is whether $L^2(G)$ is the completion of $C_{rf}(G)$, or whether something is missing. One might worry that there is an infinite dimensional irreducible representation of G whose matrix elements would be in $L^2(G)$, but orthogonal to $C_{rf}(G)$. The content of the following theorem is that this doesn't happen. For compact Lie groups, the situation is as close to that of finite groups as one can hope, with the matrix elements of finite-dimensional representations giving a basis of $L^2(G)$.

Theorem 1 (Peter-Weyl)**.** *The matrix elements of finite dimensional irreducible representations form a complete set of orthogonal vectors in* $L^2(G)$*.*

Equivalently, this theorem says that every $f \in L^2(G)$ can be written uniquely as a series

$$f = \sum_i f_i, \;\; f_i \in End(V_i)$$

which we can also write

$$L^2(G) = \hat{\oplus}_i End(V_i) = \hat{\oplus}_i (V_i^* \otimes V_i)$$

There's also an easy corollary, which says that one can expand any conjugation invariant function in terms of characters of irreducible representations:

Corollary 1. *The characters of finite dimensional irreducible reps of G give an orthonormal basis of $L^2(G)^G$ (the conjugation invariant subspace of $L^2(G)$.*

The difficult part of the Peter-Weyl theorem is to show that $C_{rf}(G)$ is dense in $L^2(G)$. If one assumes that one's compact Lie group is a group of matrices, a subgroup of $GL(n, \mathbf{C})$ for some n, then one can use the Stone-Weierstrass theorem. One just needs to show that polynomial functions on G are in $C_{rf}(G)$, then Stone-Weierstrass implies that $C_{rf}(G)$ is dense in the continuous functions on G, and thus in $L^2(G)$.

There is a much trickier proof of Peter-Weyl that works for any compact Lie group. One reason for using it is that you can then derive the existence of a finite dimensional faithful representation of G, which implies that any compact Lie group G must be a subgroup of some $GL(n, \mathbf{C})$. This proof is based on the construction of a suitable compact self-adjoint operator (basically a conjugation-invariant approximation to convolution by a delta-function), and use of the spectral theorem for compact self-adjoint operators on a Hilbert space, which implies finite-dimensionality of the eigenspaces of the operator. For a more extensive sketch of the proof of Peter-Weyl, see chapter 9 of [1]. For more detailed proofs, see section 3.3 of [2], or chapter III of [3]. Chapter 3 of [2] provides a much more detailed discussion of harmonic analysis on compact Lie groups, along the lines outlined here.

References

[1] Segal, G., Lie Groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.

[2] Sepanski, M., *Compact Lie Groups,* Springer-Verlag, 2006.

[3] Brocker, T. and tom Dieck, T., *Representations of Compact Lie Groups,* Springer-Verlag, 1985.