Topics in Representation Theory: Finite Groups and Character Theory

This semester we'll be studying representations of Lie groups, mostly compact Lie groups. Some of the general structure theory in the compact case is quite similar to that of the case of finite groups, so we'll begin by studying them. We will not go very far into the theory of finite groups, mainly due to lack of time. For more details, see [1], [2] and [3]. In the case of connected compact Lie groups we will be able to uniformly construct and classify representations, but representations of finite groups are inherently more complicated, with no uniform way to classify them. So, for now we'll stick to general features of the theory, with a small number of examples. Much later in the course I hope to discuss Schur-Weyl theory, which, for the case of one class of finite groups, the symmetric groups, does provide a uniform way to construct and classify representations.

1 Representations of Finite Groups, Generalities

In this course we will stick to the case of complex representations, i.e. representations on complex vector spaces.

Definition 1 (Representation). *A representation (π, V) of a group G on a vector space V is a homomorphism*

$$\pi : G \to GL(V)$$

where $GL(V)$ is the group of invertible linear transformations of V.

For now we will just be considering vector spaces V of finite dimension. Note that we will sometimes refer to a representation by just specifying π or V.

If V is an n-dimensional complex vector space and we choose a basis, a representation is given explicitly by, for each $g \in G$, an n by n invertible complex matrix, with entries $[\pi(g)]_{ij}$, satisfying

$$\sum_j [\pi(g_1)]_{ij}[\pi(g_2)]_{jk} = [\pi(g_3)]_{ik}$$

when $g_1 g_2 = g_3$.

Given any representation, one can look for subrepresentations. These are smaller representations contained in the representation, i.e. proper subspaces of V left invariant by the action of the group. An *irreducible representation* will be a representation with no proper subrepresentations, if there is a proper subrepresentation, the representation is called *reducible*. The main goal of representation theory will be to understand these irreducible representations, as well as how to decompose an arbitrary representation in terms of them. A group G is said have the property of *complete reducibility* if any representation can be decomposed into a direct sum of irreducible representations.

Theorem 1 (Complete Reducibility)**.** *Finite groups G have the property of complete reducibility.*

Proof. Given any (positive-definite, Hermitian) inner product $< \cdot, \cdot >_0$ on V, one can form a G-invariant inner product by averaging over the G action

$$< v_1, v_2 >= \frac{1}{|G|} \sum_{g \in G} < \pi(g)v_1, \pi(g)v_2 >_0$$

If V is not already irreducible, one can pick a subrepresentation W. Then one can check that its orthogonal complement $W^{\perp}$ will also be a subrepresentation and $V = W \oplus W^{\perp}$.

If W or $W^{\perp}$ are not irreducible, one can decompose them into direct sums of subrepresentations, continuing this process until one has expressed V as a direct sum of irreducibles. □

Note that this proof just relies on the possibility of constructing an invariant inner product. The same argument works for compact Lie groups. Another implication of the existence of such an invariant inner product is that these representations of finite groups are unitary: the representation π is a homomorphism of G into a subgroup $U(n) \subset GL(V)$, the subgroup preserving the inner product.

We will be using the convention

$$< z, w >= \sum_i \overline{z_i} w_i$$

Note that this is the same convention used by Hall, but other books, including Serpanski, use a different convention, conjugating the second variable.

One of our main goals is to classify all representations of G. In doing this, we don't want to distinguish between representations that differ just by a change of basis, i.e. by conjugation in $GL(V)$:

Definition 2 (Equivalence of Representations)**.** *Two representations π_1 and π_2 on a vector space V are said to be equivalent if they are related by conjugation, i.e.*

$$\pi_2(g) = h(\pi_1(g))h^{-1}$$

for $h \in GL(V)$.

The set of n-dimensional isomorphism classes of representations will be given by taking the quotient of the set $Hom(G, GL(n, \mathbf{C}))$ by this conjugation action of $GL(n, \mathbf{C})$.

