Groups and Representations II: Problem Set 4

Due Monday, April 16

Problem 1: If e_i are a basis of the Clifford algebra $C(n)$, show that the $\frac{1}{2}e_ie_j$, for $i < j$, satisfy the same commutation relations as the standard basis $L_{ij} = E_{ij} - E_{ji}$ elements for the antisymmetric n by n matrices. This shows that $Lie(Spin(n))$ is isomorphic to $Lie(SO(n))$.

Problem 2: Using the Clifford algebra, derive formulae for

1. The action of the Lie algebra of $Spin(n)$ on vectors, i.e for infinitesimal rotations in the $i - j$ plane.
2. The action of the group $Spin(n)$ on vectors, i.e. the formula for a rotation by angle θ in the $i - j$ plane.

Problem 3: The $\frac{1}{2}e_{2j-1}e_{2j}$ provide a basis of a Cartan subalgebra in $Lie(Spin(2n))$. Choose a complex structure on $\mathbf{R}^{2n}$ that provides an identification with $\mathbf{C}^n$. Use this to relate the Clifford algebra and the CAR algebra corresponding to a basis of $\mathbf{C}^n$.
Use this CAR algebra to construct explicitly a Lie algebra representation of $Lie(Spin(2n))$ on $\Lambda^*(\mathbf{C}^n)$. Find the weights of this representation, show that it is the sum of two irreducible representations.

Problem 4:

1. Explicitly derive the commutation relations for the Lie algebra of $SL(2, \mathbf{R})$.
2. Show that $Sp(2, \mathbf{R})$ is isomorphic to $SL(2, \mathbf{R})$.
3. Show that the algebra of quadratic polynomials in the variables (p, q) with Poisson bracket under "quantization" becomes the Lie algebra of $SL(2, \mathbf{R})$.

Problem 5: Using the definition of the Bargmann-Fock space given in class, show that the operators $a_i^\dagger$ and a_i corresponding to a basis of $\mathbf{C}^n$ are adjoint operators on this space.