For many purposes it is a good idea to think of the set of representations of G not just as a set, but as a category. If we take the objects of this category to be the isomorphism classes of representations, the morphisms in the category are not arbitrary linear maps between the representation space, but G-equivariant maps known as *intertwining operators.*

Definition 3 (Intertwining Operators)**.** *Given two representations (π_1, V_1) and (π_2, V_2) of G, the space of intertwining operators is the space $Hom_G(V_1, V_2)$ of linear maps $\phi : V_1 \to V_2$ satisfying*

$$\phi \circ \pi_1(g) = \pi_2(g) \circ \phi$$

The structure of the category of representations of a finite group is rather simple since the intertwiners between irreducibles satisfy the following property

Theorem 2. *Given two irreducible representations V_1, V_2 of a finite group G, the intertwiners satisfy*

- *$Hom_G(V_1, V_2) = \{0\}$ (the zero map) if V_1 is not isomorphic to V_2.*
- *$Hom_G(V_1, V_2) = \mathbf{C}$ if V_1 is isomorphic to V_2.*

The first part of the theorem follows from the observation that the kernel and image of an intertwining map are both invariant subspaces. They can't be proper subspaces since V_1 and V_2 are irreducible. So the only possibility is for the map to be either zero or an isomorphism. The second part of the theorem is known as Schur's lemma, and is equivalent to the following:

Lemma 1 (Schur's Lemma)**.** *If V is an irreducible representation of a finite group G, then every linear map $\phi : V \to V$ commuting with the action of all elements of g on V is a scalar.*

Proof. One can easily see that the fact that ϕ commutes with G implies that the eigenspace V_λ of V corresponding to an eigenvalue λ of ϕ is invariant under G. By irreducibility $V_\lambda = V$, and ϕ is just multiplication by the scalar λ. □

Note that this argument crucially relies on the fact that we are dealing with complex representations. For real representations ϕ may have no real eigenvalues. The story over a general field K is that $Hom_G(V, V)$ can be seen to be an algebra over K, with multiplication given by composition of maps, and that it must be a finite dimensional division algebra. Schur's lemma corresponds to the fact that $\mathbf{C}$ is the only finite dimensional division algebra over the field $\mathbf{C}$. Over $\mathbf{R}$ there are three finite dimensional division algebras ($\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$), and all of these occur as possible automorphisms of irreducible real representations of finite groups.

Schur's lemma has a wide variety of important corollaries, including:

Corollary 1. *If G is abelian, all its irreducible representations are one-dimensional.*

Proof. For G abelian, every $g \in G$ and every representation (π, V) give elements $\pi(g) \in Hom_G(V, V)$, since the $\pi(g)$ for different g commute. If V is irreducible, these $\pi(g)$ must all be given by scalar multiplication. Then any subspace of V is an invariant subspace, implying the existence of subrepresentations and thus a contradiction if V is not one-dimensional. □

Schur's lemma is also important in the following crucial theorem, that describes how an arbitrary representation decomposes into irreducibles. It tells us that to do this we need to understand two things:

- The irreducible representations (π_i, V_i).
- The spaces of intertwining operators $Hom_G(V_i, V)$.

Sometimes the $Hom_G(V_i, V)$ have no interesting structure other than their dimension as vector spaces, these dimensions are integers called the *multiplicities* n_i of the i'th irreducible in V. Often V will have some other structure, such as the action of another group H, commuting with G. In this case the $Hom_G(V_i, V)$ will provide interesting representations of H.

Theorem 3 (Canonical Decomposition Theorem)**.** *If i is an index varying over a complete set of irreducibles (π_i, V_i) of a finite group G, and*

$$\mu_i : Hom_G(V_i, V) \otimes V_i \to V$$

is the canonical G-map given by

$$\mu_i(f \otimes v_i) = f(v_i)$$

then

$$\mu = \oplus_i \mu_i : \oplus_i Hom_G(V_i, V) \otimes V_i \to V$$

is an isomorphism of G representations (on the left side, G acts trivially on the first factor, as the irreducible representation on the second).

Proof. In general, we know that V can be decomposed into a direct sum of irreducibles V_i, this will take the form

$$V = \oplus_i n_i V_i$$

where $n_i V_i$ is direct sum of n_i copies of V_i. The domain of μ is the same representation since

$$\begin{aligned} \oplus_i Hom_G(V_i, V) \otimes V_i &= \oplus_i Hom_G(V_i, \oplus_j n_j V_j) \otimes V_i && (1) \\ &= \oplus_i (\oplus_j n_j Hom_G(V_i, V_j)) \otimes V_i && (2) \\ &= \oplus_i n_i Hom_G(V_i, V_i) \otimes V_i && (3) \\ &= \oplus_i n_i (\mathbf{C} \otimes V_i) = \oplus_i n_i V_i && (4) \end{aligned}$$

where we used Schur's lemma to get the third and fourth equalities. Again by Schur's lemma, μ is an isomorphism when V is an irreducible V_i, and this remains true when $V = n_i V_i$. A final application of Schur's lemma shows that μ must take $n_i V_i$ to $n_i V i$. □

Note that this theorem implies that the multiplicities are uniquely determined. If one knows that $V = \oplus_i n_i V_i$ and $\oplus_i m_i V_i$, one must have $n_i = m_i = dim\ Hom_G(V_i, V)$.

Given two representations (π_1, V_1) and (π_2, V_2), besides forming the direct sum representation $V_1 \oplus V_2$, one can also form tensor product representations $V_1 \otimes V_2$, as well as $Hom(V_1, V_2)$. The group acts on the tensor product in the obvious way by $\pi_1 \otimes \pi_2$. It acts on $f \in Hom(V_1, V_2)$ by

$$(\pi(g)f)(v) = \pi_2(g)(f(\pi_1(g^{-1})v))$$

This is the action that makes the following diagram commute:

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \pi_1(g)}\downarrow & & \downarrow{\scriptstyle \pi_2(g)} \\ V_1 & \xrightarrow{\pi(g)f} & V_2 \end{array}$$

An important special case of the construction $Hom(V_1, V_2)$ is to take $V = V_1, V_2 = \mathbf{C}$, giving the dual representation $(\pi_{V^*}, V^* = Hom(V, \mathbf{C})$, with $\pi_{V^*}(g) = \pi_V(g^{-1})$.

One can define an algebraic gadget that captures much of the structure of the set of irreducible representations, this is the *representation ring*

Definition 4. *Let $R(G)$ be the free abelian group generated by the equivalence classes of irreducible representations of G. $R(G)$ is a ring under the multiplication induced by taking the tensor product of representations.*

An element of $R(G)$ is given by formal linear combinations $\sum_i n_i[V_i]$, where the n_i are integers (possibly zero or negative). These elements are also known as *virtual representations.* The representation ring comes with a natural inner product in which the irreducible representations form an orthonormal basis, using

$$< V_1, V_2 >= dim\ Hom_G(V_1, V_2)$$

The decomposition of any representation V into irreducibles can be computed from knowledge of these inner products. If $V = \sum_i n_i V_i$,

$$n_i =< V_i, V >= dim\ Hom_G(V_i, V)$$

To understand the ring structure of $R(G)$, we need to know how the product of irreducible decomposes into irreducibles, i.e. the structure constants n_{ij}^k defined by

$$V_i \otimes V_j = \sum_k n_{ij}^k V_k$$

Note that the n_{ij}^k can be computed in terms of the inner product

$$n_{ij}^k =< V_k, V_i \otimes V_j >= dim\ Hom_G(V_k, V_i \otimes V_k)$$

2 Character Theory

We would like to have some concrete way of easily distinguishing inequivalent representations, and computing the $dim\ Hom_G(V_i, V)$ that tell us how an arbitrary representation decomposed into irreducibles. This can be done by associating to a representation a function on G called its *character*. To motivate this definition, consider the following proposition:

Lemma 2. *The operator* $e_1^V = \frac{1}{|G|}\sum_{g\in G}\pi(g) : V \to V$ *is idempotent* ($(e_1^V)^2 = e_1^V$)*, equal to the identity on* V^G *(the* G*-invariant component of* V*), and zero on the rest of* V*.*

Proof. We'll show that the image of e_1^V is G-invariant:
For $v \in V,\ g \in G$

$$\begin{aligned} \pi(g)e_1^V v &= \frac{1}{|G|}\pi(g)\sum_{h\in G}\pi(h)v & (5)\\ &= \frac{1}{|G|}\sum_{h\in G}\pi(g)\pi(h)v & (6)\\ &= \frac{1}{|G|}\sum_{g'}\pi(g')v & (7)\\ &= e_1^V v & (8)\end{aligned}$$

and if G acts trivially on v,

$$e_1^V v = \frac{1}{|G|}\sum_{g\in G}(1)v = v$$

□

Using this, one way to compute the multiplicity n of the trivial representation in a representation V is to just take the trace of the operator e_V.

$$n = dim\ V^G = dim\ Hom_G(\mathbf{C}, V) = Tr(e_1^V)$$

Picking a basis of V, the trace is just the trace of the matrix e_1^V, but the trace of a matrix is independent of the basis, it is a conjugation invariant function on invertible matrices, satisfying

$$Tr(UMU^{-1}) = Tr(M)$$

This motivates to some extent the following definition, which associates to any representation a conjugation invariant function on the group, called the *character* of the representation.

Definition 5. *The character of a representation* (π, V) *is the complex function* $\chi_V : G \to \mathbf{C}$ *given by*

$$\chi(g) = Tr(\pi(g))$$

Note that our lemma above tells us that:

$$dim\ V^G = Tr(e_1^V) = \frac{1}{|G|}\sum_{g\in G} Tr(\pi(g)) = \frac{1}{|G|}\sum_{g\in G}\chi(g)$$

Using the properties of the matrix trace, we can quickly see the following facts:

1. $\chi_V(Id) = dim\ V$
2. $\chi_V(hgh^{-1}) = \chi_V(g)$
3. Equivalent representations have the same character.
4. $\chi_{V_1\oplus V_2} = \chi_{V_1} + \chi_{V_2}$
5. $\chi_{V_1\otimes V_2} = \chi_{V_1}\chi_{V_2}$
6. $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$

These are all straightforward from properties of the matrix trace, with 6) following from the fact that we are working with unitary representations and thus unitary matrices. Using these properties of the trace we also have:

$$\chi_{Hom(V,W)} = \chi_{V^*\otimes W} = \overline{\chi_V}\chi_W$$

Finally, we can use these properties of the trace, together with our formula for the dimension of the invariant subspace of a representation to get an explicit formula for the multiplicity of an irreducible V_i, assuming that we know the character of the irreducible χ_{V_i}

$$\begin{aligned} n_i &= dim\ Hom_G(V_i, V) && (9)\\ &= dim\ (Hom(V_i, V))^G && (10)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{Hom(V_i,V)} && (11)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V_i^*\otimes V} && (12)\\ &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}}\chi_V && (13)\end{aligned}$$

The characters give us an explicit formula for the inner product on the representation ring

$$< V, W >= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V}\chi_W$$

The map $[V] \to \chi_V$ extends to an injective ring homomorphism

$$R(G) \to C(G)^G$$

from the representation ring to the ring of conjugation invariant functions (*class functions*) on G.

3 The Regular Representation

While groups act on all sorts of spaces, not just vector spaces, one reason why we restrict our attention in representation theory to group actions on vector spaces is that we can always "linearize" a group action on a space by looking at the induced action on functions on the space. If we are given a group action, acting on the left:

$$(g, m) \in G \times M \to gm \in M$$

and if $C(M)$ is a space of functions on M, we get a representation $(\pi, C(M))$ of G by defining

$$\pi(g)f(m) = f(g^{-1}m)$$

The inverse has to be there in order to make this a homomorphism. One way to see this is that given a map $\phi : M_1 \to M_2$, the induced "pullback" map on functions

$$\phi^* : f \in C(M_2) \to (\phi^* f)(m) = f(\phi(m)) \in C(M_1)$$

goes in the opposite direction. In our case:

$$(\pi(g_1)(\pi(g_2)f))(m) = (\pi(g_2)f)(g_1^{-1}m) = f(g_2^{-1}g_1^{-1}m) = f((g_1g_2)^{-1}m) = (\pi(g_1g_2)f)(m)$$

. One space that the group G always acts on is itself, and the representation one gets on functions is called the *regular* representation.

Definition 6. *The (left) regular representation* $(\pi_L, \mathbf{C}(G))$ *is the representation of* G *on complex valued functions on* G *given by*

$$\pi_L(g)f(h) = f(g^{-1}h)$$

Note that we could also work with the action of G on itself given by right multiplication. In this case we get what is called the (right) regular representation, given by

$$\pi_R(g)f(h) = f(hg)$$

One can check that using the right action, this is the correct formula to get a homomorphism. Note that the right action commutes with the left action, and what we have is actually a representation $(\pi, \mathbf{C}(G))$ of $G \times G$, with $\pi(g_1, g_2)f(g) = f(g_1^{-1}gg_2)$. For now, we will just just the left action, and consider this as a representation of G.

We would like to decompose the regular representation into irreducibles, and as we have seen, the way to do this is by using characters. First we'll compute the character of the regular representation:

Claim 1. *The character χ_L of the regular representation satisfies:*

$$\chi_L(g) = \begin{cases} 0 & if\ g \neq e \\ |G| & if\ g = e \end{cases}$$

Proof. Whenever we have a group acting by permutations on a set X, the representation π on functions on that set will satisfy

$$\chi_\pi(g) = |X^g|$$

($|X^g|$ is the number of points in the set left fixed by the action of g). To see this, consider the representation as a matrix with respect to a basis $\{\mathbf{e}_x\}$ consisting of functions that are 1 on x, 0 elsewhere. In this basis, $\pi(g)$ has diagonal elements equal to 1 exactly corresponding to those x left fixed by g. Taking the trace just counts these. Applying this argument to the left action of G on itself, we get the proposition. □

What we really want to know is, for each irreducible V_i, the multiplicity $n_i = dim\ Hom_G(V_i, \mathbf{C}(G))$. Computing these using characters we get:

$$\begin{aligned} n_i &= < V_i, \mathbf{C}(G) > \\ &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}(g)}\chi_{\mathbf{C}(G)}(g) \\ &= \frac{1}{|G|}\overline{\chi_{V_i}(e)}|G| \\ &= dim\ V_i \end{aligned}$$

We see that every irreducible V_i occurs in the left regular representation, with multiplicity given by the dimension of the representation. So, as a G representation (with G acting on the left), we have

$$\mathbf{C}(G) = \oplus_i (dim\ V_i)V_i$$

The canonical decomposition theorem tells us that

$$\mathbf{C}(G) = \oplus_i Hom_G(V_i, \mathbf{C}(G)) \otimes V_i$$

so we have learned that

$$dim\ Hom_G(V_i, \mathbf{C}(G)) = dim\ V_i$$

The space $Hom_G(V_i, \mathbf{C}(G))$ is invariant under the action of G we have been using, but recall that there is another copy of G, acting from the right, and its action commutes with the left action of G. Under this right action of G, the V_i term in the tensor product is invariant, but the space of intertwining operators gives a nontrivial representation:

Claim 2. *The right regular representation of G induces an action on $Hom_G(V_i, \mathbf{C}(G))$, equivalent to the representation of G on V_i^*.*

Proof. The details of the proof will be left as an exercise, but here is an outline: If

$$T \in Hom_G(V_i, \mathbf{C}(G))$$

show that

$$(\pi(g)T)(v) = \pi_R(Tv)$$

gives a well-defined action of G on $Hom_G(V_i, \mathbf{C}(G))$, and that this action is isomorphic with the action fo G on V_i^*, with the intertwining isomorphism given by

$$\lambda : T \in Hom_G(V_i, \mathbf{C}(G)) \to \lambda T \in V_i^*$$

where λT is defined by

$$\lambda T(v) = (T(v))(e)$$

(where e is the identity of G). □

This is a special case of something called the Frobenius reciprocity theorem, which we will come to a little bit later in the course.

We have shown that, knowing the irreducible representations of G, the space $\mathbf{C}(G)$ decomposes under the combined $G \times G$ action as

$$\mathbf{C}(G) = \oplus_i (V_i^* \otimes V_i)$$

with one copy of G acting on the V_i, the other on the V_i^*. This is the decomposition of $\mathbf{C}(G)$ into irreducibles as a representation of $G \times G$. In an exercise, you will show that irreducible representations of a product group $G \times H$ are given by tensor products $V \otimes W$, where V is an irreducible representation of G, W is an irreducible representation of H.

This still does not tell us what the irreducible representations are. Note that $V_i^* \otimes V_i = End(V_i)$. Once we do know (π_{V_i}, V_i), we can identify the corresponding subspace of $\mathbf{C}(G)$ as the subspace spanned by the matrix elements in the representation V_i, i.e. all functions of the form

$$l(\pi_{V_i}(g)v),\ v \in V_i, l \in V_i^*$$

It is a standard result found in all the referenced texts on finite group representations that, choosing an orthonormal basis in V_i and dual basis in V_i^*, the elements of the matrices representing π_{V_i} with respect to this basis are orthogonal functions on G, using the inner product

$$< f_1, f_2 >= \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

4 An Example: S_3

Finally, let's work out a simple example, the group S_3 of permutations of a set with three elements. This group has a total of six elements (the identity, three transpositions of order two, and two permutations of order three). The regular representation will be on $\mathbf{C}^6$, and it has to decompose into subrepresentations whose dimensions are squares. They can't all be one-dimensional, since the group is non-abelian, so the decomposition must go as $6 = 1 + 1 + 4$, and there must be two irreducible representations of dimension 1, and one irreducible representation of dimension 2. More explicitly, these representations are

- The trivial representation $(Id, \mathbf{C})$
- The sign representation $(\pi_{sgn}, \mathbf{C})$, given by $\pi_{sgn}(g)v = sgn(g)v$, where $sgn(g) = \pm 1$ is the sign of the permutation.
- An irreducible representation on $\mathbf{C}^2$ constructed as follows: Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis of $\mathbf{C}^3$, with G acting by taking $\mathbf{e}_i$ to $\mathbf{e}_{g(i)}$. On coordinate functions z_i the action is given by $\pi(g)z_i = z_{g^{-1}(i)}$. This action leaves invariant the complex line proportional to $(1, 1, 1)$, as well as the orthogonal subspace
$$V = \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$$
  This is our two-dimensional representation.

Finding the characters, checking orthogonality properties, etc. will be the subject of an exercise.

References

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[3] Simon, B., *Representations of Finite and Compact Groups,*American Mathematical Society, 1996.